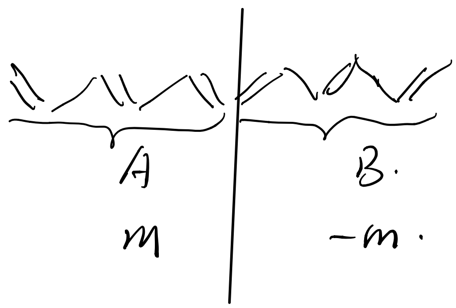
在能隙附近可以写为 $H = p\sigma_x + m\sigma_z$



Ansatz.

$\psi_n = (-1)^n \tanh(\frac{n}{L})$

L, 变分参数.

读Su的三篇文章.

推导.

PRL. 46, 738 (1981)

理解分数化.

$u_n = u\cos(\frac{2}{3}\pi n - \theta)$

$Q = 0, \pm\frac{1}{3}e, \pm\frac{2}{3}e$

Review

Jackiw-Rebbi model

1D Dirac Equation.

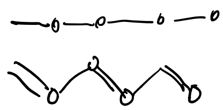
2-degenerate GS, $\pm m \Rightarrow$ topo excitation.

1: Domain

2: $m<0$ topo; $m>0$ trivial

SSH Model

意义：① 第一个 cond-mat 中的模型.



② pierls 相变.

③ 处理方法

④ 电荷分数化

$$H = p\sigma_x + m(x)\sigma_y$$

$$= \begin{pmatrix} 0 & p+im \\ p-im & 0 \end{pmatrix}$$

$$|\psi_1\rangle \xrightarrow{H} |\psi_2\rangle \xrightarrow{H} |\psi_1\rangle$$

$$\begin{cases} H^2 |\psi_1\rangle \to |\psi_1\rangle \\ H^2 |\psi_2\rangle \to |\psi_2\rangle \end{cases}$$

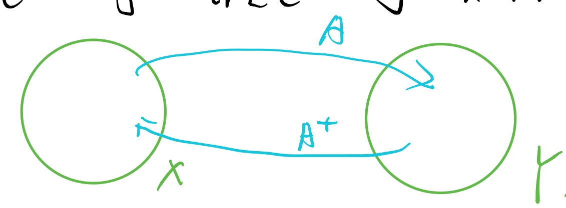
$$H^2 = -\frac{d^2}{dx^2} + \left(\begin{array}{c|c} m^2+m' & 0 \\ \hline 0 & m^2-m' \end{array}\right)$$

SUSY 最后一页

> 阅读材料：
>
> Supersymmetry and the Dirac eq. by F. Cooper (1988)

作业 $tr[e^{-tA^\dagger A}] - tr[e^{-tAA^\dagger}] = n-m$

图象：



$$A = \underbrace{\boxed{}}_{n\times m} \qquad A^{\dagger} = \underbrace{\boxed{}}_{m\times n}$$

$$\begin{cases} AA^{\dagger} : n\times n \\ A^{\dagger}A : m\times m \end{cases} \qquad \begin{cases} X \to X \\ Y \to Y \end{cases}$$

$$X \xrightarrow{A} Y \xrightarrow{A^{\dagger}} X \xrightarrow{A} Y$$

<u>性质</u> $A^{\dagger}A\phi_l = \lambda_l \phi_l$

$$AA^{\dagger}(A\phi_l) = \lambda_l (A\phi_l)$$

$AA^{\dagger}$ 与 $A^{\dagger}A$ 具有相同的本征值，且 ϕ_l 与 $A\phi_l$ 相对应

$A^{\dagger}A\phi \equiv 0$，零模

$$H^2 = -\frac{d^2}{dx^2} + m^2 \pm m'$$

$$= -\frac{d^2}{dx^2} + U_{\pm}(x) = H_{\pm}$$

$$H = \begin{pmatrix} 0 & A \\ A^{\dagger} & 0 \end{pmatrix}$$

$$H^2 = \begin{pmatrix} AA^{\dagger} & 0 \\ 0 & A^{\dagger}A \end{pmatrix}$$

$$\begin{cases} H_+ = AA^{\dagger} \\ H_- = A^{\dagger}A \end{cases}$$

零模的态. $A|\phi\rangle \equiv 0$.

Ansatz $|\psi\rangle = e^{-\int_0^x m(x')dx'}$

$$\begin{cases} \frac{d}{dx}|\psi\rangle = -m\, e^{-\int_0^x m(x')dx'} \\ \frac{d^2}{dx^2}|\psi\rangle = (m^2 - m')|\psi\rangle \end{cases}$$

or $\left(\frac{d}{dx}+m\right)|\psi\rangle = 0$

阅读材料.

① J. Zak PRL, 48, 359 (1982)

Band center

$\langle \vec{x} \rangle = \int a_n^*(\vec{r})\, \vec{r}\, a_n(\vec{r})\, d\vec{r}$

② PRL, 62, 2747 (1989)

Berry phase for Bloch Bands.

☆ Zak Phase. 特指 Bloch Band 1D

Geometry Phase $\vec{R}$, 参数空间

③ PRL, 95, 137205 (2005)

orbital magnetization

Vanderbilt, Resta

电极化 $\Delta\vec{P} = -\vec{r}\Delta\rho$

硼硅酸铜理论.

Schrödinger Equation

☆ $i\hbar\frac{\partial}{\partial t}\psi = \left\{\frac{1}{2m}(\vec{p}-e\vec{A}(t))^2 + V(x)\right\}\psi(x,t)$.

其中 $V(\vec{x}+a) = V(\vec{x})$ 周期

☆ A 缓慢变化, 可以定义每个时刻的本征值.

$\left\{\frac{1}{2m}\left(p-\frac{e}{c}A(t)\right)^2 + V(x)\right\}\psi_{nk} = \varepsilon_{nk}\psi_{nk}$

$\psi_{nk}(x) = e^{ikx}u_{nk}(x)$, $u_{nk}(x)$ 周期为 a.

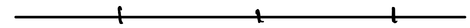
$\psi_{nk}(x+a) = e^{ika}\psi_{nk}(x)$ Bloch 定理.

twisted phase

Wannier function.

$$u_{n\vec{k}}(\vec{x}) = \sum_m e^{i\vec{k}(\vec{x}-\vec{R}_m)} a_n(\vec{x}-\vec{R}_m)$$

平移中心



$\rho(x+a)=\rho(x)$

$\psi(x+a)=e^{i\theta}\psi(x)$

$\theta \in [0, 2\pi)$

$|k| \leq \frac{\pi}{a}$

$$\Rightarrow \quad H(k,t) = e^{ikx} H e^{-ikx}$$

$$H(k,t) = \frac{1}{2m}\left(p + \hbar k - eA(t)\right)^2 + V(x)$$

$$= \frac{1}{2m}\left(p + \hbar \breve{k}(t)\right) + V(x).$$

$$\begin{cases} \breve{k}(t) = k - \frac{e}{\hbar}A(t), \\ \dot{\breve{k}}(t) = F = -eE \end{cases}$$

方程

$$H(\breve{k}(t))\, u_{n\breve{k}(t)}(x) = \varepsilon_{nk(t)}\, u_{n\breve{k}(t)}(x).$$

$u_{nk(t)}(x)$： 周期部分

$$\psi(x,t) = e^{\underbrace{i\gamma_n(t)}_{\text{几何相位}} - \frac{i}{\hbar}\int_0^t \varepsilon_{nk(t')}dt'}\, u_{nk}$$

$$\gamma_n = i\int_0^a u_{nk}^*(x)\frac{\partial}{\partial t}u_{nk}(x)\,dx$$

$\downarrow \; t \to k$

$$\gamma_n = \int_{-\pi/a}^{\pi/a} X_{nn}(k)\,dk$$

作业：

1) 证明 对于一维周期系统 $k \in S^1$

方法：

$$\left\{-\frac{d^2}{dx^2} + V(x)\right\}\varphi_{nk} = E_{nk}\varphi_{nk}$$

取 $\varphi_{nk} = \sum_m C_{nk}(G_m)\, e^{i(k+G_m)x}$.

$$[H][C] = E[C]$$

证明 $|u_{nk}\rangle = |\psi_{n,k+G}\rangle$

$$X_{nn}(k) = \int \left(\frac{2\pi}{a}\right) u^{*}_{nk}\left(i\frac{\partial}{\partial k}\right) u_{nk}\, dx$$

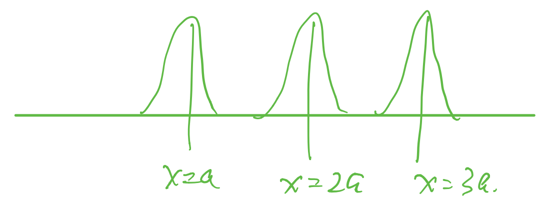
$\boxed{2)\ H_k = H_{k+G}}$

Zak 主要贡献 | Bloch $\rightarrow$ Wannier Basis

X_{nn} 明确物理意义.

令 $u_{nk}(x)\, e^{ikx} = \left(\frac{a}{2\pi}\right)^{1/2} \sum_m e^{ik(ma)}\, \underline{a_n(x-ma)}$

局域的基矢



$$u_{nk} = \sum_m a_n(x-ma)\, e^{ik(ma-x)}$$

$$u^{*}_{nk}\, i\frac{\partial}{\partial k} u_{nk} = \sum_m a_n(x-ma)(i^2)(ma-x)\, e^{ik(ma-x)}$$

$$\to \sum_{m,m'} a_n^{*}(x-m'a)\, a_n(x-ma)\,(x-ma)\, e^{-ik(m-m')a}$$

$$\Rightarrow X_{nn} = \int \sum_m |a_n(x-ma)|^2 (x-ma)\, dx$$

$$= \int \sum_m |a_n(x-ma)|^2\, x\, dx$$

12.1 $\bar{x}_n = \left(\frac{a}{2\pi}\right)^{-1} \int dx\, |a_n(x)|^2 x$

Band Center

$$\langle X_m \rangle = \int d\vec{r}\, a_n^{*}(\vec{r})\, a_n(\vec{r})\, \vec{r} = q_n$$

则 $\underline{\gamma_n = \left(\frac{2\pi}{a}\right) g_n}$

Ref: Vanderbilt 2014 PPT

证明:

$$|W_R\rangle = \sum_k e^{ik\cdot R_m} |\psi_k\rangle$$

$$= \frac{V}{(2\pi)^d} \int dk\, e^{i\vec{k}\cdot\vec{R}_m} |\psi_{nk}\rangle$$

$$|W_{n0}\rangle = \frac{V}{(2\pi)^d} \int dk\, e^{ik\cdot x} |u_{nk}(x)\rangle$$

$$\vec{x}\,|W_{n0}\rangle = \frac{V}{(2\pi)^d} \int dk\, \vec{x}\left(e^{i\vec{k}\cdot\vec{x}} |u_{n\vec{k}}\rangle\right)$$

$-i\frac{\partial}{\partial k}\left(e^{i\vec{k}\cdot\vec{x}} u_{nk}\right)$

$$= \frac{V}{(2\pi)^d} \int dk\, e^{ikx}\left(\frac{\partial}{\partial \vec{k}} |u_{n\vec{k}}\rangle\right)$$

$$\Rightarrow \underbrace{\langle W_{n0}|\vec{x}|W_{n0}\rangle}_{g_n} = \underbrace{i\frac{V}{(2\pi)^d}\int dk \langle u_{nk}| i\nabla_k | u_{nk}\rangle}_{\gamma_n.}$$

电极化

为什么? $\psi_{nk}(t) \sim e^{i\theta_{nk}(t)}\psi_{nk}$

$e^{i\vec{k}\cdot\vec{x}}\,\psi_k$

$\hbar k \to q = $ 电流 $\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0.$

动量.

又 $\nabla\cdot\vec{D} = \rho(r)$

且 $\vec{D} = \varepsilon\vec{E} + \vec{P}$

已知结果 $\Rightarrow \nabla\cdot\vec{P} = \rho$

1) $\gamma_n = \frac{2\pi}{a} q_n$

2) $\vec{\nabla}\cdot\vec{P} = \rho$ 　　P 极化，

3) $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$

$\Rightarrow \nabla(\vec{J} - \partial_t \vec{P}) = 0$

$\Downarrow$

$\vec{P} = \int_0^t \vec{J}\, dt$

$\Rightarrow$ 　$\vec{J} = \frac{\partial H}{\partial A} = \frac{\partial H}{\partial k} \leftarrow \frac{e}{m}(p + ek)$

$\langle J \rangle \sim \sum_n v_n(n) + i(\langle \partial_q \psi | \partial_t \psi \rangle$

$\langle \partial_t \psi | \partial_q \psi \rangle)$

$\vec{J} = \sum_n \int_{BZ} d\vec{k} \quad \vec{v}_{nk}$

两部分.

1) $(\partial_k \varepsilon_{nk})$

2) 几何相

$|\psi(t)\rangle = \sum_q f_q(t) |q\rangle$

$= \sum_q f_q e^{-i\Phi_q} |q\rangle$

位置： $r(t) = \sum_q \langle \psi | \vec{r} | \psi \rangle$

$= \sum_q (f_q^* i\partial_q f_q)$

$= \sum_q |f_q|^2 (\partial_q \Phi_q)$ slow

$= (\partial_q \Phi_q)$

$\Phi_q = \int_0^t \varepsilon_q(t') dt' + \gamma_q(t)$

$\partial_q \Phi_q = \int_0^t \varepsilon_q(t') dt' + \partial_q \gamma_q$

$$\vec{v} = \frac{d\vec{r}}{dt} = (\partial_q \varepsilon_q) + \partial_q \gamma_q$$

$$= \partial_q \varepsilon_q - i\underbrace{\left[\left\langle \frac{\partial\Phi}{\partial q}\middle|\frac{\partial\Phi}{\partial t}\right\rangle - \left\langle\frac{\partial\Phi}{\partial t}\middle|\frac{\partial\Phi}{\partial q}\right\rangle\right]}_{\text{Berry curvature}}$$

1) Qian Niu, Eq of Motion

2) Orbital Magnization $\vec{m} \propto \langle \vec{r} \times \vec{p} \rangle$

$$= -\frac{e}{2c}\sum_i \langle \phi_i | \vec{r} \times \vec{v} | \phi_i \rangle$$

磁性

3). Zak 文章结尾.

① 参数

② 1982年 中心反演对称 $\gamma_n = 0, \frac{\pi}{2} \Rightarrow \gamma_n = 0, \pi$

③ 推广到高维.

Motivation.

$$\gamma = \oint_{S^1} dk \underbrace{\langle \psi_k | \nabla_k | \psi_k \rangle}_{\downarrow\ \text{Connection 联络.}} = \oint \omega \qquad \omega \ \ 1\text{-form}$$

问题：① 表示复杂

② 无分类 $\Leftrightarrow$ 等价类

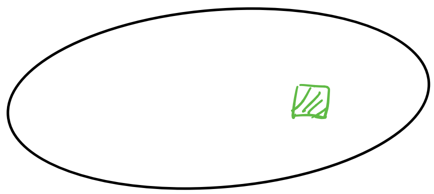
$$\oint \vec{E} \cdot d\vec{s} \qquad \oint \vec{B} \cdot d\vec{s}$$

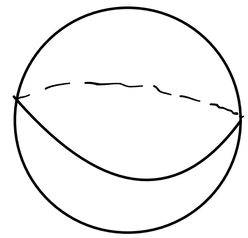
$dx_i \wedge dx_j$ 2-form.

目标：$i\oint\langle\psi|d|\psi\rangle$

$$G = \nu \frac{e^2}{h}, \quad \nu = \frac{i}{2\pi} \int \langle d m | \wedge | d m \rangle$$

流形 (manifold)





$S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$

微分流形 (manifold)

= 流形 + 微分结构 = 光滑流形 eg. $C^\infty(M)$

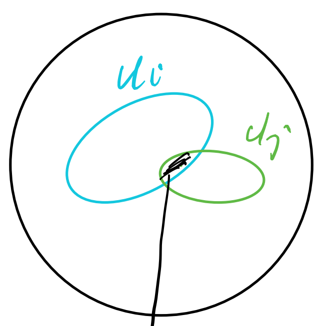
定义: m-dimensional differential manifold M

1) M是一个 Topology Space.

2). M上存在 $\{U_i, \varphi_i\}$, $\begin{cases} U_i \subset M. \\ \varphi_i \text{ 定义 } U_i \text{ 上点的坐标} \end{cases}$

3) $\bigcup_i U_i = M$.

4) 相容性.



即有 $\varphi_i(x)$

也有 $\varphi_j(x)$

存在映射 f

$$\varphi_j(p) = f \circ \varphi_i(p)$$

对 $\forall\ p \in U_i \cap U_j$ 成立.

$\Rightarrow f = \varphi_j \circ \varphi_i^{-1}$, 要求$f$是光滑函数。

$$\begin{cases} \vec{E}(\vec{x}) \Rightarrow \vec{E}(p) \\ T(\vec{x}) \Rightarrow T(p) \end{cases}$$

目的: $\begin{cases} \text{积分之与坐标选择无关.} \\ \text{光滑} \Rightarrow \text{求导} \end{cases}$

比较 $\oint\langle\psi|d|\psi\rangle$, $\oint\langle\psi|\frac{\partial}{\partial\vec{R}}|\psi\rangle d\vec{R}$

$$= \oint\langle\psi|\frac{\partial}{\partial\vec{x}}|\psi\rangle\cdot d\vec{x}$$