$H = v(k_x\sigma_x + k_y\sigma_y) + (m - Bk^2)\sigma_z$.

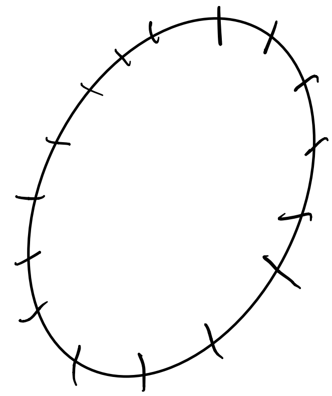
$\boxed{\text{Shen Shun Qing — Modified Dirac Equation}}$

Review 和 Summary

1, 几何相(数值方法).

$$e^{i\gamma} = \prod_i \langle \psi_i | \psi_{i+1} \rangle$$

关系 转移矩阵.



好处: 1> Gauge Invariant

2> 实验.

3> 推广到高维. , Stokes Theorem.

(Curvature)

类似于 转移矩阵.

$$\langle\psi_1|\underbrace{\psi_2\rangle\langle\psi_2|}_{P_2}\underbrace{\psi_3\rangle\langle\psi_3|}_{P_3} \cdots \langle\psi_{N-1}|\underbrace{\psi_N\rangle\langle\psi_N|}_{P_N}\psi_1\rangle$$

$= \mathrm{tr}(P_1 P_2 \cdots P_N)$, 投影子 $P_i = |\psi_i\rangle\langle\psi_i|$

$\mathrm{tr}(t_1 t_2 \cdots t_N)$ Ising Model

2, Hall 电导. $\because$ Landau Level

$\Downarrow$

Energy Gap

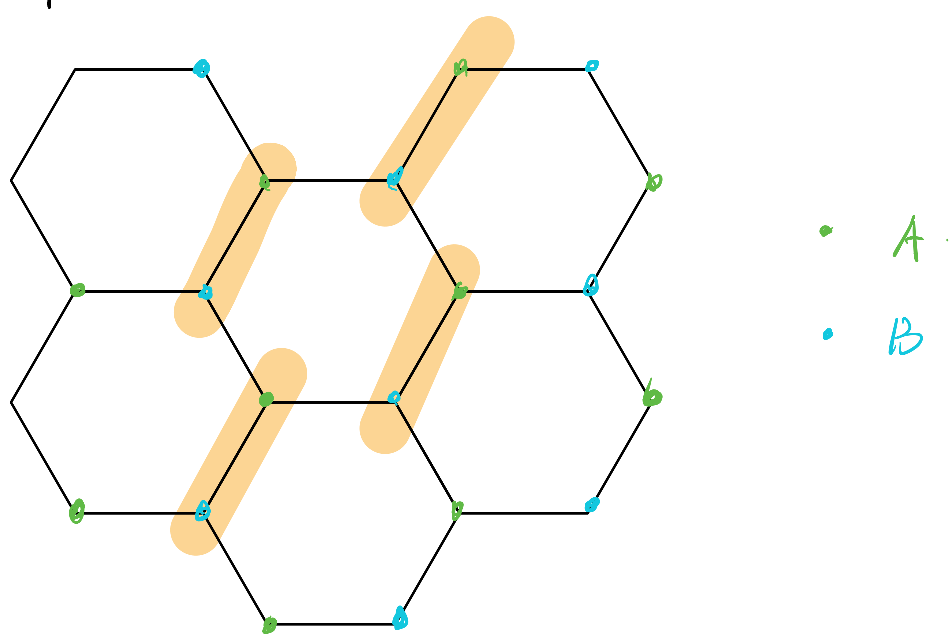
$G \propto \nu \frac{e^2}{h}$

ν 正.

Laughlin's Argument   基础是 Landau Level

Haldane 提出(1988年) 无 Landau Level 也可以有量化的电导。实验上在 ultracold atom 实现 2015年。

如何理解 Haldane Model.

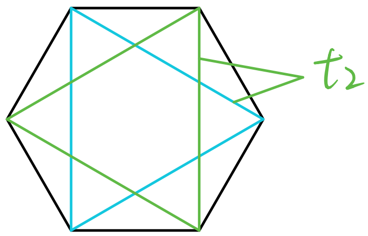


$$H = w\begin{pmatrix} 0 & k_x+ik_y \\ k_x-ik_y & 0 \end{pmatrix}, \quad \varepsilon_k = \pm v|k|$$

为什么是非对角的、只有近邻才有相互作用.

$$H_R = \sum_{ij} t_{ij} c_{iA}^\dagger c_{jB} \xrightarrow{FT} \sum_k \left(\underbrace{\quad\quad}\right) c_{kA}^\dagger c_{kB}$$

对角项来自次近邻的跃迁.



$$\left(\varepsilon_{kA}^{(t_2)}\right) C_{kA}^{\dagger} C_{kA} + \left(\varepsilon_{kB}^{(t_2)}\right) C_{kB}^{\dagger} C_{kB}$$

1) $t_2 \in \mathbb{R}$, C_{6v} 对称性.
$\varepsilon_{kA} = \varepsilon_{kB}$
$\Big\}$ NNN 项

2) t_2 形式为 $t_2 e^{\pm i\phi}$

$$\varepsilon_{kA} \propto t_2 (A\cos\phi + B\sin\phi)$$

$$\varepsilon_{kB} \propto t_2 (A\cos\phi - B\sin\phi)$$

$\cos\phi$ 是能谱的 shift, $\sin\phi$ 打开了能隙,

① effective Zeeman splitting

Shen S.Q
Extend Dirac Eq.
② Zeeman field $\propto B_k \sin\phi$,
且和 k 有关系。

$$H = (\mu + \underset{\sim}{Dk^2})\sigma_z + V(k_x\sigma_x + k_y\sigma_y)$$

one-point compactification

$k^2 \to \infty$ 时. 自旋本征态都一致.

Haldane's Paper.

观点: ① $G = ne^2/h$, $\vec{A}$ 比 $\vec{B}$ 更重要,

$\underline{C_{iA}^{\dagger} C_{jA} e^{i\phi}}$

②. 破坏 A.B 的简并性. (引入 $Bk\sin\phi$ 磁).

③ $$H = V\begin{pmatrix} 0 & k_x + ik_y \\ k_x - ik_y & 0 \end{pmatrix}$$

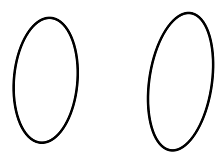
$\downarrow$ 加入电磁场

$$H(\vec{k} + e\vec{A})$$

eq2 – eq4.
Dirac 点附近的 Landau Level

Index theorem.

$$A \xrightarrow{f} B$$



Hilbert space of Dirac eq.

$\mathcal{H}_A$, $\mathcal{H}_B$ - 子空间.

$$H|\psi_A\rangle \to |\psi_B\rangle$$

$$H|\psi_B\rangle \to |\psi_A\rangle.$$

$$|\psi_A\rangle \xrightarrow{H} |\psi_B\rangle \xrightarrow{H} |\psi_A\rangle$$ $\underline{\underline{H^2}}$ 算符.

$$H^2 = \begin{pmatrix} 0 & \pi_x + i\pi_y \\ \pi_x - i\pi_y & 0 \end{pmatrix}\begin{pmatrix} 0 & \pi_x + i\pi_y \\ \pi_x - i\pi_y & 0 \end{pmatrix}.$$

$\vec{\pi} = \vec{p} + e\vec{A}$ $$= \begin{pmatrix} (\pi_x + i\pi_y)(\pi_x - i\pi_y) & 0 \\ 0 & (\pi_x - i\pi_y)(\pi_x + i\pi_y) \end{pmatrix}$$

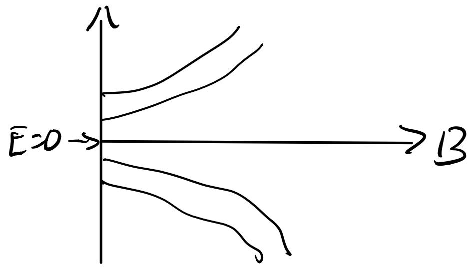
$(\pi_x + i\pi_y)(\pi_x - i\pi_y)$	$(\pi_x - i\pi_y)(\pi_x + i\pi_y)$

$$= \pi_x^2 + \pi_y^2 - i[\pi_x, \pi_y] \qquad = \pi_x^2 + \pi_y^2 + i[\pi_x, \pi_y]$$

$$= \pi_x^2 + \pi_y^2 + eB \qquad = \pi_x^2 + \pi_y^2 - eB$$

$$\varepsilon_{\alpha n \pm} = \pm \sqrt{(mc^2)^2 + n\hbar (eB) c^2} \qquad n \geq 1$$

$n=0, \quad \varepsilon_{\alpha 0} = 0$

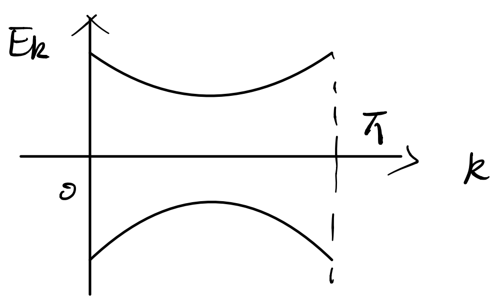


4) 能谱.

$$H = \varepsilon_s(k) + \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$$= \varepsilon_0(k) + \begin{pmatrix} d_z & d_x - i d_y \\ d_x + i d_y & -d_z \end{pmatrix} \sim$$ Berry Phese. $\mathcal{H} = \vec{B} \cdot \vec{\sigma}$

特点： $\varepsilon = \varepsilon_0 \pm |d(k)|$



Qi - Wu - Zhang PRB. 74, 085308 (2006)

$d_x = \sin(k_y)$

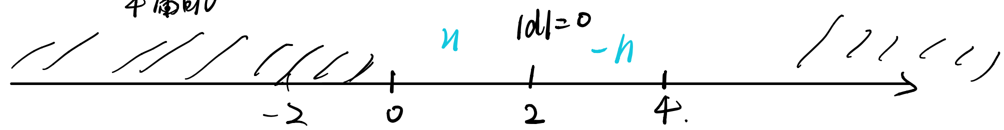
$d_y = -\sin(k_x)$

$d_z = c[2 - \cos(k_x) - \cos(k_y) - e_s]$

$|d|=0 \Rightarrow d_x=0,\ d_y=0.\ d_z=0$ ，能隙关闭

$\Downarrow$

$$\begin{cases} k_x, k_y = 0、\pi. \\ e_s = 0, 2, 4 \end{cases}$$



$G = n e^2/h$

$e_s \to \pm\infty$ 时. 能谱为两条平带

计算.

$$H = \begin{pmatrix} d_z & d_x - i d_y \\ d_x + i d_y & -d_z \end{pmatrix}$$

$H|\psi\rangle = E|\psi\rangle \qquad |\psi\rangle = \begin{pmatrix} u \\ v \end{pmatrix}$

$E = -|d|$

$d_z u + (d_x - i d_y) v = -|d| u$

$(d_z + |d|) u = (i d_y - d_x) v$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{d_x^2 + d_y^2 + (d_z + |d|)^2}} \begin{pmatrix} i d_y - d_x \\ d_z + |d| \end{pmatrix}$$

$|\psi\rangle = R \begin{pmatrix} u \\ v \end{pmatrix}$ with. $\begin{cases} u = i d_y - d_x \\ v = d_z + |d| \end{cases}$

$$\gamma = i \oint \langle\psi|\nabla_k|\psi\rangle \cdot d\vec{k} = \oint \vec{A}\cdot d\vec{k} = \int \vec{B}\cdot d\vec{S}$$

注意 $\begin{cases} \text{1) well-defined } e^{-i\gamma} = \prod \langle \psi_i | \psi_{i+1}\rangle , \\ \text{2). Berry 原么 } G = \nu \dfrac{e^2}{h} \end{cases}$

$$|\psi\rangle = R\begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow \begin{cases} R^2(|u|^2+|v|^2) = 1 \\ R \in \mathbb{R} \end{cases}$$

$$\vec{A} = i\langle\psi|\nabla_k|\psi\rangle$$

$$= i(u^*, v^*) R \nabla_k R \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= i(u^*, v^*) R \left[\nabla_k R \begin{pmatrix} u \\ v \end{pmatrix} + R \nabla_k \begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$= i(u^*, v^*)(R\nabla_k R)\begin{pmatrix} u \\ v \end{pmatrix} + i(u^*, v^*) R^2 \nabla_k \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \underbrace{i(|u|^2+|v|^2)(R\nabla_k R)} + iR^2(u^*\nabla_k u + v^*\nabla_k v)$$

$$= \underbrace{i\frac{1}{R}\nabla R}_{i\nabla(\ln R)\text{ 无贡献}} + i\frac{u^*\nabla_k u + v^*\nabla_k v}{|u|^2+|v|^2}$$

$$= i\frac{u^*\vec{\nabla}_k u + v^*\vec{\nabla}_k v}{|u|^2+|v|^2}$$

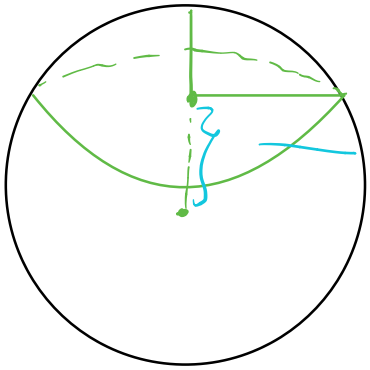
考虑 $\begin{cases} d_x = k_x \\ d_y = k_y \\ d_z = A + Bk^2 \end{cases}$

则 $\begin{cases} u = i d_y - d_x = ik_y - k_x \\ v = A + Bk^2 + \sqrt{k_x^2 + k_y^2 + (A+Bk^2)^2} \end{cases}$

再考虑 $|k| \to \infty$，令 $k_x = k\cos\theta$，$k_y = k\sin\theta$

$$H = \begin{pmatrix} A + Bk^2 & ke^{i\theta} \\ ke^{-i\theta} & -A - Bk^2 \end{pmatrix}$$

$$\Leftrightarrow \quad |\psi\rangle = \begin{pmatrix} e^{i\varphi/2}\,\tilde{u} \\ \tilde{v} \end{pmatrix}$$



$B_z = A + Bk^2.$

$$\begin{cases} A_x = \dfrac{u^* \partial_{k_x} u + v^* \partial_{k_y} v}{|u|^2 + |v|^2} \\[2ex] A_y = \dfrac{u^* \partial_{k_y} u + v^{+} \partial_{k_y} v}{|u|^2 + |v|^2} \end{cases}$$

Model: <1> $\begin{cases} d_x = k_x \\ d_y = k_y \\ d_z = A + Bk^2 \end{cases}$

<2> QWZ Model.

$\begin{cases} \text{SSH Model (1976) 分数化} \\ \text{Jackiw - Rebbi Model} \end{cases}$

Laughlin 翻译

Ref.

① PRB, 22, 2099 (1980)

② PRL. 42, 1698 (1979)

③ PRL. 46, 738 (1981)

Jackiw - Rebbi Model.

$H = p\sigma_x + m(x)\sigma_x, \quad p = -i\frac{\partial}{\partial x}.$

1) 若 $m(x) = m$, 则

$H = k\sigma_x + m\sigma_z \Rightarrow \varepsilon = \pm\sqrt{k^2 + m^2}.$

作业:

1) A_x, A_y 计算出来

Mathematica

D[f, x]

2) $\oint A_x\, dx + A_y\, dy$

$\big\downarrow$ Stokes Theorem

$\oint B\, dk_x\, dk_y$

$B = (\partial_x A_y - \partial_y A_x)$

FullSimplify

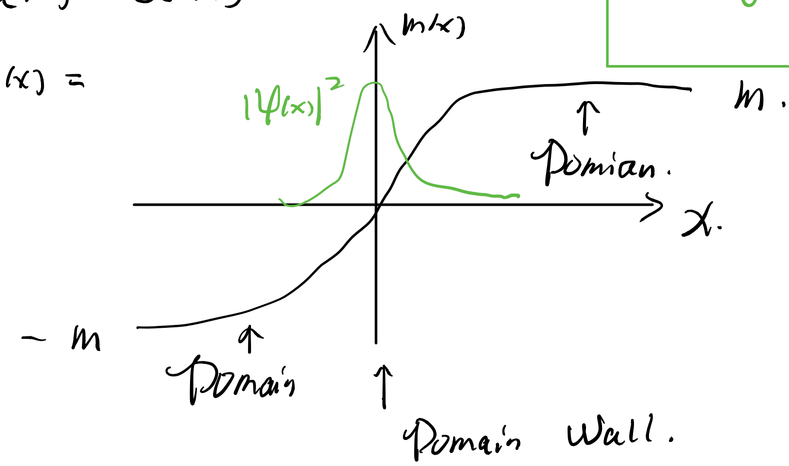
3) 求积分.

$B = B(k).$

$\mathcal{E}(m) = \mathcal{E}(-m)$

$k_x, k_y \to k, \theta$

2) 若. $m(x) =$



存在一个束缚态. $H|\psi\rangle = 0$.

解: $\begin{pmatrix} m(x) & -i\frac{\partial}{\partial x} \\ -i\frac{\partial}{\partial x} & -m(x) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$

$$\begin{cases} mu - i\frac{\partial}{\partial x} v = 0 \\ -i\frac{\partial}{\partial x} u - mv = 0 \end{cases}$$

Ansatz

$\psi(x) \sim \begin{pmatrix} a \\ b \end{pmatrix} e^{\theta(x)}$

$$\Rightarrow \begin{cases} mae^{\theta} - ibe^{\theta}\theta' = 0. \\ -iae^{\theta}\theta' - mb = 0. \end{cases}$$

$$\Rightarrow \begin{cases} ma - ib\theta' = 0 \\ -ia\theta' - mb = 0 \end{cases} \Rightarrow a^2 = -b^2$$

$\theta(x) = \int^x m(x')dx'$

$\psi(x) \sim \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{-\int^x m(x')dx'}$

$$H^2 = \begin{pmatrix} m & -i\partial_x \\ -i\partial_x & -m \end{pmatrix}\begin{pmatrix} m & -i\partial_x \\ -i\partial_x & -m \end{pmatrix}$$

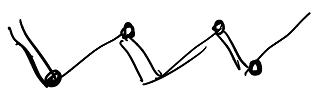
$$= \begin{pmatrix} m^2 - \frac{\partial^2}{\partial x^2} & g\delta(x) \\ g\delta(x) & m^2 - (\frac{\partial^2}{\partial x^2}) \end{pmatrix}$$

$$= (m^2 - \frac{\partial^2}{\partial x^2}) + g\delta(x)\sigma^x$$

or

$$= m^2 - \frac{\partial^2}{\partial x^2} \pm g\delta(x).$$

SSH Model. (Pereils 不稳定性.)



2个简并基态. ⇒ Soliton, in Gap.

$$H = -\sum_n (t_{n+1,n} C^\dagger_{n+1} C_n + h.c)$$

$$+ \sum_n \frac{1}{2} k^2 (u_{n+1} - u_n)^2 + \frac{1}{2} M \dot{u}_n^2$$

$$t_{n+1,n} = t_0 - \alpha(u_{n+1} - u_n)$$

怎么做

1) 量子力学处理.

2) 经典处理. $u_n = (-1)^n u$ u常数.

$\dot{u} = 0$

$$t_{n+1,n} = t_0 - \alpha(u_{n+1} - u_n)$$

$$= t_0 + 2\alpha u(-1)^n$$

$$= \begin{cases} t_0 + t_1 \\ t_0 - t_1. \end{cases}$$

$$H = -\underbrace{\sum_n t_{n+1,n} C^\dagger_{n+1} C_n}_{A_0 + A_1 u} + \sum_n \frac{1}{2} k (u_{n+1} - u_n)^2$$

$$2ku^2 \times 4N$$

$$= 2Nku^2.$$

$$H = \sum_i \left[(t_0 + t_1) C^\dagger_{iA} C_{iB} + (t_0 - t_1) C^\dagger_{iB} C_{i+1A} + h.c \right]$$

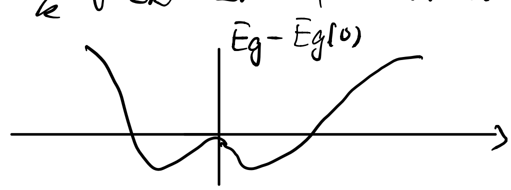
$$\Big\downarrow \text{FT}$$

$$H = \sum_k H_k \qquad H_k = \begin{bmatrix} 0 & \gamma \\ \gamma^* & 0 \end{bmatrix}$$

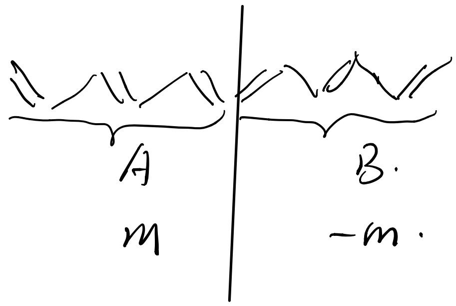
$$E_k = \pm \sqrt{\varepsilon_k^2 + \Delta_k^2}$$

$$\begin{cases} \varepsilon_k = -2t_0 \cos(ka) \\ \Delta_k = 2u \sin(ka) \end{cases} \qquad \Delta_k \propto u$$

$$E_g = -\sum_k \sqrt{\varepsilon_k^2 + \Delta_k^2} + 2Nku^2$$



在能隙附近可以写为 $H = p\sigma_x + m\sigma_z$



Ansatz.

$\varphi_n = (-1)^n \tanh(\frac{n}{l})$

l，变分参数.

读Su的三篇文章.

推导.

PRL. 46, 738 (1981)

理解分数化.

$u_n = u\cos(\frac{2}{3}\pi n - \theta)$

$Q = 0, \pm\frac{1}{3}e, \pm\frac{2}{3}e$