Let $X \in \mathfrak{X}(M)$ be a vector field on M. X assigns a vector $X|_p \in T_pM$ at each point $p \in M$. From our viewpoint, X is looked upon as a smooth map $M \to TM$. This map is not utterly arbitrary since a point p must be mapped to a point $u \in TM$ such that $\pi(u) = p$. We define a **section** (or a **cross section**) of TM as a smooth map $s : M \to TM$ such that $\pi \circ s = \mathrm{id}_M$. If a section $s_i : U_i \to TU_i$ is defined only on a chart U_i, it is called a **local section**.

9.2 Fibre bundles

The tangent bundle in the previous section is an example of a more general framework called a fibre bundle. Definitions are now in order.

9.2.1 Definitions

Definition 9.1. A (differentiable) fibre bundle (E, π, M, F, G) consists of the following elements:

(i) A differentiable manifold E called the **total space**.
(ii) A differentiable manifold M called the **base space**.
(iii) A differentiable manifold F called the **fibre** (or **typical fibre**).
(iv) A surjection $\pi : E \to M$ called the **projection**. The inverse image $\pi^{-1}(p) = F_p \cong F$ is called the fibre at p.
(v) A Lie group G called the **structure group**, which acts on F on the left.
(vi) A set of open covering $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The map ϕ_i is called the **local trivialization** since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ *onto* the direct product $U_i \times F$.
(vii) If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ be an element of G. Then ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as (figure 9.2)

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{9.4}$$

The maps t_{ij} are called the **transition functions**.

[*Remarks*: We often use a shorthand notation $E \xrightarrow{\pi} M$ or simply E to denote a fibre bundle (E, π, M, F, G).

Strictly speaking, the definition of a fibre bundle should be independent of the special covering $\{U_i\}$ of M. In the mathematical literature, this definition is employed to define a **coordinate bundle** $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$. Two coordinate bundles $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$ and $(E, \pi, M, F, G, \{V_i\}, \{\psi_i\})$ are said to be equivalent if $(E, \pi, M, F, G, \{U_i\} \cup \{V_j\}, \{\phi_i\} \cup \{\psi_j\})$ is again a coordinate bundle. A fibre bundle is defined as an equivalence class of coordinate bundles. In practical applications in physics, however, we always employ a certain

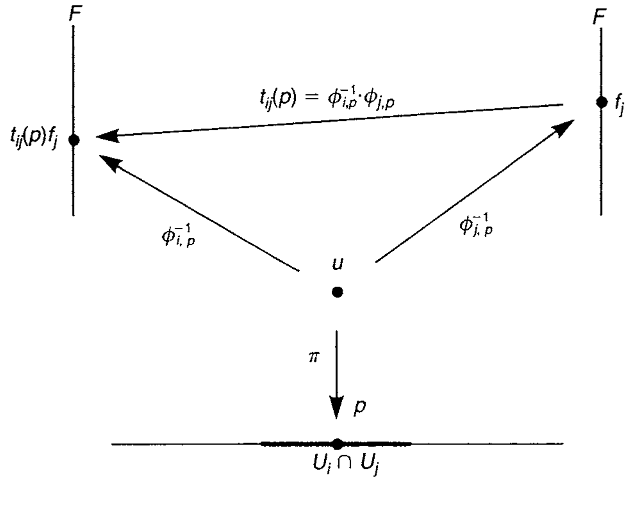

Figure 9.2. On the overlap $U_i \cap U_j$, two elements $f_i, f_j \in F$ are assigned to $u \in \pi^{-1}(p)$, $p \in U_i \cap U_j$. They are related by $t_{ij}(p)$ as $f_i = t_{ij}(p) f_j$.

definite covering and make no distinction between a coordinate bundle and a fibre bundle.]

We need to clarify several points. Let us take a chart U_i of the base space M. $\pi^{-1}(U_i)$ is a direct product diffeomorphic to $U_i \times F$, $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$ being the diffeomorphism. If $U_i \cap U_j \neq \emptyset$, we have two maps ϕ_i and ϕ_j on $U_i \cap U_j$. Let us take a point u such that $\pi(u) = p \in U_i \cap U_j$. We then assign two elements of F, one by ϕ_i^{-1} and the other by ϕ_j^{-1},

$$\phi_i^{-1}(u) = (p, f_i), \qquad \phi_j^{-1}(u) = (p, f_j) \tag{9.5}$$

see figure 9.2. There exists a map $t_{ij} : U_i \cap U_j \to G$ which relates f_i and f_j as $f_i = t_{ij}(p) f_j$. This is also written as (9.4).

We require that the transition functions satisfy the following consistency conditions:

$$t_{ii}(p) = \text{identity map} \qquad (p \in U_i) \tag{9.6a}$$

$$t_{ij}(p) = t_{ji}(p)^{-1} \qquad (p \in U_i \cap U_j) \tag{9.6b}$$

$$t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p) \qquad (p \in U_i \cap U_j \cap U_k). \tag{9.6c}$$

Unless these conditions are satisfied, local pieces of a fibre bundle cannot be glued together consistently. If all the transition functions can be taken to be identity maps, the fibre bundle is called a **trivial bundle**. A trivial bundle is a direct product $M \times F$.

Given a fibre bundle $E \xrightarrow{\pi} M$, the possible set of transition functions is obviously far from unique. Let $\{U_i\}$ be a covering of M and $\{\phi_i\}$ and $\{\tilde{\phi}_i\}$ be two sets of local trivializations giving rise to the same fibre bundle. The transition functions of respective local trivializations are

$$t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} \tag{9.7a}$$

$$\tilde{t}_{ij}(p) = \tilde{\phi}_{i,p}^{-1} \circ \tilde{\phi}_{j,p}. \tag{9.7b}$$

Define a map $g_i(p) : F \to F$ at each point $p \in M$ by

$$g_i(p) \equiv \phi_{i,p}^{-1} \circ \tilde{\phi}_{i,p}. \tag{9.8}$$

We require that $g_i(p)$ be a homeomorphism which belongs to G. This requirement must certainly be fulfilled if $\{\phi_i\}$ and $\{\tilde{\phi}_i\}$ describe the same fibre bundle. It is easily seen from (9.7) and (9.8) that

$$\tilde{t}_{ij}(p) = g_i(p)^{-1} \circ t_{ij}(p) \circ g_j(p). \tag{9.9}$$

In the practical situations which we shall encounter later, t_{ij} are the gauge transformations required for pasting local charts together, while g_i corresponds to the gauge degrees of freedom within a chart U_i. If the bundle is trivial, we may put all the transition functions to be identity maps. Then the most general form of the transition functions is

$$t_{ij}(p) = g_i(p)^{-1} g_j(p). \tag{9.10}$$

Let $E \xrightarrow{\pi} M$ be a fibre bundle. A **section** (or a **cross section**) $s : M \to E$ is a smooth map which satisfies $\pi \circ s = \mathrm{id}_M$. Clearly, $s(p) = s|_p$ is an element of $F_p = \pi^{-1}(p)$. The set of sections on M is denoted by $\Gamma(M, F)$. If $U \subset M$, we may talk of a **local section** which is defined only on U. $\Gamma(U, F)$ denotes the set of local sections on U. For example, $\Gamma(M, TM)$ is identified with the set of vector fields $\mathcal{X}(M)$. It should be noted that not all fibre bundles admit global sections.

Example 9.1. Let E be a fibre bundle $E \xrightarrow{\pi} S^1$ with a typical fibre $F = [-1, 1]$. Let $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$ be an open covering of S^1 and let $A = (0, \pi)$ and $B = (\pi, 2\pi)$ be the intersection $U_1 \cap U_2$, see figure 9.3. The local trivializations ϕ_1 and ϕ_2 are given by

$$\phi_1^{-1}(u) = (\theta, t), \qquad \phi_2^{-1}(u) = (\theta, t)$$

for $\theta \in A$ and $t \in F$. The transition function $t_{12}(\theta)$, $\theta \in A$, is the identity map $t_{12}(\theta) : t \mapsto t$. We have two choices on B;

(I) $\phi_1^{-1}(u) = (\theta, t)$, $\phi_2^{-1}(u) = (\theta, t)$
(II) $\phi_1^{-1}(u) = (\theta, t)$, $\phi_2^{-1}(u) = (\theta, -t)$

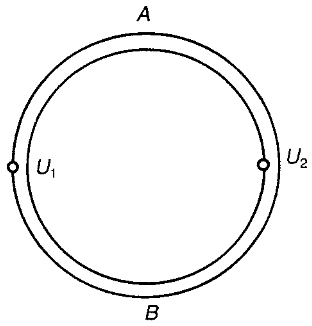

Figure 9.3. The base space S^1 and two charts U_1 and U_2 over which the fibre bundle is trivial.

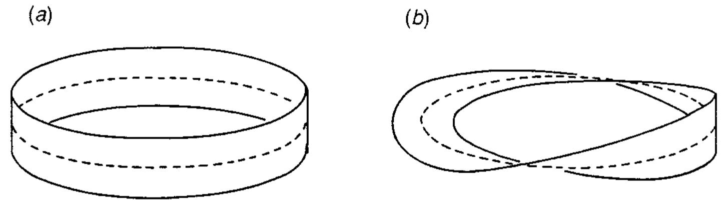

Figure 9.4. Two fibre bundles over S^1: (*a*) is the cylinder which is a trivial bundle $S^1 \times I$; (*b*) is the Möbius strip.

For case (I), we find that $t_{12}(\theta)$ is the identity map and two pieces of the local bundles are glued together to form a cylinder (figure 9.4(*a*)). For case (II), we have $t_{12}(\theta) : t \mapsto -t, \theta \in B$, and obtain the Möbius strip (figure 9.4(*b*)). Thus, a cylinder has the trivial structure group $G = \{e\}$ where e is the identity map of F onto F while the Möbius strip has $G = \{e, g\}$ where $g : t \mapsto -t$. Since $g^2 = e$, we find $G \cong \mathbb{Z}_2$. A cylinder is a trivial bundle $S^1 \times F$, while the Möbius strip is not. [*Remark*: The group $\mathbb{Z}_2$ is not a Lie group. This is the only occasion we use a discrete group for the structure group.]

9.2.2 Reconstruction of fibre bundles

What is the minimal information required to construct a fibre bundle? We now show that for given M, $\{U_i\}$, $t_{ij}(p)$, F and G, we can reconstruct the fibre bundle (E, π, M, F, G). This amounts to finding a unique π, E and ϕ_i from given data. Let us define

$$X \equiv \bigcup_i U_i \times F. \tag{9.11}$$

Introduce an equivalence relation $\sim$ between $(p, f) \in U_i \times F$ and $(q, f') \in U_j \times F$ by $(p, f) \sim (q, f')$ if and only if $p = q$ and $f' = t_{ij}(p)f$. A fibre

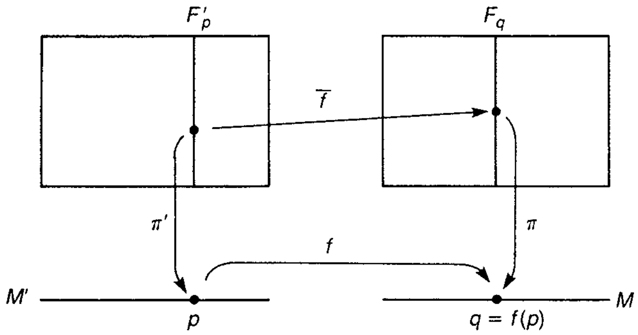

Figure 9.5. A bundle map $\bar{f} : E' \to E$ induces a map $f : M' \to M$.

bundle E is then defined as

$$E = X/\sim . \tag{9.12}$$

Denote an element of E by $[(p, f)]$. The projection is given by

$$\pi : [(p, f)] \mapsto p. \tag{9.13}$$

The local trivialization $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ is given by

$$\phi_i : (p, f) \mapsto [(p, f)]. \tag{9.14}$$

The reader should verify that E, π and $\{\phi_i\}$ thus defined satisfy all the axioms of fibre bundles. Thus, the given data reconstruct a fibre bundle E uniquely.

This procedure may be employed to construct a new fibre bundle from an old one. Let (E, π, M, F, G) be a fibre bundle. Associated with this bundle is a new bundle whose base space is M, transition function $t_{ij}(p)$, structure group G and fibre F' on which G acts. Examples of associated bundles will be given later.

9.2.3 Bundle maps

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be fibre bundles. A *smooth* map $\bar{f} : E' \to E$ is called a **bundle map** if it maps each fibre F'_p of E' onto F_q of E. Then $\bar{f}$ naturally induces a smooth map $f : M' \to M$ such that $f(p) = q$ (figure 9.5). Observe that the diagram

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M \end{array} \qquad \left(\begin{array}{ccc} u & \xrightarrow{\bar{f}} & \bar{f}(u) \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ p & \xrightarrow{f} & q \end{array} \right) \tag{9.15}$$

commutes. [*Caution*: A smooth map $\bar{f} : E' \to E$ is not necessarily a bundle map. It may map $u, v \in F'_p$ of E' to $\bar{f}(u)$ and $\bar{f}(v)$ on different fibres of E so that $\pi(\bar{f}(u)) \neq \pi(\bar{f}(v))$.]

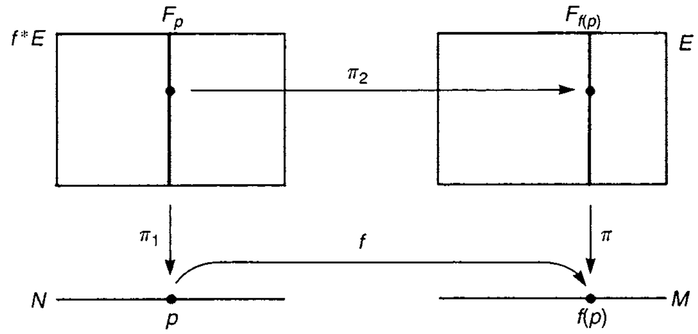

Figure 9.6. Given a fibre bundle $E \stackrel{\pi}{\longrightarrow} M$, a map $f : N \to M$ defines a pullback bundle f^*E over N.

9.2.4 Equivalent bundles

Two bundles $E' \stackrel{\pi'}{\longrightarrow} M$ and $E \stackrel{\pi}{\longrightarrow} M$ are equivalent if there exists a bundle map $\bar{f} : E' \to E$ such that $f : M \to M$ is the identity map and $\bar{f}$ is a diffeomorphism:

$$\begin{array}{ccc} E' & \stackrel{\bar{f}}{\longrightarrow} & E \\ \pi' \big\downarrow & & \big\downarrow \pi \\ M & \stackrel{\mathrm{id}_M}{\longrightarrow} & M. \end{array} \tag{9.16}$$

This definition of equivalent bundles is in harmony with that given in the remarks following definition 9.1.

9.2.5 Pullback bundles

Let $E \stackrel{\pi}{\longrightarrow} M$ be a fibre bundle with typical fibre F. If a map $f : N \to M$ is given, the pair (E, f) defines a new fibre bundle over N with the same fibre F (figure 9.6). Let f^*E be a subspace of $N \times E$, which consists of points (p, u) such that $f(p) = \pi(u)$. $f^*E \equiv \{(p, u) \in N \times E | f(p) = \pi(u)\}$ is called the **pullback** of E by f. The fibre F_p of f^*E is just a copy of the fibre $F_{f(p)}$ of E. If we define $f^*E \stackrel{\pi_1}{\longrightarrow} N$ by $\pi_1 : (p, u) \mapsto p$ and $f^*E \stackrel{\pi_2}{\longrightarrow} E$ by $(p, u) \mapsto u$, the pullback f^*E may be endowed with the structure of a fibre bundle and we obtain the following bundle map,

$$\begin{array}{ccc} f^*E & \stackrel{\pi_2}{\longrightarrow} & E \\ \pi_1 \big\downarrow & & \big\downarrow \pi \\ N & \stackrel{f}{\longrightarrow} & M \end{array} \qquad \left(\begin{array}{ccc} (p, u) & \stackrel{\pi_2}{\longrightarrow} & u \\ \pi_1 \big\downarrow & & \big\downarrow \pi \\ p & \stackrel{f}{\longrightarrow} & f(p) \end{array} \right). \tag{9.17}$$

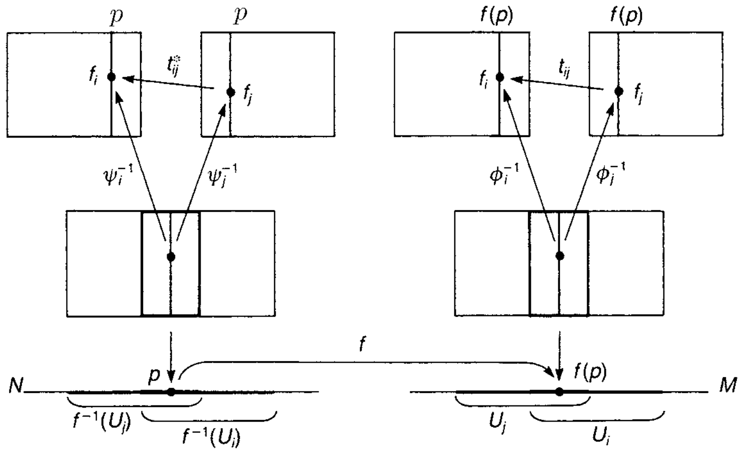

Figure 9.7. The transition function t^*_{ij} of the pullback bundle f^*E is a pullback of the transition function t_{ij} of E.

The commutativity of the diagram follows since $\pi(\pi_2(p, u)) = \pi(u) = f(p) = f(\pi_1(p, u))$ for $(p, u) \in f^*E$. In particular, if $N = M$ and $f = \mathrm{id}_M$, then two fibre bundles f^*E and E are equivalent.

Let $\{U_i\}$ be a covering of M and $\{\phi_i\}$ be local trivializations. $\{f^{-1}(U_i)\}$ defines a covering of N such that f^*E is locally trivial. Take $u \in E$ such that $\pi(u) = f(p) \in U_i$ for some $p \in N$. If $\phi_i^{-1}(u) = (f(p), f_i)$ we find $\psi_i^{-1}(p, u) = (p, f_i)$ where ψ_i is the local trivialization of f^*E. The transition function t_{ij} at $f(p) \in U_i \cap U_j$ maps f_j to $f_i = t_{ij}(f(p))f_j$. The corresponding transition function t^*_{ij} of f^*E at $p \in f^{-1}(U_i) \cap f^{-1}(U_j)$ also maps f_j to f_i; see figure 9.7. This shows that

$$t^*_{ij}(p) = t_{ij}(f(p)). \tag{9.18}$$

Example 9.2. Let M and N be differentiable manifolds with $\dim M = \dim N = m$. Let $f : N \to M$ be a smooth map. The map f induces a map $\pi_2 : TN \to TM$ such that the following diagram commutes:

$$\begin{array}{ccc} TN & \xrightarrow{\pi_2} & TM \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi} \\ N & \xrightarrow{f} & M. \end{array} \tag{9.19}$$

Let $W = W^\nu \partial/\partial y^\nu$ be a vector of T_pN and $V = V^\mu \partial/\partial x^\mu$ be the corresponding vector of $T_{f(p)}M$. If TN is a pullback bundle $f^*(TM)$, π_2 maps T_pN to $T_{f(p)}M$ diffeomorphically. This is possible if and only if π_2 has the maximal rank m at each point of TN. Let $\varphi(f(p)) = (f^1(y), \ldots, f^m(y))$ be the coordinates of $f(p)$ in a chart (U, φ) of M, where $y = \varphi(p)$ are the coordinates of p in a chart (V, ψ) of N. The maximal rank condition is given by $\det(\partial f^\mu(y)/\partial y^\nu) \neq 0$ for any $p \in N$.

9.2.6 Homotopy axiom

Let f and g be maps from M' to M. They are said to be **homotopic** if there exists a smooth map $F : M' \times [0, 1] \to M$ such that $F(p, 0) = f(p)$ and $F(p, 1) = g(p)$ for any $p \in M'$, see section 4.2.

Theorem 9.1. Let $E \xrightarrow{\pi} M$ be a fibre bundle with fibre F and let f and g be homotopic maps from N to M. Then f^*E and g^*E are equivalent bundles over N.

The proof is found in Steenrod (1951). Let M be a manifold which is contractible to a point. Then there exists a homotopy $F : M \times I \to M$ such that

$$F(p, 0) = p \qquad F(p, 1) = p_0$$

where $p_0 \in M$ is a fixed point. Let $E \xrightarrow{\pi} M$ be a fibre bundle over M and consider pullback bundles h_0^*E and h_1^*E, where $h_t(p) \equiv F(p, t)$. The fibre bundle h_1^*E is a pullback of a fibre bundle $\{p_0\} \times F$ and hence is a trivial bundle: $h_1^*E \simeq M \times F$. However, $h_0^*E = E$ since h_0 is the identity map. According to theorem 9.1, $h_0^*E = E$ is equivalent to $h_1^*E = M \times F$, hence E is a trivial bundle. For example, the tangent bundle $T\mathbb{R}^m$ is trivial. We have obtained the following corollary.

Corollary 9.1. Let $E \xrightarrow{\pi} M$ be a fibre bundle. E is trivial if M is contractible to a point.

9.3 Vector bundles

9.3.1 Definitions and examples

A **vector bundle** $E \xrightarrow{\pi} M$ is a fibre bundle whose fibre is a vector space. Let F be $\mathbb{R}^k$ and M be an m-dimensional manifold. It is common to call k the **fibre dimension** and denote it by $\dim E$, although the total space E is $m + k$ dimensional. The transition functions belong to $\mathrm{GL}(k, \mathbb{R})$, since it maps a vector space onto another vector space of the same dimension isomorphically. If F is a complex vector space $\mathbb{C}^k$, the structure group is $\mathrm{GL}(k, \mathbb{C})$.

Example 9.3. A tangent bundle TM over an m-dimensional manifold M is a vector bundle whose typical fibre is $\mathbb{R}^m$, see section 9.1. Let u be a point in TM such that $\pi(u) = p \in U_i \cap U_j$, where $\{U_i\}$ covers M. Let $x^\mu = \varphi_i(p)$