Review

① wedge product $dx\wedge dy$

统一 Jacobi 行列式. 比较: ① 热力学.

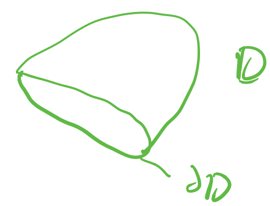
② $df = u\,dx + v\,dy$

$d(df) = 0$

$dx\wedge dy\wedge dz$.

统一 Stokes 定理中. $\langle \partial D, \omega\rangle = \langle D, d\omega\rangle$

古典微积分的高峰.



② 不变量.

1) $\frac{1}{2\pi i}\oint \frac{dz}{z} \in \mathbb{Z}$

2) $\oint \vec{E}\cdot d\vec{S} = Q_+ - Q_-$

$\oint \vec{B}\cdot d\vec{S} = Q$ Monopole   , k-space

3) $I = \oint p\cdot dq$ Adiabatic Invariant.

$= nh$ 几何意义.

量子化 vs 几何（问题）

4) 几何相.

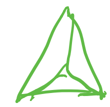
$\gamma = \oint \langle \phi(\vec{R})|\nabla_R|\phi(\vec{R})\rangle \cdot d\vec{R}$ Berry (1983)

$= \oint \vec{P}_R \cdot d\vec{R}$

5) 其它 ① 哥尼斯堡七桥问题.

② 染色问题

③ Euler 数. $\chi(K) = V - L + F = ?$ (点, line, face)



$4 - 6 + 4 = \underline{2}$

④ 立体角. 4π

$$V(r) = \frac{1}{4\pi\varepsilon_0 r}$$

Outline

Topo 相关.

1、放弃距离，用邻域/近邻表示 (Neighborhood)

用集合的关系描述Topology形变中点的关系

2、放弃距离 ⇒ 坐标 (x) ⇒ <u>point</u>

3、如何理解形变，

形变理解为流形到流形之间的<u>映射</u> (Mapping)

$p' = f(p)$ $\quad p' \in M'$

坐标有时不是定义物理的必要量。 $\quad p \in M$

物理中相关的东西.

1、$L(q_i, \dot{q}_i)$ or $H(p_i, q_i)$

相空间 (p_i, q_i) 上的函数 $\quad H: M \to \mathbb{R}$

2, $\psi(x) \Rightarrow \psi: X \to C^N$

3. $\vec{E}(\vec{x}), \vec{B}(\vec{x})$, $X \to R^3$

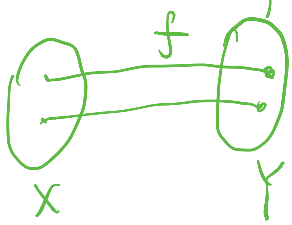
4. $H_k = e^{-i\vec{k}\cdot\vec{x}} H e^{i\vec{k}\cdot\vec{x}}$ $H: BZ \to$ Hilbert Space operator.

关心所有可能的k.

数学定义与术语. (Ref. Nakahara 书第二章)

1. Map 映射.

$f: X \to Y$. , $\begin{cases} X: \text{set} \\ Y: \text{set} \end{cases}$

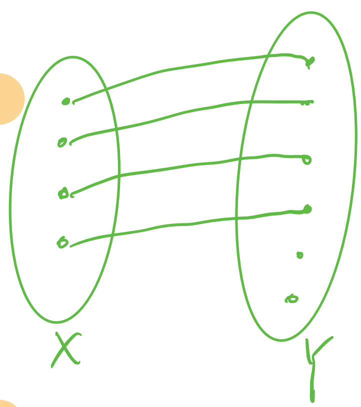


$\begin{cases} \text{单射.} \\ \text{满射.} \\ \text{双射} \end{cases}$

1-1 映射 $\Rightarrow f^{-1}$ 存在.

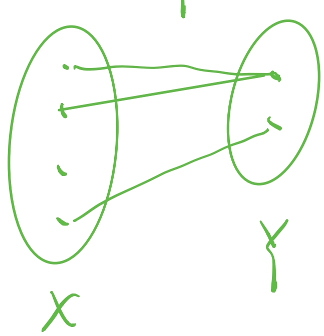
morphism 态射.

单射.
injection



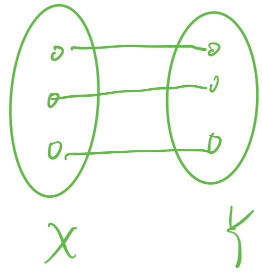
$\forall\, x, x' \in X$, 若有 $f(x) = f(x')$
当且仅当 $x = x'$ 时成立。

满射.
surjection



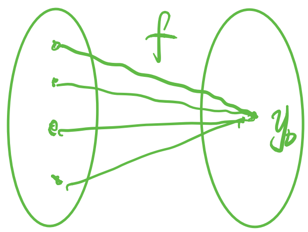
对于 $\forall\, y \in Y$.
都存在一个 $x \in X$ 有
$y = f(x)$

双射. 即是单射，又是满射.
bijection



可逆的映射。

Const Map. $C: X \to Y$



$f(x) = y_0$

Inclusion Map. $i: A \to X$ 其中 $A \subset X$

$i \in A$, $i(x) = x \in X$

$i: A \hookrightarrow X$

Identical map.

$id_X: X \to X$

$id_X(x) = x$

复合映射 $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\Rightarrow$ $z = g(f(x))$

$g \circ f = g(f(x))$

等价关系 (Equivalent Relation)

$A \sim B$. A等价于B

举例：

$[a] = \{a + n\pi \mid n \in \mathbb{Z}\}$

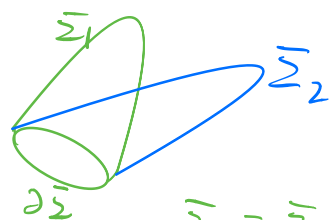
$R/\sim = [0, \pi)$

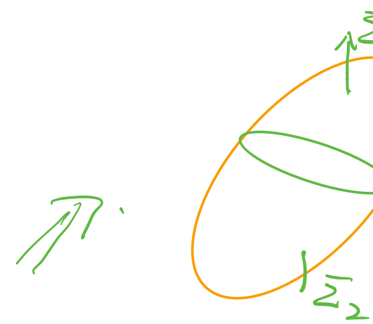
① 概念、Bloch 能带

$|k| \le (\frac{\pi}{a})$ 第一布里渊区.

k 与 $k+G$ 是等价的.

② stokes 定理.





$\Sigma = \Sigma_1 - \Sigma_2$

$\Rightarrow \partial\Sigma = 0$ 无边界

$$\Rightarrow \left.\begin{aligned} \langle \partial\Sigma_1, \omega\rangle &= \langle \Sigma_1, d\omega\rangle \\ \langle \partial\Sigma_2, \omega\rangle &= \langle \Sigma_2, d\omega\rangle \end{aligned}\right\}$$

$\Rightarrow$ 所有具有同一边界的流形上的积分完全相同.

可以只关注代表（典型的流形）

不同的类实际上与不同的相相联系

Bool Algerbra (mod 2)

$$[0] = \{0 + 2n \mid n \in \mathbb{Z}\}$$

$$[1] = \{1 + 2n \mid n \in \mathbb{Z}\}$$

所有的求和

$$\begin{cases} [0] + [0] = [0] \\ [1] + [1] = [0] \\ [0] + [1] = [1] \end{cases}$$

磁场中的规范势.

$$\vec{A}' = \vec{A} + \nabla\phi.$$

给出相同的可观测量

$$\vec{B} = \nabla \times \vec{A}' = \nabla \times \vec{A}$$

How to define equivalent relation

1: $a \sim a$

2: $a \sim b \Rightarrow b \sim a$

3: $a \sim b,\ b \sim c \Rightarrow a \sim c$

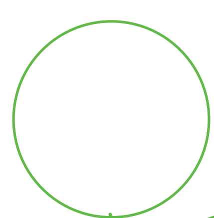
$\Longleftrightarrow$ $[a] = \{x \in X \mid x \sim a\}$

商空间: quotient space.

$$\{[x]\} = X/\sim$$

应用:

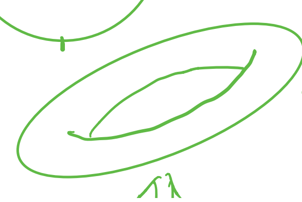
<1> $\mathbb{R}/\mathbb{Z} = S^1$. S^1是曲线.



圆

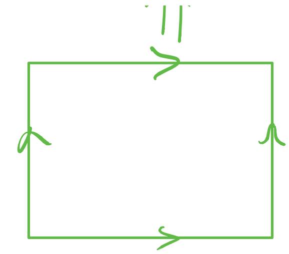
<2> BZ 1d $\sim S^1$

2d $\sim T^2 = S^1 \times S^1$

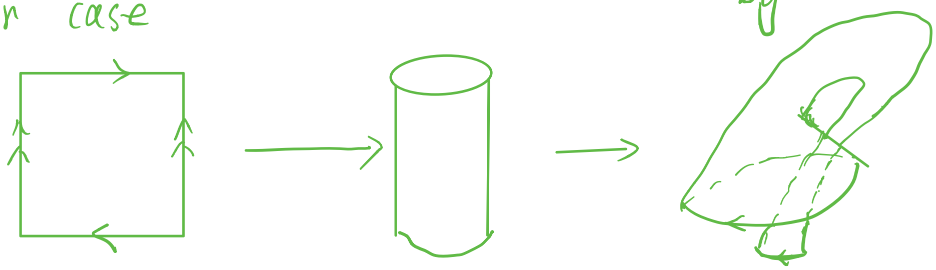


轮胎面.

Torus



Another case



Klein - Battle.

线性空间与矢量空间

V. 线性空间 $\Rightarrow$ $v = v^i e_i$ e_i是基矢.

线性变换 $f: V \to V$

有 $f(av_1 + bv_2) = af(v_1) + bf(v_2)$

反线性变换.

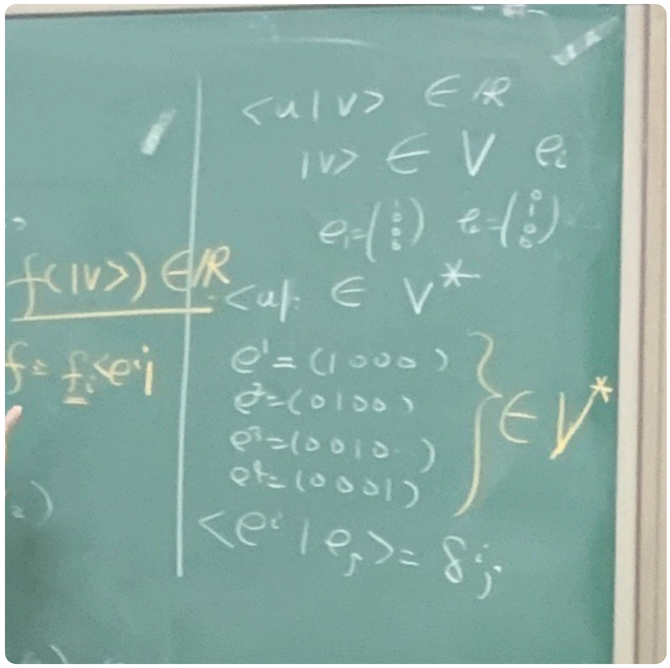
K: conjugate operator.

Some Mapping

$g: V \to \mathbb{R}$ 术语

eg. $\langle u|v\rangle \in \mathbb{R}$ or $\mathbb{C}$ inner product.

$g(v) \in \mathbb{R}$, $g \in Hom(V, \mathbb{R})$

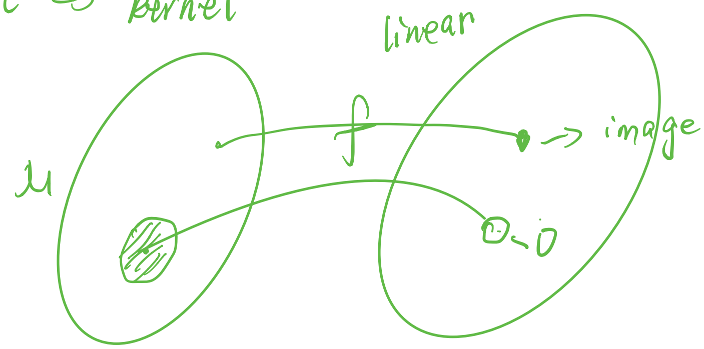


$V^* = Hom(V, R)$

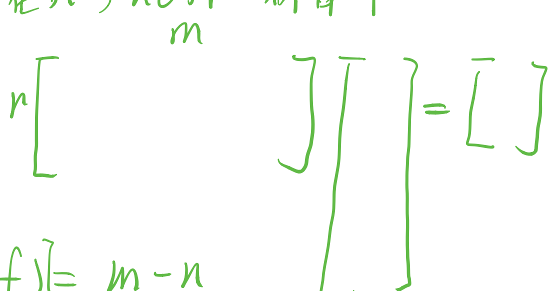
对偶.

Im 与 Ker

image ⇒ kernel



存在H, $h \in H$ 都有 $f(h)=0$



$\dim[kernel(f)] = m-n$

$\dim[image(f)] = n$

$\dim(M) = m$

$\Rightarrow \dim(M) = \dim(\ker(f)) + \dim(\operatorname{im}(f))$