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一、$I(u)=\int_{\sin u}^{\cos u} e^{x^2-xu}dx$，求 $I'(u)$.

6'

$$I'(u)=-\int_{\sin u}^{\cos x} xe^{x^2-xu}dx-\sin u\, e^{\cos^2 u-u\cos u}-\cos u\, e^{\sin^2 u-u\sin u}.$$

二、$v=\left(\frac{y}{z}-\frac{1}{y},\ \frac{x}{z}+\frac{x}{y^2},\ 1-\frac{xy}{z^2}\right)$，$y>0, z>0$.

(1) 证明 v 有势场 4' (2) 求势函数 4' (3) $\int_{(1,1,1)}^{(1,2,3)} v\cdot\tau\, ds$.

6'

$=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$

(1) $\mathrm{rot}\, V=(0,0,0)$

(2) $Pdx+Qdy+Rdz=\left(\frac{y}{z}dx+\frac{x}{z}dy-\frac{xy}{z^2}dz\right)+\left(-\frac{1}{y}dx+\frac{x}{y^2}dy\right)+(dz)$

$=d\left(\frac{xy}{z}-\frac{x}{y}+z\right)$ 势函数 $\frac{xy}{z}-\frac{x}{y}+z+C$.

(3) $\int_{(1,1,1)}^{(1,2,3)} Pdx+Qdy+Rdz=\frac{xy}{z}-\frac{x}{y}+z\Big|_{(1,1,1)}^{(1,2,3)}=\frac{13}{6}$.

三、10' $x^2+y^2\le\pi$ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\sin(x^2+y^2)$，求 $\oint_{\partial D}\frac{\partial u}{\partial n}ds$.

$I=\oint_{\partial D}\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\cdot n\,ds=\oint_{\partial D}\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)(+dy,-dx)$

Green公式 $\iint_D\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)dxdy=\iint_D\sin(x^2+y^2)dxdy=\int_0^{2\pi}\int_0^{\sqrt{\pi}} r\sin r^2\, dr\, d\theta=2\pi$

四、12' $I=\iint_S 2(1+x)dydz+yzdxdy$，$S: y=\sqrt{x}$ $(0\le x\le 1)$ 绕 x 轴旋转生成.（法向与x轴正向成钝角）

$S: x=y^2+z^2$. $D: y^2+z^2\le 1, x=0$. $\Sigma: x=1, (y,z)\in D$. 前侧 $\Omega: S\cup\Sigma$ 围成

$\iint_S 2(1+x)dydz+yzdxdy=\left(\iint_{S\cup\Sigma}-\iint_\Sigma\right)2(1+x)dydz+yzdxdy$.

Gauss公式 $\iiint_\Omega(2+y)dxdydz-\iint_\Sigma 4dydz$

$=\int_0^1 dx\iint_{y^2+z^2\le x}(2+y)dydz-4\pi=-3\pi$

五、10' $v=(y^2+z^2)\vec i+(z^2+x^2)\vec j+(x^2+y^2)\vec k$，计算 $I=\oint_L(y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz$.

$L: x^2+y^2+z^2=4x$ 与 $x^2+y^2=2x$ $(z\ge 0)$ 从z轴正向看顺时针

$\vec n=-\left(\frac{x-2}{2},\frac{y}{2},\frac{z}{2}\right)$. $D: x^2+y^2\le 2x$. $S: x^2+y^2+z^2=4x$ 被 $x^2+y^2=2x$ 截下部分

Stokes $I=-\iint_S 2(z-y)dS=-2\iint_S z\,dS=-2\iint_D\sqrt{4x-x^2-y^2}\cdot\sqrt{1+z_x^2+z_y^2}\,dxdy=-4\pi$.

or $\begin{cases}x=1+\cos t\\ y=\sin t\\ z=\sqrt{2+2\cos t}\end{cases}$，$0\le t\le 2\pi$

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六、(1) 6' $f(x)=\frac{\pi}{2}-x$, $x\in[0,\pi]$ 展成余弦级数，(2) 10' 求 $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$, $\sum_{n=1}^{\infty}\frac{1}{n^2}$, $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

(1) $f(x)\sim\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$. 由 Dirichlet thm. $f(x)\to S_n(x)$, $x\in[0,\pi]$.

(2). $f(0)=\frac{\pi}{2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}\times\frac{4}{3}=\frac{\pi^2}{6}$

由 Parseval 等式 $\Rightarrow\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{96}\times\frac{16}{15}=\frac{\pi^4}{90}$.

七、(1) 4' $\varphi(\alpha)=\int_0^{\infty}\frac{x^\alpha}{1+x^2}dx$ 收敛范围，(2) 6' 计算收敛时 $\varphi(\alpha)$

(3) 6' 证明：$\varphi(\alpha)$ 在 $[-\alpha_0,\alpha_0]$ 上一致收敛，$\alpha_0\in(0,1)$.

证：(1) $\varphi(\alpha)=\int_0^1\frac{x^\alpha}{1+x^2}dx+\int_1^{+\infty}\frac{x^\alpha}{1+x^2}dx$

$x\to0^+$, $\frac{x^\alpha}{1+x^2}\sim x^\alpha$ 收敛 $\Leftrightarrow\alpha>-1$

$x\to+\infty$, $\frac{x^\alpha}{1+x^2}\sim x^{\alpha-2}$ 收敛 $\Leftrightarrow 2-\alpha>1\Leftrightarrow\alpha<1$. $\therefore\alpha\in(-1,1)$

(2) 令 $x^2=z$, $\varphi(\alpha)=\frac{1}{2}\int_0^{+\infty}\frac{z^{\frac{\alpha-1}{2}}}{1+z}dz=\frac{1}{2}B(\frac{1+\alpha}{2},\frac{1-\alpha}{2})$ 令 $\frac{z}{1+z}$ 换 $\frac{1}{1+z}$

$=\frac{1}{2}\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{1-\alpha}{2})/\Gamma(1)=\frac{\pi}{2\sin\frac{1+\alpha}{2}\pi}$

(3) $0<\alpha_0<1$, $\alpha\in[-\alpha_0,\alpha_0]$, $0<x\le1$ 时, $\frac{x^\alpha}{1+x^2}\le x^{-\alpha_0}=\frac{1}{x^{\alpha_0}}$ 收敛 由 Weierstrass 判别法 $\Rightarrow\int_0^1\frac{x^\alpha}{1+x^2}dx$ 一致收敛

$x\ge1$ 时, $\frac{x^\alpha}{1+x^2}\le\frac{x^{\alpha_0}}{1+x^2}<\frac{1}{x^{2-\alpha_0}}$ 收敛 $\Rightarrow\int_1^{+\infty}\frac{x^\alpha}{1+x^2}dx$ 一致收敛

八、8' P、Q 二阶连续可导，在 L 上 $\begin{cases}x=x_0+r\cos\theta\\y=y_0+r\sin\theta\end{cases}$, $(\theta\in[0,\pi])$ 都有 $\int_L P(x,y)dx+Q(x,y)dy=0$. 证：$P=0$, $\frac{\partial Q}{\partial x}=0$

L 直径 AB：D = L∪AB 内部.

$\int_{L\cup AB}Pdx+Qdy=(\int_L+\int_{AB})P(x,y)dx+Q(x,y)dy=\int_{AB}P(x,y)dx+Q(x,y)dy$

由 Green 公式和积分中值定理，$\int_{L\cup AB}Pdx+Qdy=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})|_M\cdot\frac{\pi}{2}r^2$, $M\in D$.

$\int_{AB}P(x,y)dx+Q(x,y)dy=\int_{x_0-r}^{x_0+r}P(x,y_0)dx=2rP(\xi,x_0)$, $\xi\in(x_0-r,x_0+r)$.

$r\to0$ 时，$M\to(x_0,y_0)\Rightarrow P(x_0,y_0)=0$ 由 (x_0,y_0) 任意性 $\Rightarrow P(x,y)=0\Rightarrow\frac{\partial Q}{\partial x}|_M=0\Rightarrow\frac{\partial Q}{\partial x}=0$.

九、(1) 10' $\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}\sin x\,dx$ (2) 10' 用 Euler 积分 算 $\int_0^1\frac{x^{n-1}}{\sqrt{1-x^m}}dx$.

(1) $I=\int_0^{+\infty}\int_a^b e^{-ux}\sin x\,du\,dx$ 收敛

① $f(x,u)$ 在 $[a,b]$ 上一致收敛. $f(x,u)\le e^{-ax}$

$I_1=\int_0^{+\infty}e^{-ux}\sin x\,dx=\frac{1}{u}\int_0^{+\infty}e^{-ux}\cos x\,dx=\frac{1}{u^2}-\frac{1}{u^2}I_1$

$\Rightarrow I_1=\frac{1}{u^2+1}\Rightarrow I=\arctan b-\arctan a$

(2) 令 $1-x^m=t\Rightarrow x=(1-t)^{\frac{1}{m}}$

$\Rightarrow I=\int_1^0(1-t)^{\frac{n-1}{m}}\cdot t^{-\frac{1}{2}}\cdot\frac{1}{m}(1-t)^{\frac{1}{m}-1}dt$

$=-\int_0^1\frac{1}{m}(1-t)^{\frac{n}{m}-1}\cdot t^{-\frac{1}{2}}dt=-\frac{1}{m}B(\frac{1}{2},\frac{n}{m})$

$-\frac{\sqrt{\pi}}{m}\frac{\Gamma(\frac{n}{m})}{\Gamma(\frac{1}{2}+\frac{n}{m})}$

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