Linear Algebra- Review And Beyond

Lecture 1

In this lecture, we will talk about the most basic and important concept of linear algebra — **vector space**. After the basics of vector space, I will introduce dual space, annihilator, module, infinite dimensional vector space and various structures on vector space.

1 Basics of vector space

For beginners, the term "space" is not easy to understand. When we say "space" in mathematics, we usually mean a set with some certain structures, which is far different from the meaning in physics. And most students tend to think that a vector is an $n - tuple$. However, in fact we even don't require vector to be specific. In linear algebra, vector is just the basic object we deal with.

Definition(vector space) V is a non-empty set with elements called "vector", and F is a field (with multiplication identity 1) with elements called "scalar". We have two operations: addition between vectors,

$$+ : V \times V \to V, (a, b) \mapsto a + b$$

multiplication between scalars and vectors

$$\cdot : F \times V \to V, (\lambda, a) \mapsto \lambda a$$

We say V is a vector space on F if

- $a + b = b + a, \forall a, b \in V$
- $a + (b + c) = a + b + b, \forall a, b, c \in V$
- $\exists 0 \in V, \forall a \in V, 0 + a = a + 0 = a$
- $\forall a \in V, \exists - a \in V, a + (-a) = (-a) + a = 0$
- $\forall a \in V, \forall \lambda, \mu \in F, \lambda\mu a = \lambda(\mu a)$
- $\forall a \in V, 1a = a$
- $\forall \lambda \in F, \forall a, b \in V, \lambda(a + b) = \lambda a + \lambda b$
- $\forall \lambda, \mu \in F, \forall a \in V, (\lambda + \mu)a = \lambda a + \mu a$

Remark

- This definition is too long. However, if we are careful enough, we can discover that the first 4 requirements have nothing to do with F. More accurately, the first 4 requirements can be replaced by 1: V is an abelian group under the vector addition. The last 4 requirements insure that the multiplication is "consistent" with the addition. In fact, we will see these requirements come from the definition of "module".
- Some books require F to be a number field. That is more acceptable for beginners. However, the definition with any field F is more general. When we talk about module, we even don't require F to be a field — a commutative ring is enough.
- Some students may ignore the role that F plays. This is insane! When we talk about a vector space, the scalar field is important. When you say V is a vector space, unless we all know the scalar field, then you are wrong.
- For now, we only have **linear** structure here. That is to say, we talk nothing about inner product, topology, metric and so on.

Fact V is a vector space on F

- The zero vector and inverse vector are unique (because V is a group!)
- $\lambda \in F, a \in V$, $\lambda a = 0$ if and only if $a = 0$ or $\lambda = 0$
- $\lambda \in F, a \in V$, then $(-\lambda)a = \lambda(-a) = -\lambda a$

Example If we have a field F, then F^n is a vector space on F. Take $F = \mathbb{R}, \mathbb{C}$, we get the most common vector space. You must know that $\mathbb{R}^n$ is not a vector space on $\mathbb{C}$. However, you should verify that $\mathbb{C}^n$ is also a vector space on $\mathbb{R}$. As vector spaces, $\mathbb{C}^n$ on $\mathbb{C}$ is different from $\mathbb{C}^n$ on $\mathbb{R}$.

Example F is a field. All polynomials with degree less than n is a vector space.

Example All functions on $\mathbb{R}$ with scalar field $\mathbb{R}$ is a vector space.

Definition(Subspace) V is a vector space on F. If $U \subset V$ and the addition and multiplication is closed in U, then we say U is a subspace of V.

Remark If U is a subspace of V, then U is a vector space on F itself.

Exercise Suppose U, W are two subspaces of vector space V on F. Then $U + W, U \cap W$ are also subspaces.

Exercise Give an example of a nonempty subset U of a vector space V on F such that U is closed under addition and under taking additive inverses, but U is not a subspace of V.

Exercise Give an example of a nonempty subset U of a vector space V on F such that U is closed under scalar multiplication, but U is not a subspace of V.

Exercise Prove that the union of two subspaces is still a subspace if and only if one of the subspaces is contained in the other.

Definition(Linear Combination) V is a vector space on F, $v_1, v_2, ..., v_j \in V$, then we say $\sum_{i=1}^{j} \lambda_i v_i, \lambda_i \in F$ is a linear combination of $v_1, v_2, ..., v_j$.

Definition(Span) V is a vector space on F, $v_1, v_2, ..., v_j \in V$. All linear combinations of $v_1, v_2, ..., v_j$ is a subspace. We say this subspace is spanned by $v_1, v_2, .., v_j$.

Definition(Linearly independent) V is a vector space on F, $v_1, v_2, ..., v_j \in V$. If $\sum_{i=1}^{j} \lambda_i v_i = 0, \lambda_i \in F$ implies $\lambda_i = 0$, then $v_1, v_2, ..., v_j$ are linearly independent.

Definition(Linearly independent) V is a vector space on F, $v_1, v_2, ..., v_j \in V$. If $\sum_{i=1}^{j} \lambda_i v_i = 0, \lambda_i \in F$ with some $\lambda_i \neq 0$, then $v_1, v_2, ..., v_j$ are linearly dependent.

Fact

- A vector set containing zero vector must be linearly dependent.
- $S' \subset S$, if S' is linearly dependent, then so is S; if S is linearly independent, so is S'.

Theorem Non-zero vectors $v_1, v_2, .., v_j$ is linearly dependent if and only if $\exists m \leq j$ such that v_m is the linear combination of $v_1, v_2, ..., v_m$ if and only if there is some vector v_m can be represented by others.

Remark

- Note that in the above 4 definitions, the sum is finite sum. Thus, for a vector set with infinite elements, we can't use the definition directly. In fact, for a infinite vector set S, we say S is linearly independent if and only if any finite subset of S is linear independent.
- Linearly independent and linear dependent is the most basic concepts in linear algebra. It comes from the idea that we want to determine how many equations indeed in a linear equations system. If a vector set is linearly independent, we think that every member of them can't be replaced by others.

- Linearly independent and linearly dependent is the property inside a vector set. It's meaningless to say two vector sets are linearly independent or linearly dependent.
- The theorem tells us that a vector set is linearly dependent if and only if there is some vector v_m can be represented by others. However, not all vectors in this vector set can be represented by others. The key point is existence.

Exercise In R^n, we have different $x_1, x_2, ..., x_m, m \leq n$. Prove $\{(1, x_i, x_i^2, ..., x_i^{m-1}), 1 \leq i \leq m\}$ is linearly independent.

Exercise Suppose V is the set of all continuous real-valued functions on $[0, 1]$, then V is a real vector space. $S = \{f_i | f_i(x) = x^i, i = 1, 2, ...\}$ is linearly independent.

Definition(basis and dimension) V is a vector space on F. $S \subset V$ is a linearly independent vector set and $span(S) = V$, then we say S is a basis of V. The cardinality of S is called the dimension of V, and we use $\dim V$ to denote.

Remark The definition above is well-defined as we have the following theorem. The proof of the following theorem is an important and standard trick in linear algebra which is called **Steinitz exchange**.

Theorem V is a vector space on F. $v_1, v_2, ..., v_n$ is linearly independent and $s_1, s_2, ..., s_m$ spans V, then we have $n \leq m$.

Remark Basis of a vector space is extremely important. If we want to study a vector space, it's enough to study the basis. In this way, it plays a similar role as primes play in integers.

Example As we talked above, $\mathbb{C}^n$ on $\mathbb{C}$ and $\mathbb{C}^n$ on $\mathbb{R}$ are both vector spaces. Find their dimensions.

Example What's the dimension of $\mathbb{R}^n$ on $\mathbb{Q}$? What's the dimension of all real-coefficient polynomials on $\mathbb{R}$?

Example(Odd-Even Town) This is an extremely beautiful application of the theorem above. The problem names "Odd-Even Town". Imagine there is a town with n people, and there are many clubs in this town. We require that:

- Any club must have an odd number of members.
- Any two club must have an even number of members in common.

Question:What's the maximum number of clubs?

Definition(Isomorphism) We have two vector spaces V_1, V_2 on the same field F and a bijection between them. If the bijection contains vector addition and scalar multiplication, then the mapping is an isomorphism.

As long as we have the concepts of basis and dimension, we can introduce the most important result in linear algebra:

Theorem All linear vector space of dimension n on field F is isomorphic. Especially, they are all isomorphic to F^n.

Remark

- Note that we require they have the same scalar field.
- This theorem tells us that n-dimension vector space on field F is unique to some extent , that is F^n. However, you must notice that the isomorphism we say here is just linear isomorphic. In fact, we talk nothing about linear structure of vector space. Here comes an example:

Example Consider V as the vector space of all real polynomials on $\mathbb{R}$ with degree less than n, then it must be isomorphic to $\mathbb{R}^n$. However, as we can imagine , the definition of inner product and distance in $\mathbb{R}^n$ is natural while you can't extend these to V.

You may have guessed that there are some relations between linearly independent vector set and basis. In fact, any linearly independent vector set can be extended to a basis. we have the following:

Theorem V is a vector space. Suppose $v_1, v_2, ..., v_m$ is linearly independent, then we can find $u_1, u_2, .., u_k$ such that $v_1, ..., v_m, u_1, ..., u_k$ is a basis of V.

It's obvious that a vector space with low dimension is easier to study. So for a vector space, if we can decompose it to various smaller subspace, then many questions become easy. But not every decomposition is good. Thus, we need the concept of direct sum.

Lemma If S, T are two subspaces of V, then $\dim S + \dim T = \dim(S + T) + \dim(S \cap T)$.

Remark The proof of this lemma is also a standard method in linear algebra —- extension.

Definition(Direct Sum) V is a vector space, and $U_1, ..., U_m$ are vector subspaces of V. If $\forall v \in V$ can be **uniquely** represented by $v = \sum_{i=1}^{m} u_i$ where $u_i \in U_m$, then we say V is the direct sum of $U_1, U_2, ..., U_m$, we denote this by $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$.

Theorem V is a vector space, and $U_1, ..., U_m$ are vector subspaces of V and $W = \sum_{i=1}^{m} U_i$. The following are equal:

- $W = U_1 \oplus U_2 \oplus \cdots \oplus U_m$.
- $\sum_{i=1}^{m} u_i = 0, u_i \in U_i$ implies $u_i = 0, 1 \leq i \leq m$.
- $U_i \cap (\sum_{j=1, j\neq i}^{m} U_j) = \emptyset$.
- $U_i \cap (\sum_{j=1}^{i-1} U_j) = \emptyset$.
- $\dim(\sum_{i=1}^{m} U_i) = \sum_{i=1}^{m} \dim U_i$.
- Suppose M_i is a basis of U_i, then $\cup_{i=1}^{m} M_i$ is a basis of W.

Exercise Prove or give a counterexample,if U_1, U_2, W are subspaces of V:

- If $U_1 + W = U_2 + W$, then $U_1 = U_2$
- If $U_1 \oplus W = U_2 \oplus W$, then $U_1 = U_2$

Definition(quotient space) V is a vector space on F, and S is a subspace of V. Then $V/S = \{v + S | v \in V\}$ is a vector space on F with addition:

$$(u + S) + (v + S) = (u + v) + S, \ where\ u, v \in V$$

and scalar multiplication

$$\lambda(u + S) = \lambda u + S, \lambda \in F, a \in V$$

Remark As we mentioned in lecture 0, quotient space is the extension of the concept of congruence. In V/S, we see two vectors the same if their difference is in S. That is to say, we ignore the different part in S.

Exercise V is a vector space and S is a subspace. Prove that $\dim S + \dim V/S = \dim V$.

2 More about vector space

2.1 Dual space

Now, we have the concept of vector space, then we can study functions on vector space. In linear algebra, most things we care are all linear. So we just care about linear function on a vector space.

Definition(Linear Functional) V is a vector space on F, a map $l : V \to F$ is a linear functional if $\forall a, b \in F, x, y \in V, l(ax + by) = al(x) + bl(y)$

Definition(Dual Space) All linear functional on a vector space V is still a vector space, with the same scalar field F. We call this vector space the dual space of V, and denote this by V^*.

Remark Suppose V has a basis $\{e_1, e_2, ..., e_n\}$, then $\forall v \in V$ has the form $v = \sum_{i=1}^n a_i e_i$. By linearity, $l(v) = l(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i l(e_i)$. So we just need to consider the result of l acts on the basis.

Exercise Define e_i^* by $e_i^*(e_j) = \delta_{ij}$, then e_i^* is a linear functional. You can see this as a projection to the direction of e_i. You can also see the e_i^* takes out the coordinate of a vector on the ith position: if $v = \sum_{j=1}^n a_j e_j$, then $e_i^*(v) = e_i^*(\sum_{j=1}^n a_j e_j) = \sum_{j=1}^n a_j e_i^*(e_j) = a_i$.

Remark As above, $l(v) = \sum_{i=1}^n a_i l(e_i) = \sum_{i=1}^n l(e_i) e_i^*(v)$, so $l = \sum_{i=1}^n l(e_i) e_i^*$ is a linear combination of $\{e_1^*, e_2^*, ..., e_n^*\}$.

Definition(Dual basis) V is a vector space on F, with a basis $\{e_1, e_2, ..., e_n\}$, then $\{e_1^*, e_2^*, ..., e_n^*\}$ is a basis for V^*. This basis is called the dual basis.

Remark We have known that all vector spaces on the same scalar field with the same degree are isomorphic, so V is isomorphic to V^*. Thus, any linear functional on V must have the form $l = \sum_{i=1}^n a_i e_i^*$, if $v = \sum_{i=1}^n b_i e_i$, then $l(v) = \sum_{i=1}^n a_i b_i$! You may guess there is some connection between linear functional and inner product.

Exercise Let P be the space of all real polynomials of degree $\leq n$ on $\mathbb{R}$, and $t_1, t_2, ..., t_n$ are n different points.

- Check $l_j(p) = p(t_j)$ is a linear functional.
- Check $l_1, l_2, ..., l_n$ is a basis for P^*.
- Find a basis $\{e_1, e_2, ..., e_n\}$ of P, such that $l_i = e_i^*$.

Example I is an interval on $\mathbb{R}$, we have n different points $t_1, t_2, ..., t_n$, there are $m_1, m_2, ..., m_n$ such that

$$\int_I p(x)dx = \sum_{i=1}^n m_i p(t_i)$$

holds for any polynomial $p(x)$ with degree $< n$.(Use the result of last exercise)

We can repeat the things we do in V to V^*, then we get a double dual space V^{**}, and a double dual basis $\{e_i^{**}\}$. But in fact, we get nothing new because V^{**} is the same as V.

Fact

- $e_i : V^* \to F, e_i(l) \mapsto l(e_i)$ is a linear functional on V^*, which means for $l(e_i)$, we can regard this as l acts on e_i, and we can regard this as e_i acts on l as well. Especially, $e_i(e_j^*) = \delta_{ij}$. So $V \subset V^{**}$.
- On the other hand ,for $l \in V^*$, we must have $l = \sum_{i=1}^n a_i e_i^*$, then for any $l' \in V^{**}$, $l'(l) = l'(\sum_{i=1}^n a_i e_i^*) = \sum_{i=1}^n a_i l'(e_i^*) = \sum_{i=1}^n e_i(l) l'(e_i^*)$, so $l' = \sum_{i=1}^n l'(e_i^*) e_i$. Thus, $\{e_i\}$ is a basis for V^{**}.

Remark By the fact above, we know that V^* is the dual space of V while V is the dual space of V^*.

Definition(Annihilator) Suppose V is a vector space, $M \subset V$ is a non-empty set. $M^\perp = \{f \in V^* | f(M) = 0\}$ is the annihilator of M.

Fact

- $M^\perp$ is a subspace of V^*.
- Suppose $\dim V = n$, $\dim(SpanM) + \dim M^\perp = n$. (Prove by basis extension)
- Suppose $M \subset N$, then $N^\perp \subset M^\perp$.
- $(M^\perp)^\perp = Span(M)$.

Fact V is a vector space and $V = S \oplus T$, then $S^\perp = T^*, S^* = T^\perp, V^* = S^\perp \oplus T^\perp$.

Exercise Suppose W is the subspace spanned by $(1, 0, -1, 2)$ and $(2, 3, 1, 1)$ in $\mathbb{R}^4$, try to find $M^\perp$.

2.2 Regard vector space as module

In this section, we want to extend the definition of vector space to module. Module is one of the most important structure in algebra. You can regard module as a commutative ring acting on an abelian group.

Definition(Module) R is a commutative ring, M is an abelian group with addition $+$. We have another operation $\cdot : R \times M \to M, (r, m) \mapsto rm$ satisfying:

- $\forall u, v \in M, r \in R, r(u + v) = ru + rv$
- $\forall r, s \in R, u \in M, (r + s)u = ru + su$
- $\forall r, s \in R, u \in M, (rs)u = r(su)$

- $\forall u \in M, 1u = u$

Then we say M is a R-module.

Example Take $R = \mathbb{Z}$, then $\mathbb{Z}$-module is just an abelian group.

Example When R is a field F, F-module is just a vector space.

2.3 Infinite Dimensional vector space

Most vector spaces in linear algebra are finite dimensional. However, there are many infinite dimensional vector spaces and most of them are extremely useful in functional analysis.

Definition(Infinite dimensional Vector space) A vector space is infinite dimensional if it can be spanned by any finite subset.

Example Let V be the set of all polynomials on $\mathbb{R}$, then V is infinite dimensional.

Remark Vector spaces of functions with certain property are usually infinite dimensional. A main topic of functional analysis is to study the space of functions. In finite dimensional vector space, the concept of basis is very useful. However, in infinite dimensional vector space, we can't find a countable basis.

Definition(Hamel Basis) V is a vector space. $H \subset V$ is a Hamel basis if H is linearly independent and $\forall v \in V$, there are finitely many $e_i \in H, 1 \leq i \leq n$ such that $v = \sum_{i=1}^{n} \lambda_i e_i$, and the representation is unique.

Theorem Any vector space has a Hamel basis.(Prove by Zorn's Lemma)

Exercise Try to find a hamel basis for V, where V is the set of all polynomials on $\mathbb{R}$.

3 Structures on vector space

We talk nothing but linear structure till now. In fact, a vector space for analysis is too simple that we even can't talk anything about continuity. Thus, we must have other structures on vector space. I will introduce metric, norm and inner product structures. In general, if we weaken the requirements on scalar field F, we go far in algebra; if we add extra structure to a real or complex vector space, and don't assume the dimension to be finite, then we go far in analysis.

Note In the following, we always assume X is a vector space on F where $F = \mathbb{R}$ or $\mathbb{C}$.

Definition(Metric) A function $d : X \times X \to R$ on a vector space X is a metric, if

- $\forall x, y \in X, d(x, y) \geq 0$ and $d(x, y) = 0$ implies $x = y$.
- $\forall x, y \in X, d(x, y) = d(y, x)$.
- **triangle inequality:** $\forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z)$.

We call this vector space a metric vector space.

Remark You may feel familiar with this concept. For example, in $\mathbb{R}$, the absolute value is a metric. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the Euclidean distance is a metric. For the triangle inequality, you can regard it as the sum of length of any two edges is greater than the length of the other edge in a triangle.

Exercise Let X be all continuous functions on $[0, 1]$. Check the followings are all metric:

- $\forall f, g \in X,\ d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$.
- $\forall f, g \in X,\ d(f, g) = \int_0^1 |f(x) - g(x)| dx$.
- $\forall f, g \in X,\ d(f, g) = (\int_0^1 (|f(x) - g(x)|)^2 dx)^{1/2}$.

Definition(Translation Invariant) If d is a metric on X, and $\forall x, y, z \in X$, we have $d(x, y) = d(x + z, y + z)$.

Definition(Homogeneous) If d is a metric on X, and $\forall \lambda \in F, x, y \in X, d(\lambda x, \lambda y) = |\lambda| d(x, y)$.

Example On $\mathbb{R}$, we define $d(x, y) = |x| + |y|$ is a metric, but it's not translation invariant.

Example On $\mathbb{R}$, we define $d(x, y) = \frac{|x-y|}{1+|x-y|}$ is a metric, but it's not homogeneous.

Definition(Compatible Metric) We say a metric is compatible if it's translation invariant and homogeneous.

Remark This definition describes a metric with nice properties. In a vector space, translation invariant and homogeneous metric is compatible with the linear structure.

Definition(Norm) A function $|| \cdot || : X \to R$ on a vector space X is a norm, if

- $\forall x \in X, ||x|| \geq 0$ and $||x|| = 0$ implies $x = 0$.
- $\forall x \in X, \lambda \in F ||\lambda x|| = |\lambda| ||x||$.
- **triangle inequality:** $\forall x, y \in X, ||x + y|| \leq ||x|| + ||y||$.

We call this vector space a normed vector space.

Fact

- If we have a norm $|| \cdot ||$ on vector space X, then we can define a metric: $\forall x, y \in X, d(x, y) = ||x - y||$. You should verify that this metric is compatible.
- If we have a compatible metric d on vector space X, then we can define a norm :$\forall x \in X, ||x|| = d(x, 0)$.

Theorem A compatible metric vector space $\Leftrightarrow$ A normed vector space.

Definition(Real Inner product) V is a real vector space, $<, >: V \times V \to R$ is an inner product if

- $\forall v \in V, < v, v > \geq 0$ and $< v, v >= 0$ if and only if $v = 0$.
- $\forall u, v \in V, < u, v >=< v, u >$.
- $\forall u, v, w \in V, a, b \in \mathbb{R}, < au + bv, w >= a < u, w > +b < v, w >, < w, au + bv >= a < w, u > +b < w, v >$.

We call this space an Euclidean space.

Definition(Complex Inner product) V is a complex vector space, $<, >: V \times V \to R$ is an inner product if

- $\forall v \in V, < v, v > \geq 0$ and $< v, v >= 0$ if and only if $v = 0$.
- $\forall u, v \in V, < u, v >= \overline{< v, u >}$.
- $\forall u, v, w \in V, a, b \in \mathbb{C}, < au + bv, w >= a < u, w > +b < v, w >, < w, au + bv >= \bar{a} < w, u > +\bar{b} < w, v >$.

We call this space an unitary space.

Fact If V is a vector space with inner product $<, >$, then we define $\forall x \in V, ||x|| = \sqrt{(}< x, x >)$ is a norm on V. Thus, an inner product space must be a norm space, a norm space must be a metric space.

Example $\mathbb{R}^n$ is an inner product vector space.

Example Let V be all continuous functions on $[0,1]$, then $< f, g >= \int_0^1 f(x)g(x)dx$ is an inner product.

Fact In an inner product vector space X, we always have

- $\forall x, y \in X, ||x+y||^2 + ||x-y||^2 = 2||x||^2 + x||y||^2$.
- **Cauchy-Schwarz Inequality** $\forall x, y \in X, |< x, y >|^2 \leq ||x||^2||y||^2$ with equality holds if and only if x, y are linearly dependent.

We have seen various structures on vector space. However, what's the weakest structure we need to do analysis on vector space? That's topological structure. As I mentioned in lecture 0, we can talk about continuity as long as we have topology.

Definition(Topological Vector Space) Suppose X is a vector space with topology $\mathcal{F}$, we say X is a topological vector space if the vector addition and scalar multiplication is continuous under this topology.

Another important problem in analysis is limit. However, as I mentioned in lecture 0, we can't talk about limit if the space is not complete. So we need the concept of completeness in vector space.

Definition(Cauchy sequence) X is a topological vector space, a sequence $\{x_n\}$ is said to be a Cauchy sequence if for any open neighborhood U of 0, $\exists N$ such that $\forall m, n > N, x_m - x_n \in U$.

Remark You may feel uncomfortable with this definition. You have already know that if we have a metric, then we have a topology induced by the metric with open sets $\{y \in X | d(x, y) < r\}, x \in X, r \in \mathbb{R}$. So if we describe the definition in the language of metric, it becomes:

Definition(Cauchy sequence) X is a metric vector space with metric d, a sequence $\{x_n\}$ is said to be a Cauchy sequence if for any $\epsilon > 0$, $\exists N$ such that $\forall m, n > N, d(x_m, x_n) \leq \epsilon$.

Definition(Completeness) A metric vector space X with metric d is complete if any Cauchy sequence $\{x_n\}$, there is $x_0 \in X$ such that $d(x_n, x_0) < \epsilon$ for any $\epsilon > 0$ and n large enough.

Remark A complete normed vector space is named a **Banach Space** and a complete inner product vector space is named a **Hilbert space**.

Example $\mathbb{R}^n$ is a Hilbert space.

Example Let X be all continuous functions on $[0, 1]$. X with norm $||f(x)|| = max_{x\in[0,1]}|f(x)|$ is a Banach space. However, X with norm $||f(x)|| = \int_0^1 |f(x)|dx$ is not complete, thus it's just a normed vector space.