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What is the simplest Quantum Field Theory?

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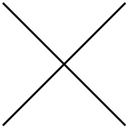
Lagrangian Method: Lagrangian–>Feynman Rules–>Amplitudes

- Focus on concrete scattering procedures.
- i.e. scalar field as the simplest QFT.
- Terribly complicated for higher spin fields, with gauge redundancy.

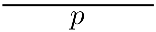
Amplitude Method: Amplitudes for a theory!

The simplest theory is scalar fields theory:

- i.e. $\lambda\varphi^4$ theory
- $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}\lambda\varphi^4$
- No spinor indices, no Lorentz indices.
- Quite simple amplitudes:



$= -i\lambda$



$= -\frac{i}{p^2 - m^2 + i\varepsilon}$

Figure: Feynman Rules for $\lambda\varphi^4$ theory

Really?

BCFW: On-shell recursion relations of amplitudes.

- Analytic continuation for momentum $p(z)$ and amplitudes $M(z)$.
- BCFW Shift: $p_1 \to p_1(z) = p_1 + zq, \quad p_2 \to p_2(z) = p_2 - zq$.
- On-shell: keep $p_1^2(z) = p_2^2(z) = 0$, thus $q^2 = 0$ means q is complex.

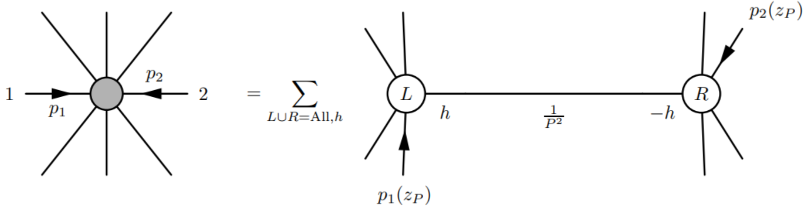


Figure: BCFW recursion relation

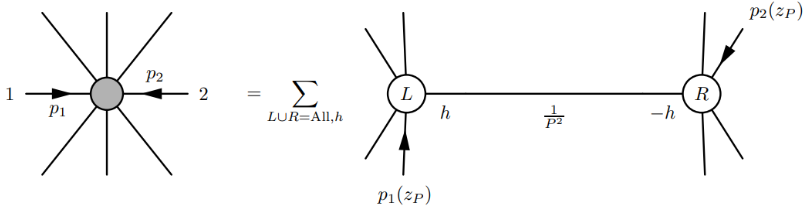


Figure: BCFW recursion relation

Now we can focus on the certain propagator:

$$\frac{1}{P^2(z)} = \frac{1}{(p_1(z) + \sum_{i\in L, i\neq 1} p_i)^2} = \frac{1}{P^2(0) + 2zq\cdot P}$$

- A single pole at $z_P = -\frac{P^2}{2q\cdot P}$

Traverse the whole diagram, we find several single poles of $M(z)$:

$$z_{P,j} = -\frac{P_j^2}{2q \cdot P_j} \qquad \text{(j traverse the diagram except 1,2)}$$

And we easily derive residues:

$$\mathrm{Res}[P^2(z)M(z), z_{P,j}] = \sum_{h=\pm} M_L(\{p_1(z_P), h_1\}, \{-P(z_P), h\}, L) \times M_R(\{p_2(z_P), h_2\}, \{P(z_P), -h\}, R)$$

According to Cauthy's residues theorem, we know:

$$M(0) = \sum_{i \text{ traverse}, h=\pm} M_L(\{p_1(z_P), h_1\}, \{-P(z_P), h\}, L) \times \frac{1}{P^2} \times M_R(\{p_2(z_P), h_2\}, \{P(z_P), -h\}, R) + \text{Residue at} \infty$$

From BCFW recursion, we decompose an n-point amplitudes as smaller:

$$\text{Amplitudes} = \sum_{\text{edges,heliciy}} \text{Left subgraph amplitude} \times \frac{1}{\text{edge momentum}^2} \times \text{Right subgraph amplitude} + \text{Residue at } \infty$$

Naively we take the residue at ∞ as 0 to get a wonderful relation. However, this can be wrong! According to:

- J. Bedford, A. Brandhuber, B. J. Spence and G. Travaglini [arXiv:hep-th/0502146]
- F. Cachazo and P. Svrcek, [arXiv:hep-th/0502160]
- P. Benincasa, C. Boucher-Veronneau and F. Cachazo [arXiv:hep-th/0702032].

Conclusion:

- $M^{\text{anything},-}_{\text{Yang-Mills}} \to \frac{1}{z}, \qquad M^{\text{anything},-}_{\text{Gravity}} \to \frac{1}{z^2}$
- $M^{\text{anything},-}_{\lambda\varphi^4} \to z^0$

Surprisingly, the so-called "simplest" $\lambda\varphi^4$ forbids BCFW recursion! Why?

Recall our diagram decomposition:

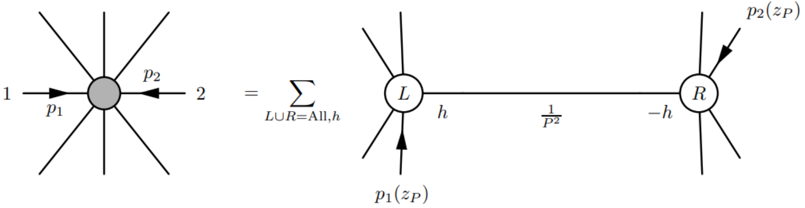


Figure: BCFW recursion relation

We have natually assumed that p_1 and p_2 is separated on either side of the certain factorization channels,P.
It is highly non-trivial that ensuring these channels factorize correctly guarantees that all channels factorize correctly
Encoded in the statement that $M(z) \to 0$ when $z \to \infty$.

Simplicity and structure is not everywhere!

1. Amplitudes of "simplest" $\lambda\varphi^4$ cannot be recursed.
2. Amplitudes of "most complicated" YM,Gravity have many hidden symmetries.
3. Easy Lagrangians $\neq$ easy amplitudes!

Now that we want amplitudes eventually, why don't we calculate amplitudes straightly?

Today we will focus on $\mathcal{N} = 4s$ maximally supersymmetric tree amplitudes, and find what SUSY gives us beyond our tradition Lagrangian methods.

Recall QFT, with Poincare $\text{Poin}(1,3)$ symmetry generated by P^μ and $M^{\mu\nu}$. Poincare Algebra writes:

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, M_{\rho\sigma}] &= \mathrm{i}(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}) \\
[M_{\mu\nu}, P_\rho] &= -\mathrm{i}g_{\mu\nu}P_\rho + \mathrm{i}g_{\rho\nu}P_\mu
\end{aligned}$$

SUSY extended these relations, to be the so-called SUSY Algebra.

Explicitly, we introduce undotted spinors Q^I_α and dotted spinors $\bar{Q}^I_{\dot{\alpha}}$:

$$[P_\mu, Q^I_\alpha] = [P_\mu, \bar{Q}^I_{\dot{\alpha}}] = 0$$
$$[M_{\mu\nu}, Q^I_\alpha] = \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta$$
$$[M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = \mathrm{i}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}}$$

and they satisfy their conjugate relations:

$$\{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta} Z^{IJ}$$
$$\{Q^{I\dot{\alpha}}, Q^{J\dot{\beta}}\} = \varepsilon^{\dot{\alpha}\dot{\beta}} Z^{*IJ}$$
$$\{Q^I_\alpha, Q^{J\dot{\beta}}\} = 2(\sigma^\mu)\alpha\dot{\beta} P_\mu \delta^{IJ}$$

This is SUSY algebra, and we name Q and $\bar{Q}$ as "supercharge".

With s the highest spin of the theory, we have $s = 1$ for $\mathcal{N} = 4$ SYM and $s = 2$ for $\mathcal{N} = 8$ SUGRA.
These maximally SUSY allow us to construct a supermultiplet, whose CPT conjugate is just itself.

- All states in SUSY are related by continuous SUSY transformations.
- All amplitudes in maximally SUSY should be labelled by **smooth Grassmann parameters**.

Let's try to write amplitudes. We care more about massless amplitudes, so we take $Z^{IJ} = 0$.

Supercharge Q^I_α and $Q^{\dot\alpha}_I$ implies 2 kinds of symmetries.

- Spinor indices α and $\dot\alpha$, implies Lorentz symmetry.
- SUSY indices I and J, implies SUSY $SU(\mathcal{N})$ R-symmetry.
- An object with an (upper) lower I index is in the (anti-) fundamental representation of $SU(\mathcal{N})$

Thus, we use supercharge to create coherent states:

$$|\bar\eta, \lambda, \bar\lambda\rangle = \mathrm{e}^{\bar Q^{I\dot\alpha}\bar\omega_{\dot\alpha}\bar\eta_I}\,|{+s}, \lambda, \bar\lambda\rangle$$
$$|\eta, \lambda, \bar\lambda\rangle = \mathrm{e}^{Q_{I\alpha}\omega^\alpha\eta^I}\,|{-s}, \lambda, \bar\lambda\rangle$$

with $\langle\omega\lambda\rangle = [\bar\omega\bar\lambda] = 1$, and $\eta_I, \bar\eta^I$ two Grassmann variables.

Q and $\bar{Q}$ are applied to spin states:

$$Q_{I\alpha}\left|-s\right\rangle=\lambda_{\alpha}\left|-s+\frac{1}{2}\right\rangle_{I},\quad \bar{Q}^{I\dot{\alpha}}\left|+s\right\rangle=\bar{\lambda}^{\dot{\alpha}}\left|+s-\frac{1}{2}\right\rangle^{I}$$

Of course, Q and $\bar{Q}$ conjugates:

$$Q\left|+\right\rangle=\bar{Q}\left|-\right\rangle=0$$

ω is fixed up to a shift: $\omega_{\alpha}\sim\omega_{\alpha}+c\lambda_{\alpha}$,thus $\left|\eta\right\rangle\sim\left|\eta+c_{I}\lambda_{\alpha}\right\rangle$. We can fix this redundancy by denoting $\eta_{I\alpha}=\omega_{\alpha}\eta_{I}$

Note that states labelled by η and $\bar{\eta}$ can diagnalize Q and $\bar{Q}$:

$$Q_{I\alpha}\left|\bar{\eta}\right\rangle = \bar{\eta}_I \lambda_\alpha \left|\bar{\eta}\right\rangle , \quad \bar{Q}^{I\dot{\alpha}}\left|\eta\right\rangle = \bar{\lambda}^{\dot{\alpha}}\eta^I \left|\eta\right\rangle$$

When we use $|eta\rangle$, we call the amplitude is in the η representation, while $|\bar{\eta}\rangle$ for $\bar{\eta}$ representation. They are related via a Grassmann Fourier Transform:

$$\left|\bar{\eta}\right\rangle = \int \mathrm{d}^{\mathcal{N}}\eta \mathrm{e}^{\eta\bar{\eta}}\left|\eta\right\rangle , \quad \left|\eta\right\rangle = \int \mathrm{d}^{\mathcal{N}}\bar{\eta} \mathrm{e}^{\bar{\eta}\eta}\left|\bar{\eta}\right\rangle$$

Recall the definition of coherent states, we have:

$$\mathrm{e}^{Q_{I\alpha}\zeta^{I\alpha}}\left|\eta\right\rangle = \left|\eta + \langle\zeta\lambda\rangle\right\rangle, \mathrm{e}^{Q_{I\alpha}\zeta^{I\alpha}}\left|\bar{\eta}\right\rangle = \mathrm{e}^{\bar{\eta}_J\langle\lambda\zeta^J\rangle}\left|\bar{\eta}\right\rangle$$

- Q shifts η and rephases $\bar{\eta}$.
- $\bar{Q}$ will do the opposite.

These SUSY transformations change Grassmann parameters only, while $\lambda, \bar{\lambda}$ stay invariant.

Scattering amplitudes are smooth functions of Grassmann variables:

$$M(\{\eta_i, \lambda_i, \bar{\lambda}_i\}, \{\bar{\eta}_{\bar{i}}, \lambda_{\bar{i}}, \bar{\lambda}_{\bar{i}}\})$$

Recall that little group transformations of spinor is $t = \Lambda^{-2s}$,we have:

$$M\left(\{t_i\eta_i, t_i\lambda_i, t_i^{-1}\bar{\lambda}_i\}; \{t_{\bar{i}}^{-1}\eta_{\bar{i}}, t_{\bar{i}}\lambda_{\bar{i}}, t_{\bar{i}}^{-1}\bar{\lambda}_{\bar{i}}\}\right)$$
$$= \prod_{i,\bar{i}} t_i^{2s} t_{\bar{i}}^{-2s} M\left(\{\eta_i, \lambda_i, \bar{\lambda}_i\}; \{\bar{\eta}_{\bar{i}}, \lambda_{\bar{i}}, \bar{\lambda}_{\bar{i}}\}\right)$$

Note that $t_i\lambda_i = \lambda_i, t_i^{-1}\bar{\lambda}_i = \bar{\lambda}_i$, we know:

$$M(\eta_i; \bar{\eta}_i) = \mathrm{e}^{\sum_j [\bar{\lambda}_j \bar{\zeta}]\eta_j + \sum_{\bar{j}} \langle\lambda_{\bar{j}}\zeta\rangle \bar{\eta}_{\bar{j}}} M(\eta_i + \langle\lambda_i\zeta\rangle; \bar{\eta}_{\bar{i}} + [\bar{\lambda}_{\bar{i}}\bar{\zeta}])$$

Shift and rephase imply special structures of amplitude M.

First we consider shift. An obvious trick is that we take a certain Grassmann variable:

$$M(\eta_i) = M(\eta_i + \langle \lambda_i, \zeta \rangle)$$

For instance, we take

$$\zeta_{I\alpha} = \frac{\eta_{2I}\lambda_{1\alpha} - \eta_{1I}\lambda_{2\alpha}}{\langle 12 \rangle} \quad (1)$$

Surprisingly, $\eta_1 \to 0, \eta_2 \to 0$.
With this trick,we can set 2 Grassmann variables to 0.

Then we consider rephase. Recall that in QFT, we have:

$$M(p_i) = \mathrm{e}^{ix\sum_j p_j} M(p_i)$$

which reflects spacetime translation invariance of the amplitude, as:

$$M(p_i) = \delta^{(4)}(\sum_j p_j)\hat{M}(p_i)$$

Now that we have: $M(\eta_i) = \mathrm{e}^{\bar{\zeta}\sum_j \bar{\lambda}_j \eta_j} M(\eta_i)$

caused by $\bar{Q}$ SUSY transformations, we know all amplitudes must be proportional to:

$$M(\eta_i) = \delta^{2\mathcal{N}}(\sum_i \bar{\lambda}_i \eta_i)\hat{M}(\eta_i)$$

Moreover, we have to consider a hidden relation.

- Since maximally SUSY is a CPT invariant theory, our choice by η and $\bar{\eta}$ can only reflect its PT invariance:

$$\int \prod_i \mathrm{d}^{\mathcal{N}} \eta_i \mathrm{e}^{\bar{\eta}_i \eta_i} M(\eta_i) = M(\bar{\eta}_i)$$

We have to argue that there should be $\lambda \leftrightarrow \bar{\lambda}$. However, with a PT reversal, $\langle \lambda_i \lambda_j \rangle \leftrightarrow [\bar{\lambda}_i \bar{\lambda}_j]$

- We can treat the new amplitude still in η representation, only to evaluate with $\eta_i \to \bar{\eta}_i$

In this section, we will see the vanishing of $M^{++\cdots+}$ and $M^{++\cdots+-}$.
First we take the amplitude:

$$M^{++\cdots+} = \int \mathrm{d}^{\mathcal{N}}\eta_1 \ldots \mathrm{d}^{\mathcal{N}}\eta_n M(\eta_1, \ldots, \eta_n)$$

use Q SUSY, we let η_1 vanish:

$$M^{++\cdots+} = \int \mathrm{d}^{\mathcal{N}}\eta_1 \ldots \mathrm{d}^{\mathcal{N}}\eta_n M(0, \eta_2', \ldots, \eta_n')$$

For Grassmann variables, $\int \mathrm{d}\eta(*) = \frac{\partial}{\partial\eta}(*)$,so:

$$M^{++\cdots+} = 0$$

Then, similar tricks for $M^{++\cdots+-}$:

$$M^{++\cdots+-} = \int \mathrm{d}^{\mathcal{N}}\eta_1 \ldots \mathrm{d}^{\mathcal{N}}\eta_{n-1}\mathrm{d}^{\mathcal{N}}\bar{\eta}_n M(\eta_1, \ldots, \eta_{n-1}, \bar{\eta}_n)$$
$$= \int \mathrm{d}^{\mathcal{N}}\eta_1 \ldots \mathrm{d}^{\mathcal{N}}\eta_{n-1}\mathrm{d}^{\mathcal{N}}\bar{\eta}_n \mathrm{e}^{\bar{\eta}_n(A\eta_1+B\eta_2)} M(0, 0, \ldots, \eta_{n-1}, \bar{\eta}_n)$$

This time we cancel η_1, η_2. A, B can be calculated as equation(1), but we don't need to write it explicitly. We just take $\eta_a = A\eta_1 + B\eta_2, \eta_b = C\eta_1 + D\eta_2$, A, B, C, D are constants independent of η_i. Hence:

$$\mathrm{d}^{\mathcal{N}}\eta_1\mathrm{d}^{\mathcal{N}}\eta_2 \to \mathcal{J}\mathrm{d}^{\mathcal{N}}\eta_a\mathrm{d}^{\mathcal{N}}\eta_b$$

Just integrate over η_b and get $M^{++\cdots+-} = 0$

For spin s,MHV amplitude is defined as:

$$M^{++\cdots+--}=\int \mathrm{d}^{\mathcal{N}}\eta_1\mathrm{d}^{\mathcal{N}}\eta_2\ldots\mathrm{d}^{\mathcal{N}}\eta_{n-2}\mathrm{d}^{\mathcal{N}}\bar{\eta}_{n-1}\mathrm{d}^{\mathcal{N}}\eta_n M(\eta_1,\eta_2,\ldots,\eta_{n-2},\bar{\eta}_{n-1},\bar{\eta}_n)$$

As showed above, we have η_1,η_2 vanished:

$$\begin{aligned}M(\eta_1,\eta_2,\ldots,\eta_{n-2},\bar{\eta}_{n-1},\bar{\eta}_n) =& \mathrm{e}^{\sum_{i=n-1}^{n}\bar{\eta}_i(A_i\eta_1+B_i\eta_2)}\\ &\times M(0,0,\eta_3',\ldots,\eta_{n-2}',\bar{\eta}_{n-1},\bar{\eta}_n)\end{aligned}$$

This time, we have to get A_i, B_i:

$$A_{n-1}=\frac{\langle 2(n-1)\rangle}{\langle 12\rangle},\quad A_n=\frac{\langle 2n\rangle}{\langle 12\rangle}$$

$$B_{n-1}=\frac{\langle (n-1)1\rangle}{\langle 12\rangle},\quad B_n=\frac{\langle n1\rangle}{\langle 12\rangle}$$

Also take: $\eta_{n-1} = A_1\eta_1 + B_1\eta_2, \eta_n = A_2\eta_1 + B_2\eta_2$ with a Jacobian:

$$\mathcal{J} = \left(\frac{\langle 2(n-1)\rangle \langle n1\rangle - \langle 2n\rangle \langle (n-1)1\rangle}{\langle 12\rangle^2}\right)^{\mathcal{N}} = \left(\frac{\langle (n-1)n\rangle}{\langle 12\rangle}\right)^{\mathcal{N}}$$

the last step depend on Schouten Identity.

- It seems that Jacobian should be $\mathcal{J}^{-1}$. If you think so, please read this article.

Thus, the integral can be written as:

$$M^{++\cdots+--} = \left(\frac{\langle (n-1)n\rangle}{\langle 12\rangle}\right)^{\mathcal{N}} \int \mathrm{d}^{\mathcal{N}}\eta_{n-1}\mathrm{d}^{\mathcal{N}}\eta_n\mathrm{d}^{\mathcal{N}}\eta_3\ldots\mathrm{d}^{\mathcal{N}}\eta_{n-2}$$

$$\int \mathrm{d}^{\mathcal{N}}\bar{\eta}_{n-1}\mathrm{d}^{\mathcal{N}}\bar{\eta}_n \mathrm{e}^{\bar{\eta}_{n-1}\eta_{n-1}}\mathrm{e}^{\bar{\eta}_n\eta_n} M(0,0,\eta_3,\ldots,\eta_{n-2},\bar{\eta}_{n-1},\bar{\eta}_n)$$

Integrate over $\bar{\eta}_{n-1}, \bar{\eta}_n$ to have a partly η representation:

$$M^{++\cdots+--} = \left(\frac{\langle(n-1)n\rangle}{\langle 12\rangle}\right)^{\mathcal{N}} \int \mathrm{d}^{\mathcal{N}}\eta_{n-1}\mathrm{d}^{\mathcal{N}}\eta_n\mathrm{d}^{\mathcal{N}}\eta_3\ldots\mathrm{d}^{\mathcal{N}}\eta_{n-2} \\ M(0,0,\eta_3,\ldots,\eta_{n-2},\eta_{n-1},\eta_n)$$

We have (or haven't?) know that for Grassmann varibles η:

$$\int \mathrm{d}\eta 1 = 0, \quad \int \mathrm{d}\eta\eta = 1$$

So, we can treat $\delta(\eta-\eta')$ as $(\eta-\eta')$. Exactly, for Grassmann odd δ:

$$\delta^{(2)}(\lambda^\alpha\theta_1 + \mu^\alpha\theta_2) = \delta(\theta_1)\delta(\theta_2)\langle\lambda\mu\rangle$$

Having used these properties to resume η_1, η_2,we conclude that:

$$M^{++\cdots+--} = \left(\frac{\langle (n-1)n\rangle}{\langle 12\rangle}\right)^{\mathcal{N}} M^{--+\cdots++}$$

This is the well-known form of the Ward identities for MHV amplitudes. It implies:

$$M(i-,j-) = \langle ij\rangle^{\mathcal{N}}\, \hat{M}_{\text{MHV}}(\lambda_i, \bar{\lambda}_i)$$

- For $\mathcal{N}=4$ SYM, $\hat{M}_{\text{MHV}} = \dfrac{1}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle}$ is the famous Parke-Taylor amplitude.
- While for $\mathcal{N}=8$ SUGRA, $\hat{M}$ is more complicated–not holomorphic, depending on both $\lambda_i, \bar{\lambda}_i$ as well.

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- When reseaching the MHV amplitude, we set $\eta_{1,2} \to 0$ by equation (1).
- But for 3-point amplitudes, $\langle 12 \rangle$ maybe 0, so that our cancellation is invalid.

In fact, we know in 3-point problem, either all $\langle ij \rangle = 0$ or all $[ij] = 0$.

- So we can always find the non-zero term, and set it to zero.

With 3-point YM and GR amplitudes, we can fix the amplitude by SUSY:

$$M(\eta_i) = \frac{\Delta(\eta_i)}{([12][23][31])^s} + \frac{\bar{\Delta}(\eta_i)}{(\langle 12\rangle \langle 23\rangle \langle 31\rangle)^s}$$

Here:

$$\Delta(\eta_i) = \delta^{2\mathcal{N}}(\sum_i \bar{\lambda}_i \eta_i)$$

$$\bar{\Delta}(\eta_i) = \int \mathrm{d}^{\mathcal{N}} \bar{\eta} \mathrm{e}^{\bar{\eta}\eta} \delta^{2\mathcal{N}}(\sum_i \lambda_i \bar{\eta}_i)$$

- The denominator is fixed by the required little group transformation of the amplitude. (As Parke-Taylor amplitude)
- δ function part can be verified by integrating over η_i
- i.e. $M^{++-} = \int \mathrm{d}^{\mathcal{N}}\eta_1 \mathrm{d}^{\mathcal{N}}\eta_2 M(\eta_1, \eta_2, 0) = \frac{[12]^{4s}}{([12][23][31])^s}$

The same methods can be applied to determin the full 4-pt amplitude.

- A special case is $M(\eta_1, \eta_2, \bar{\eta}_3, \bar{\eta}_4)$
- Set $\eta_{1,2} \to 0$, picking up the phase factor with $e^{\bar{\eta}_{n-1}*}$, $e^{\bar{\eta}_n*}$, then no additional phase when shifting $\bar{\eta}_{3,4}$.

Do the same calculations, with (in $\mathcal{N} = 8$ SUGRA):

$$M(\eta_1, \eta_2, \bar{\eta}_3, \bar{\eta}_4) = \frac{(\langle 12\rangle [34])^4}{stu} \exp\left[\begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} \frac{\langle 23\rangle}{\langle 12\rangle} & \frac{\langle 24\rangle}{\langle 12\rangle} \\ \frac{\langle 31\rangle}{\langle 12\rangle} & \frac{\langle 41\rangle}{\langle 12\rangle} \end{pmatrix} \begin{pmatrix} \bar{\eta}_3 \\ \bar{\eta}_4 \end{pmatrix} \right]$$

Set all these to zero, as expected for Lorentz invariance,we have:

$$M^{--++} = (\langle 12\rangle [34])^4 \times (\frac{1}{stu} + \text{polyn.}(s, t, u))$$

We've known in normal YM or GR theory, amplitudes with certain helicities may cause $z \to \infty$ divergence.
However, in maximally SUSY theories, **all** amplitudes vanish at infinite complex momentum.

- The key point is that, we stimultanously shift η:
- $\lambda_1 \to \lambda_1 + z\lambda_2, \quad \bar{\lambda}_2 \to \bar{\lambda}_2 - z\bar{\lambda}_1 \Rightarrow \eta_1 \to \eta_1 + z\eta_2$

When we use Q SUSY to send $\eta_1(z), \eta_2 \to 0$, our translation parameter:

$$\zeta = \frac{\lambda_2\eta_1(z) - \lambda_1(z)\eta_2}{\langle 1(z)2(z)\rangle} = \frac{\lambda_2\eta_1 - \lambda_1\eta_2}{\langle 12\rangle}$$

- ζ is manifestly z independent.

Therefore we have:

$$
\begin{aligned}
& M(\{\eta_1(z), \lambda_1(z), \bar{\lambda}_1\}, \{\eta_2, \lambda_2, \bar{\lambda}_2(z)\}, \eta_i) \\
= & M(\{0, \lambda_1(z), \bar{\lambda}_1\}, \{0, \lambda_2, \bar{\lambda}_2(z), \eta_i + \langle \zeta i \rangle\}) \\
\sim & M^{--+\cdots+}
\end{aligned}
$$

Thus we construct a 2 $(-s)$ amplitude, which surely converge as $1/z^s$ at large z. Thus, recursion relation can be very safe.

We just write the SUSY Recursion realtion as :

$$M = \sum_i \int d^{\mathcal{N}}\eta M_L(\{\eta_1(z_{P_i}), \lambda_1(z_{P_i}), \bar{\lambda}_1\}, \eta)\frac{1}{P^2} \times M_R(\{\eta_2(z_{P_i}), \lambda_2, \bar{\lambda}_2(z_{P_i})\}, \eta)$$

Careful! M_L and M_R are functions of z_{P_i}. That is to say, a given amplitude is determined by a recursion relation involving lower-point amplitudes with **different external states**.

- Examples in Johannes M. Henn & Jan C. Plefka, Scattering Amplitudes in Gauge Theory.

Recall that for the for the usual BCFW recursion relations in YM and Gravity, there is a natural asymmetry between particles 1,2.

- Exactly, we always try to shift λ_1 and $\bar{\lambda}_2$ to guarantee convergency.
- Thus particle 2 should have negative helicity, while particle 1 won't.
- But in SUSY, we've proved that all amplitudes converge.

So, deform λ_2 and $\bar{\lambda}_2$ are both valid, implying a brand new relation which cannot be directly derived by PT invariance:

$$\sum_{L,R} \int d^{\mathcal{N}}\eta M_L\left(\left\{\eta_1\left(z_{P_L}\right), \lambda_1\left(z_{P_L}\right), \bar{\lambda}_1\right\}, \eta, \eta_L\right) \frac{1}{P_L^2} M_R\left(\left\{\eta_2, \lambda_2, \bar{\lambda}_2\left(z_{P_L}\right)\right\}, \eta, \eta_R\right) =$$

$$\sum_{L,R} \int d^{\mathcal{N}}\eta M_L\left(\left\{\eta_1, \lambda_1, \bar{\lambda}_1\left(z_{P_R}\right)\right\}, \eta, \eta_L\right) \frac{1}{P_R^2} M_R\left(\left\{\eta_2\left(z_{P_R}\right), \lambda_2\left(z_{P_R}\right), \bar{\lambda}_2\right\}, \eta, \eta_R\right)$$

Since $\mathcal{N}=8$ amplitudes converge as $1/z^2$,faster than $\mathcal{N}=4$, we have another surprising relation concluded from SUSY recursion:

$$0=\sum_{L}\int d^8\eta M_L\left(\left\{\eta_1\left(z_P\right),\lambda_1\left(z_P\right),\bar{\lambda}_1\right\},\eta\right)\frac{z_P}{P^2}M_R\left(\left\{\eta_2,\lambda_2,\bar{\lambda}_2\left(z_P\right)\right\},\eta\right)$$

- This relation can be derived by consider $M(z)$'s residue.

For pure GR, a similar relation writes:

$$0=\sum_{L,h}M_L\left(\left\{p_1\left(z_P\right),h_1\right\},\left\{-P\left(z_P\right),h\right\}\right)\frac{z_P}{P^2}M_R\left(\left\{p_2\left(z_P\right),h_2\right\},\left\{P\left(z_P\right),-h\right\}\right)$$

- Still invalid in pure Yang-Mills.

A unique technique for SUSY is called Quadruple cut, allowing us to calculate one-loop SYM amplitudes with a linear combination of several tree level amplitudes.

- Consider a n-point tree amplitudes with an extra soft gluon emitted—IR divergent.
- But the 1-loop correction to origin M_n (also IR divergent) can perfectly cancel this divergence.

$$M_{\text{IR}}^{\text{1-loop}} = -\frac{1}{\varepsilon^2}\sum_{i=1}^{n}(-s_{i,i+1})^{-\varepsilon}M^{\text{tree}}$$

Compute in terms of linear combinations of products of tree level amplitudes!

However, there EXISTs something different between $\mathcal{N} = 4$ and $\mathcal{N} = 8$.

In $\mathcal{N} = 4$ SYM, all relations derived from quadruple cut are familiar recursion relations.

- Actually, the origin idea of BCF recursion was inspired by the IR singular behavior of $\mathcal{N} = 4$ SYM

While in $\mathcal{N} = 8$ SUGRA, quadruple cut may give brand new equations, independent of recursion relations.

In this article, we take a look at the SUSY amplitudes, and their recursion relations.

- Grassmann representation of amplitudes.
- Accidental symmetry by deforming Grassmann variables to zero.
- Unique recursion relations different from BCFW.
- IR divergences and differences between SYM and SUGRA.
- ...

So, it's time to answer the question: What is the simplest quantum field theory?

What is the simplest quantum field theory?

- CPT invariance.
- No explicit gauge redundancy.
- Smooth helicity parameters, instead of discrete little group indices.
- Beautiful recursion relations at tree level, with good behavior at large z.
- Good IR and UV behavior at loop level, better with complete divergence cancellation.

Thank you for your listening!