Thm: $f\in H(D)$ D单连通域 z_0 为D内一固定点 令 $F(z)=\int_{z_0}^{z} f(z)dz$ 则

$F(z)\in H(D)$ $F'(z)=f(z)$

Pf: 设 $z\in D$ 因为f在z连续 $\forall\varepsilon>0$ $\exists\delta>0$ 当 $|z-\xi|<\delta$ 时 $\xi\in D$ 且 $|f(z)-f(\xi)|<\varepsilon$

若 $0<|\Delta z|<\delta$ 则 $\dfrac{F(z+\Delta z)-F(z)}{\Delta z}=\dfrac{\int_z^{z+\Delta z}f(z)dz}{\Delta z}$ (积分路径取直线段)

故 $\left|\dfrac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right|=\left|\dfrac{1}{\Delta z}\int_z^{z+\Delta z}f(\xi)d\xi-f(z)\right|=\left|\dfrac{1}{\Delta z}\int_z^{z+\Delta z}(f(\xi)-f(z))d\xi\right|$

$\le\dfrac{1}{\Delta z}\varepsilon\Delta z=\varepsilon$ 从而 $\lim\limits_{\Delta z\to0}\dfrac{F(z+\Delta z)-F(z)}{\Delta z}=f(z)$ 即 $F'(z)=f(z)$.

Thm (N-L公式) 设 $f\in H(D)$ D单连通域 则 $\int_{z_0}^{z}f(z)dz=F(z)-F(z_0)$

这里F是f的任一原函数.

Pf: 易知 $F'=f$ 则 $F'(z)-\left(\int_{z_0}^{z}f(z)dz\right)'=f(z)-f(z)=0$.

故 $\left(F-\int_{z_0}^{z}f(z)dz\right)'=0$ 故 $F-\int_{z_0}^{z}f(z)dz=C$ (常数).

从而 $F=\int_{z_0}^{z}f(z)dz+C$.

Rmk (放在N-L公式前) 事实上证明了 若 $f\in C(D)$ 域D不一定单连通 且对D中任意闭曲线C均有 $\int_C f(z)dz=0$ 则 $F(z)=\int_{z_0}^{z}f(z)dz\in H(D)$, $F'=f$.

Thm (Morera) 若 $f\in C(D)$ 且对D中任意闭曲线C有 $\int_C f(z)dz=0$ 则 $f\in H(D)$

Pf: 由上一Rmk知 $F'=f$ $F(z)=\int_{z_0}^{z}f(z)dz$ $F\in H(D)$

因解析函数有任意阶导数 $f\in H(D)$.

调和函数 (Harmonic Function)

Def: Laplace算子: $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ (注意f是实值函数!)

有二阶连续偏导且满足Laplace方程 ($\Delta f=0$) 的函数f称为调和函数

设 $f\in H(D)$ f任意阶可导 故 实部虚部有任意阶偏导数

记 $f=u+iv$ 有 $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$ 于是

$\Delta u=\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right)+\dfrac{\partial}{\partial y}\left(-\dfrac{\partial v}{\partial x}\right)=0$

$\Delta v=\dfrac{\partial^2v}{\partial x^2}+\dfrac{\partial^2v}{\partial y^2}=\dfrac{\partial}{\partial x}\left(-\dfrac{\partial u}{\partial y}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)=0$

若u和v是D上一对调和函数 且满足CR方程则称v是u的共轭调和函数

Thm: 若 $f=u+iv\in H(D)$ 则 u, v 为D上共轭调和函数

Thm: 若u是单连通域D上共轭调和函数 则 $\exists v$ 使 $u+iv\in H(D)$

Pf: 因为 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. 令 $P=\frac{\partial u}{\partial y}$ $Q=\frac{\partial u}{\partial x}$ 则 $\frac{\partial Q}{\partial x}=\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}=-\frac{\partial P}{\partial y}$

因而曲线积分 $\int_{(x_0,y_0)}^{(x,y)} Pdx+Qdy$ 与路径无关 ((x_0,y_0)为D任一固定点)

令 $v(x,y)=\int_{(x_0,y_0)}^{(x,y)}\left(-\frac{\partial u}{\partial y}\right)dx+\left(\frac{\partial u}{\partial x}\right)dy+C$ (常数)

可验证 u 与 v 满足 CR 方程…

Rmk: 给定v 则 $u(x,y)=\int_{(x_0,y_0)}^{(x,y)}\frac{\partial v}{\partial y}dx-\frac{\partial v}{\partial x}dy$ s.t. $u+iv\in H(D)$

例: 求解析函数 f 使虚部为 $v=2x^2-2y^2+x$ 且 $f(0)=-1$

解1: $\Delta v=0$ $u(x,y)=\int_{(0,0)}^{(x,y)}-4y\,dx-(4x+1)dy=-4xy-y+C_1$

则 $f=(-4xy-y+C_1)+i(2x^2-2y^2+x)$ $f(0)=1=C_1$

解2: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=-4y$ ① $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-4x-1$ ②

由①知 $u=-4xy+C(y)$ 代入②知 $C'(y)=-1 \Rightarrow C(y)=-y+C$

再由 $f(0)=1$ 知 $C=1$ 得 $u=-4xy-y+1$