Prop: $\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz|$ ← 记为$|dz|$，即 $\int_C |f(z)|ds$ 是 $\widehat{z_{k-1}z_k}$ 的弧长.

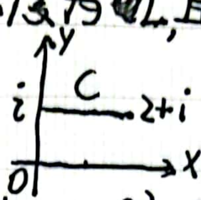
Pf: $\left|\sum_{k=1}^{n} f(\zeta_k)\Delta z_k\right| \le \sum_{k=1}^{n}|f(\zeta_k)||\Delta z_k| \le \sum_{k=1}^{n}|f(\zeta_k)|\Delta s_k$

两边取极限 直接得证

Thm (长大不等式) 设曲线C长度为L,且 $|f(z)|\le M$ 则 $\left|\int_C f(z)dz\right| \le LM$

Pf: 用上述Prop可直接得证.

例 试证 $\left|\int_C \frac{dz}{z^2}\right| \le 2$



解1: $C: z(x)=x+i,\ 0\le x\le 2$ $\int_C \frac{dz}{z^2} = \int_0^2 \frac{dx}{(x+i)^2} = ?$ 不好算 放弃了.

解2: $\left|\int_C \frac{dz}{z^2}\right| \le 2\times 1 = 2$

Thm (Cauchy积分定理) 设D是闭路C所围成单连通域 f在闭域$C\cup D$上解析 则 $\int_C f(z)dz=0$ 注: 在闭域解析指在闭域的开邻域上解析.

Pf: 只证特殊情形—设f'在$\bar{D}=C\cup D$上连续 于是 $\frac{\partial u}{\partial x}$ $\frac{\partial u}{\partial y}$ $\frac{\partial v}{\partial x}$ $\frac{\partial v}{\partial y}$ 连续

由Green公式 (若$P(x,y)$ $Q(x,y)$在Ω上连续且有连续偏导那么有 $\int_{\partial\Omega} Pdx+Qdy = \iint_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$) $\int_C f(z)dz = \int_C u(x,y)dx - v(x,y)dy$

$+i\int_C v(x,y)dx + u(x,y)dy = \iint_D\left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy + i\iint_D\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dxdy \overset{\text{C-R方程}}{=} 0$

~~Prop~~ Thm 设f在单连通域D内解析. C是D内任一闭曲线. 则 $\int_C f(z)dz=0$

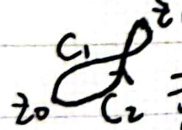
闭曲线可分解为几个简单闭曲线的和 证明不打算讲了

Prop 设f在单连通域D内解析, 则f在D内积分与路径无关.

即对D内任意两点 z_0 z_1, C为D内任一条接起点z_0终点z_1的曲线.

积分 $\int_C f(z)dz$ 不依赖于C而只由z_0,z_1决定. 故该积分可记为 $\int_{z_0}^{z_1} f(z)dz$.

Pf:



C_1与C_2^- 相接而成一条闭曲线C

于是 $0=\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz \Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

Prop 多连通域的Cauchy积分定理 设有n+1条简单闭曲线 $C_0, C_1, \dots, C_n$, 每条都在其余各条外部,都在C_0内部, $C=C_0+C_1^-+\dots+C_n^-$ (约定沿C正向走时区域在左手边) 则 $\int_C f(z)dz=0 \Leftrightarrow \int_{C_0}+\int_{C_1^-}+\dots+\int_{C_n^-} f(z)dz=0$

$\Leftrightarrow \int_{C_0} f(z)dz - \int_{C_1} f(z)dz - \dots - \int_{C_n} f(z)dz = 0 \Leftrightarrow \int_{C_0} f(z)dz = \int_{C_1} f(z)dz + \dots + \int_{C_n} f(z)dz$

e.g. C闭路 a是C内任一点. 求 $\int_C \frac{1}{(z-a)^n}dz$, $n\in\mathbb{Z}$

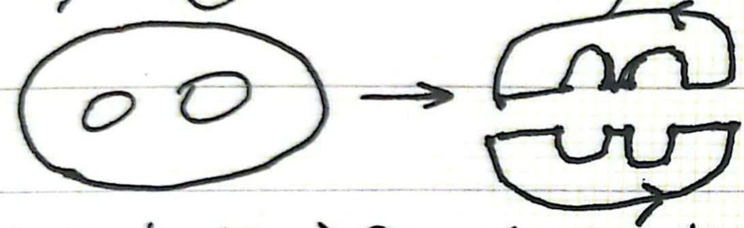
$$\int_C \frac{1}{(z-a)^n}dz = \begin{cases} 0, & n=0 \\ \int_{|z-a|=\rho}\frac{1}{(z-a)^n}dz = \begin{cases} 2\pi i, & n=1 \\ 0, & n\ne 1 \end{cases} \end{cases}$$

Thm (Cauchy积分公式) 设f在闭路(或复闭路) C及其所围区域D内解析

则对D内任一点z有 $f(z)=\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z}d\zeta$.

Rmk 解析函数值由边界上的值确定

补: 多连通域的Cauchy积分定理. 证明思路:



对两个简单闭曲线分别运用单连通域的Cauchy积分定理再把两式相加重合部分被消去.

现在开始证明Cauchy积分定理: 令 Γ_ρ 为 $\{\zeta : |\zeta-z|=\rho\}$ 则

$\frac{f(\zeta)}{\zeta-z}$ (关于ζ的函数) 在由C与Γ_ρ围成闭域上解析. 由多连通域的

Cauchy积分定理 $\int_C \frac{f(\zeta)}{\zeta-z}d\zeta=\int_{\Gamma_\rho}\frac{f(\zeta)}{\zeta-z}d\zeta$