

非线性偏微分方程数值解

成绩： 平时 50~60%
 纸质作业 程序作业
 期末 40~50%
 大程序 期末报告

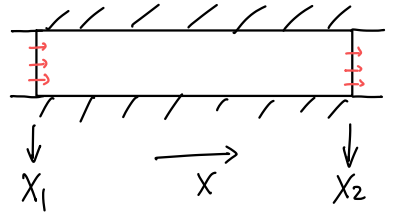
李顺

Part I Conservation Law

eg: Flow in a tube (1D)

x distance. t time.

ρ density. v velocity. ($v > 0 \Rightarrow$)



($x_1 < x_2$). ($0 < t_1 < t_2$)

total mass in $[x_1, x_2]$ at time t $\int_{x_1}^{x_2} \rho(x, t) dx$

mass flux at (x, t) . $\rho(x, t) V(x, t)$

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t) dx = \rho(x_1, t) V(x_1, t) - \rho(x_2, t) V(x_2, t)$$

integrate t in $[t_1, t_2]$

$$\int_{x_1}^{x_2} \rho(x, t_2) dx - \int_{x_1}^{x_2} \rho(x, t_1) dx = \int_{t_1}^{t_2} \rho(x_1, t) V(x_1, t) dt - \int_{t_1}^{t_2} \rho(x_2, t) V(x_2, t) dt$$

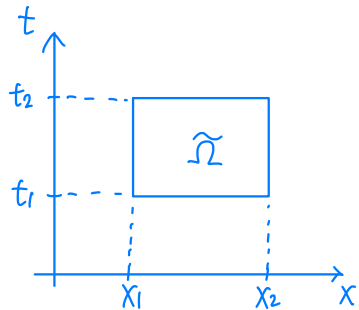
Assume that $\rho(x, t) \cdot V(x, t) \in C^1$.

$$\text{LHS} = \int_{x_1}^{x_2} dx \int_{t_1}^{t_2} \frac{\partial}{\partial t} \rho(x, t) dt$$

$$\text{RHS} = - \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} \frac{\partial}{\partial x} (\rho(x, t) V(x, t)) dx$$

$$\Rightarrow \int_{\tilde{\Omega}} \left(\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} (\rho(x, t) V(x, t)) \right) = 0$$

i.e. $\boxed{\rho_t + (\rho v)_x = 0}$



$$\forall \tilde{\Omega} = [x_1, x_2] \times [t_1, t_2]$$

- remark: (1) if $\rho(x,t) \in C_c^1$, compact supported in space, then
 $\frac{d}{dt} \int_{\mathbb{R}} \rho(x,t) dx = 0$. i.e. "conservation of mass"
- (2) if $\rho(x,t), V(x,t)$ are periodic in space
 $\frac{d}{dt} \int_0^T \rho(x,t) dx = 0$.
- (3) if $V(x,t)$ is known, or $V = V(\rho)$. let $f(\rho) = \rho V$.
 $\rho_t + f(\rho)_x = 0$

conservation of $\begin{cases} \text{mass:} & \rho_t + (\rho V)_x = 0 \\ \text{momentum:} & (\rho V)_t + (\rho V^2 + p)_x = 0 \\ \text{energy:} & E_t + ((E+p)V)_x = 0 \end{cases}$

let $\vec{u}(x,t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho V \\ E \end{pmatrix} \Rightarrow \vec{u}_t + \vec{F}(\vec{u})_x = 0$

$\vec{F} = \begin{pmatrix} \rho V \\ \rho V^2 + p \\ (E+p)V \end{pmatrix} = \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + p(\vec{u}) \\ \frac{u_2}{u_1} (u_3 + p(\vec{u})) \end{pmatrix} = \vec{F}(\vec{u}) \quad \text{where } p = p(\vec{u})$

Euler equations in 1D: $E = \frac{1}{2} \rho V^2 + \frac{p}{\gamma-1} \quad \gamma: \text{constant } 1.4$

def (hyperbolic conservation law)

(1) 1D scalar case $u_t + f(u)_x = 0$

$$\left(\begin{array}{l} u: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ conserved quantity} \\ f: \mathbb{R} \rightarrow \mathbb{R} \text{ flux function} \end{array} \right)$$

$$\text{or: } \int_{x_1}^{x_2} u(x, t_2) - u(x, t_1) dx + \int_{t_1}^{t_2} f(u(x_2, t)) - f(u(x_1, t)) dt = 0$$

$$\text{or: } \frac{d}{dt} \int_{x_1}^{x_2} u(x, t) dx = f(u(x_1, t)) - f(u(x_2, t))$$

(2) 1D hyperbolic system. $\vec{u}_t + \vec{f}(\vec{u})_x = \vec{0}$

$$\left(\begin{array}{l} \vec{u}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^m \\ \vec{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m \end{array} \right)$$

remark: it's hyperbolic iff the Jacobian matrix

$J(\vec{u}) = \frac{\partial \vec{f}}{\partial \vec{u}} \in \mathbb{R}^{m \times m}$ is diagonalizable with real eigenvalues.

$$J(\vec{u}) R(\vec{u}) = R(\vec{u}) \Lambda(\vec{u})$$

eg: 1D Euler equations. $c = \sqrt{\frac{\gamma p}{\rho}}$ $H = \frac{(E+p)}{\rho}$

eigenvalues: $\Lambda(\vec{u}) = \text{diag}(v-c, v, v+c)$

eigenvectors: $R(\vec{u}) = \begin{pmatrix} 1 & 1 & 1 \\ v-c & v & v+c \\ H-vc & \frac{1}{2}v^2 & H+vc \end{pmatrix}$

(13) 2D hyperbolic conservation law.

$$\vec{U}_t + \vec{f}_x(\vec{U}) + \vec{g}_y(\vec{U}) = 0$$

$$\begin{pmatrix} \vec{U} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^m \\ \vec{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m \\ \vec{g} : \mathbb{R}^m \rightarrow \mathbb{R}^m \end{pmatrix}$$

remark: it's hyperbolic iff $\forall \alpha, \beta \in \mathbb{R}$
 $\alpha \vec{f}(\vec{U}) + \beta \vec{g}(\vec{U}) \in \mathbb{R}^{m \times m}$ ★

should be diagonalizable with real eigenvalues.

eg: 2D Euler equations

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ u \\ v \\ E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho v^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho v \\ \rho uv \\ \rho u^2 + p \\ v(E+p) \end{pmatrix} = 0$$

$$E = \frac{\rho}{2}(u^2 + v^2) + \frac{p}{\gamma - 1}$$

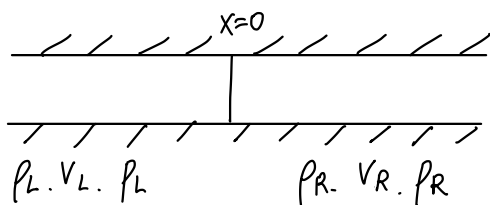
def: $U_t + \nabla \cdot F(U) = 0$

$$U : \mathbb{R}^m \rightarrow \mathbb{R}^n \quad F : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

it's hyperbolic iff $\forall \vec{n} \in \mathbb{R}^n$.

$\vec{n} \cdot \nabla F = \frac{\partial F}{\partial n}$ should be diagonalizable with real eigenvalues.

eg: shock tube

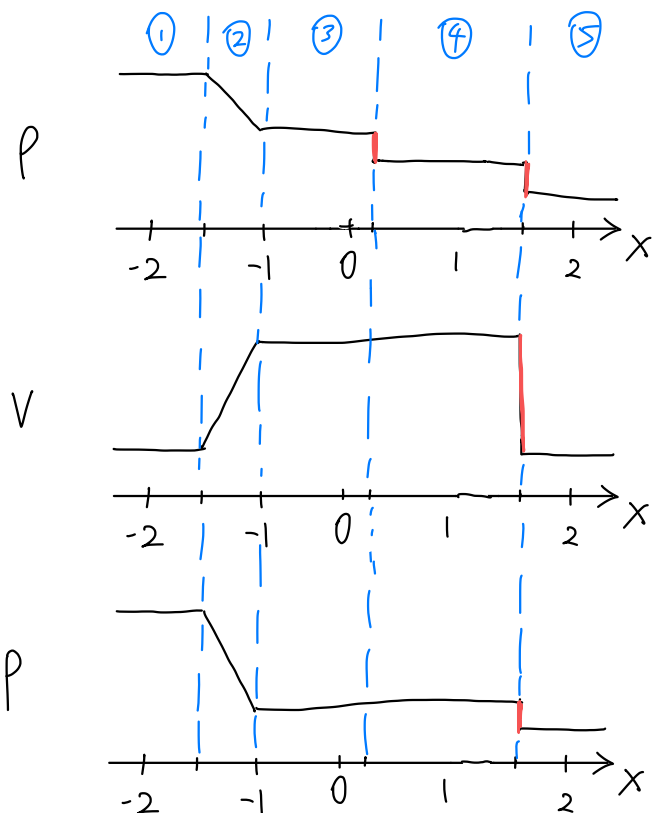


$$v_L = v_R = 0$$

$$p_L > p_R$$

$$p_L > p_R$$

Riemann problem: $\vec{u}(x,0) = \begin{cases} \vec{u}_L & x < 0 \\ \vec{u}_R & x > 0 \end{cases}$

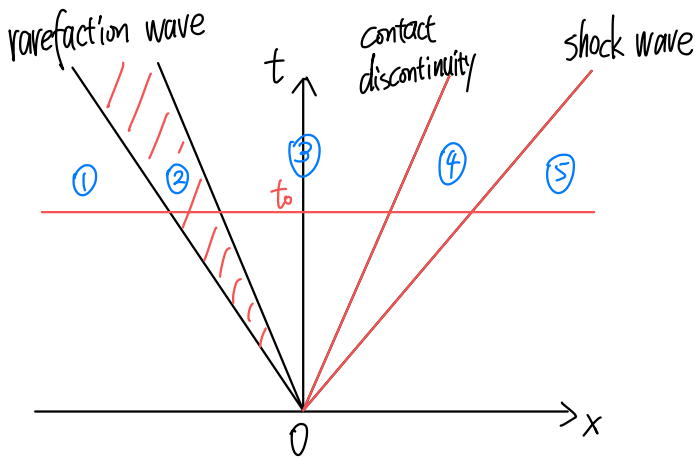


① : $\vec{u}_L$ ⑤ : $\vec{u}_R$

② : all variables are continuous.
smooth transition
rarefaction wave (稀疏波)
($u \pm c$)

③ → ④ : p is discontinuous
but v, p are continuous.
contact discontinuity (接触间断)
(u)

④ → ⑤ : all variables are discontinuous
shock wave (激波)
($u \pm c$)



The waves move at a constant speed.

$$u(x,t) = \text{constant}$$

if $\frac{x}{t} = \text{constant}$ in (x,t) plane.

$\Rightarrow$ similarity solution :

$$u(x,t) = u(\alpha x, \alpha t) \quad \forall \alpha > 0. \quad \text{i.e.} \quad u(x,t) = w\left(\frac{x}{t}\right)$$

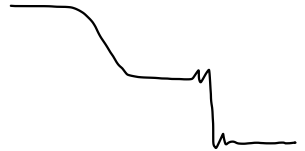
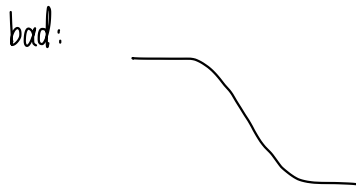
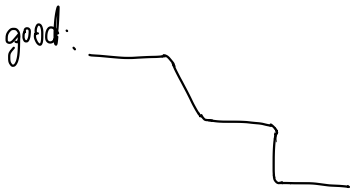
Mathematical difficulties:

$\left\{ \begin{array}{l} \text{differential form} \rightarrow \text{"strong solution"} \\ \text{integral form} \rightarrow \text{"weak solution"} \end{array} \right. \quad \text{not smooth} \quad \left\{ \begin{array}{l} \text{discontinuous condition} \\ \text{discontinuous solution.} \end{array} \right.$

not unique $\rightarrow \left\{ \begin{array}{l} \text{Viscosity solution } u_t + f(u)_x = \varepsilon u_{xx} \\ \text{entropy solution} \end{array} \right.$

Numerical difficulties:

1. shock tracking
2. shock capturing

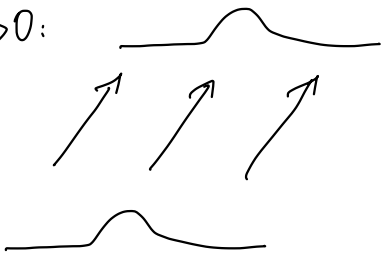


The linear advection equation. $f(u) = au$

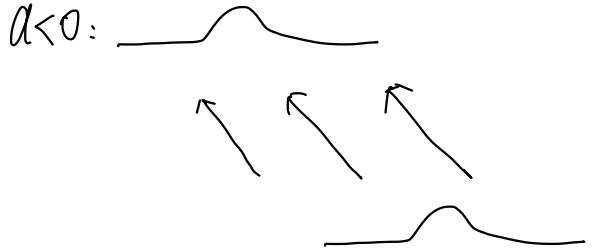
$$\begin{cases} u_t + a u_x = 0 \\ u(x, 0) = u_0(x) \end{cases} \quad x \in \mathbb{R} \quad t \in \mathbb{R}^+ \quad (\text{Cauchy Problem})$$

$\Rightarrow$ exact solution $u(x, t) = u_0(x - at)$.

$a > 0$:



$a < 0$:



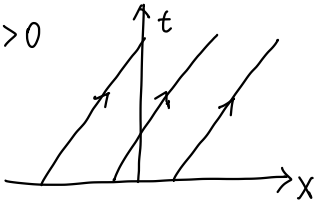
characteristic

$$X = X(t)$$

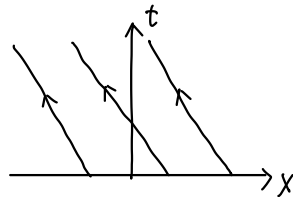
$$0 = \frac{d}{dt} u(X(t), t) = u_x X'(t) + u_t = 0 \Rightarrow X'(t) = a$$

$$\begin{cases} X'(t) = a \\ X(0) = X_0 \end{cases} \Rightarrow X(t) = X_0 + at$$

$a > 0$



$a < 0$



remark: The spatial smoothness is not required.

But if $u_0(x)$ is nondifferentiable. how to define the solution?

Approach 1: integral form

$$\begin{aligned}\int_{x_1}^{x_2} u(x, t_2) dx &= \int_{x_1}^{x_2} u_0(x - at_2) dx = \int_{x_1 - at_2}^{x_2 - at_2} u_0(y) dy \\&= \int_{x_1 - at_1}^{x_2 - at_1} u_0(y) dy + \int_{\underline{x_1 - at_2}}^{x_1 - at_1} u_0(y) dy + \int_{x_2 - at_1}^{\underline{x_2 - at_2}} u_0(y) dy \\&= \int_{x_1}^{x_2} u(x, t_1) dx + a \int_{t_1}^{t_2} u_0(x_1 - at) dt - a \int_{t_1}^{t_2} u_0(x_2 - at) dt \\&= \int_{x_1}^{x_2} u(x, t_1) dx + a \int_{t_1}^{t_2} u(x_1, t) - u(x_2, t) dt \quad \# \end{aligned}$$

Approach 2: vanishing viscosity solution

$$\begin{cases} u_t^\varepsilon + a u_x^\varepsilon = \varepsilon u_{xx} & (\varepsilon > 0) \\ u^\varepsilon(x, 0) = u_0(x) \end{cases}$$

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x, t) \stackrel{?}{=} u_0(x - at)$$

$$\text{let } \begin{cases} V^\varepsilon(x, t) = U^\varepsilon(x + at, t) \\ V_t^\varepsilon = U_t^\varepsilon + a U_x^\varepsilon \\ V_{xx}^\varepsilon = U_{xx}^\varepsilon \end{cases} \Rightarrow \begin{cases} V_t^\varepsilon = \varepsilon V_{xx}^\varepsilon \quad (\text{heat equation}) \\ V^\varepsilon(x, t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{4\varepsilon t}\right) U_0(y) dy \end{cases}$$

$$\text{let } \varepsilon \rightarrow 0. \quad V^\varepsilon(x, t) \rightarrow \int_{\mathbb{R}} U_0(y) \delta(x) dy = U_0(x)$$

$$U^\varepsilon(x, t) = V^\varepsilon(x - at, t) \xrightarrow{\mathbb{R}} U_0(x - at, t)$$

#

Variable coefficient advection equation:

$$U_t + (a(x) U)_x = 0$$

$$\text{or. } U_t + a(x) U_x = -a'(x) U$$

$$\text{Characteristic } \begin{cases} X = X(t) \\ X(0) = X_0 \end{cases}$$

$$U_t + b(x) U_x = 0$$

$$\begin{cases} X'(t) = b(X(t)) \\ X(0) = X_0 \end{cases}$$

$$\frac{d}{dt} U(X(t), t) = 0$$

$$\begin{aligned} \frac{d}{dt} U(X(t), t) &= U_t + \boxed{U_x X'(t)} \\ &= U_t + \boxed{a(x) U_x} = \underline{-a'(X(t)) U(X(t), t)} \end{aligned}$$

$$\Rightarrow \begin{cases} X'(t) = a(X(t)) \\ X(0) = X_0 \end{cases}$$

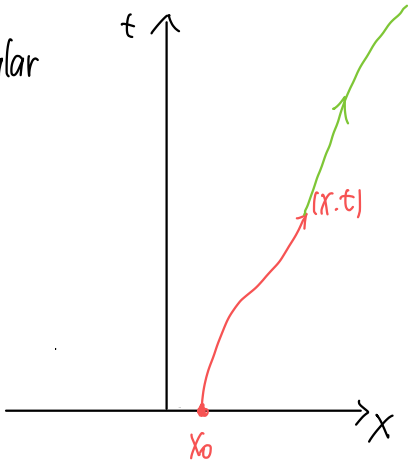
$$\text{ODE } X = X(t)$$

$$\Rightarrow \begin{cases} \frac{d}{dt} U(X(t), t) = -a'(X(t)) U(X(t), t) \\ U(X(0), 0) = U(X_0, 0) = U_0(X_0) \end{cases}$$

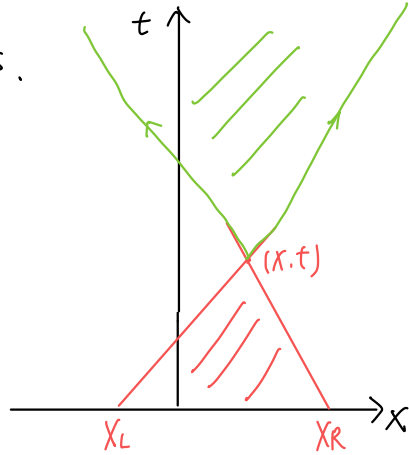
$$\text{ODE } U = U(X(t), t)$$

Domain of dependence & influence.

Scalar



Equations.



In general:

domain of dependence. $D(x, t) \subseteq \{x' : |x' - x| \leq M(a, t)\}$

domain of influence $\subseteq \{x' : |x' - x| \leq M(a, t)\}$

$$M = M(a(x), t) = \max |a| \cdot t.$$

Nonlinear equation

$$u_t + f(u)_x = 0 \quad \text{or} \quad u_t + f'(u) u_x = 0$$

Characteristic

$$X(t): \begin{cases} \frac{d}{dt} X(t) = f'(u(X(t), t)) \\ X(0) = X_0 \end{cases}$$

$$\frac{d}{dt} u(X(t), t) = u_x X'(t) + u_t = 0$$

$$\Rightarrow u(X(t), t) = u_0(X_0) = \text{constant}$$

$$\Rightarrow \frac{d}{dt} X(t) = f'(u_0(X_0)) = \text{constant}$$

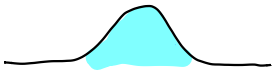
eg: Burgers' equation $f(u) = \frac{1}{2} u^2$
 $u_t + u u_x = 0$



$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{X'(t)}$$

$$\tan \theta = X'(t) = f'(u_0(X_0))$$

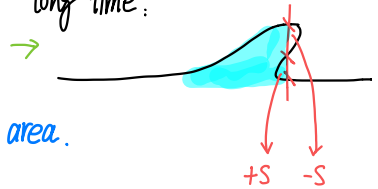
init:



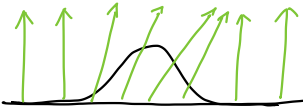
small time:



long time:



conservation of area.



(1) For small time, we can use the characteristics to determine the solution $u(x, t)$ (characteristics do not cross)

$$(X, t) \longrightarrow \begin{cases} X = X_0 + u_0(X_0) t \\ u(X, t) = u_0(X_0) \end{cases} \quad X_0 = ?$$

(2) For long time, some characteristics have crossed. (triple-valued function)

eg: $u_t + uu_x = 0$. with smooth initial data $u_0(x)$

for which $u'_0(x)$ is somewhere negative. Then the wave will break at time T_b .

$$T_b = \frac{-1}{\min u'_0(x)}$$

HW: $T_b = \frac{-1}{\min u'_0(x)}$ Ex 3.3
(3.21. Tuesday)

Ex 3.3

解: 特征线方程: $X = X(t) = \xi + u_0(\xi)t$ 其中 u_0 是初值. 特征线经过 $(\xi, 0)$

(1) 若两特征线 $\begin{cases} X = \xi_1 + u_0(\xi_1)t \\ X = \xi_2 + u_0(\xi_2)t \end{cases}$ 相交. $\Rightarrow t = \frac{-1}{\left(\frac{u_0(\xi_1) - u_0(\xi_2)}{\xi_1 - \xi_2}\right)} = \frac{-1}{u'_0(\xi)}$
 ξ 在 ξ_1 与 ξ_2 之间

因此相交的必要条件 $\min u'_0 < 0$. 相交时间 $t \geq \frac{-1}{\min u'_0}$

(2) 沿着 $X = \xi + u_0(\xi)t$ 特征线. 反解出 $\xi = \xi(X, t)$ 满足

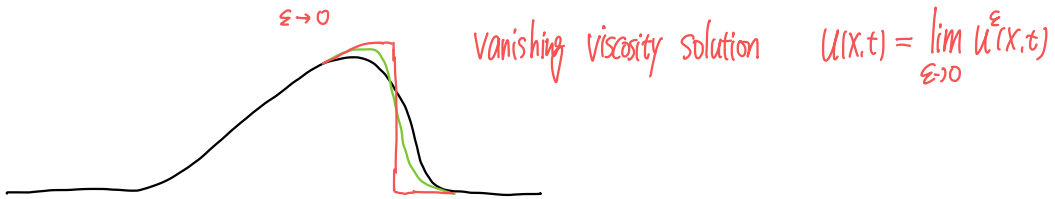
$$1 = \xi_x + u'_0(\xi)\xi_x t \quad \xi_x = \frac{1}{1 + u'_0(\xi)t}$$

$$\text{有 } u(X, t) = u_0(\xi(X, t)) \quad u_x = u'_0(\xi)\xi_x = \frac{u'_0(\xi)}{1 + u'_0(\xi)t}$$

当 $\xi = \xi_0$ 取到 $\min u'_0(\xi)$, 并且 $t \rightarrow \frac{-1}{\min u'_0} = \frac{-1}{u'_0(\xi_0)}$ 此时 $|u_x| \rightarrow +\infty$ 产生无穷斜率 #

因此恰有 $T_b = \frac{-1}{\min u'_0}$. 此时产生无穷斜率, 即间断

Consider $u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon \quad (\varepsilon > 0)$



If the initial data is smooth and $\varepsilon \ll 1$.

- Before the wave breaks.

$\varepsilon u_{xx}^\varepsilon$ is negligible compare to other terms, and the solution for both PDEs look nearly identical.

- As the wave begin to break.

u_{xx} grows much faster than u_x . εu_{xx} term is comparable to other terms and begins to play a role.

- At the time T_b when characteristics first cross.

$u(x,t)$ has an infinite slope — the wave breaks and a shock form.

Weak solution

choose test function $\phi \in \underline{C}_0'(\mathbb{R} \times \mathbb{R}^+)$

$$u_t + f(u)_x = 0$$

$$\Rightarrow \int_0^{+\infty} dt \int_{-\infty}^{+\infty} [\phi u_t + \phi f(u)_x] dx = 0$$

with initial data

$$(*) \quad \boxed{\int_0^{+\infty} dt \int_{-\infty}^{+\infty} [\phi_t u + \phi_x f(u)] dx = - \int_{-\infty}^{+\infty} \underline{u(x, 0)} \phi(x, 0) dx} \quad \forall \phi \in C_0'(\mathbb{R} \times \mathbb{R}^+)$$

Note: $\Updownarrow$

$$(**) \quad \int_{x_1}^{x_2} u(x, t_2) dx = \int_{x_1}^{x_2} u(x, t_1) dx + \int_{t_1}^{t_2} f(u(x_1, t)) dt - \int_{t_1}^{t_2} f(u(x_2, t)) dt$$

$\forall [x_1, x_2] \times [t_1, t_2]$

The two integral form are equivalent.

$(*) \Rightarrow (**)$

let $\phi^\varepsilon(x, t) = \begin{cases} 1 & \text{in } [x_1, x_2] \times [t_1, t_2] \\ 0 & \text{in } ([x_1 - \varepsilon, x_2 + \varepsilon] \times [t_1 - \varepsilon, t_2 + \varepsilon])^c \end{cases}$

1

0

$\phi^\varepsilon \longrightarrow \chi_{[x_1, x_2] \times [t_1, t_2]}$

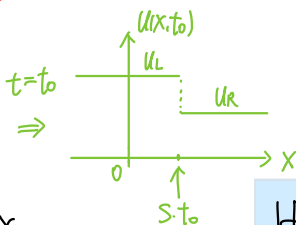
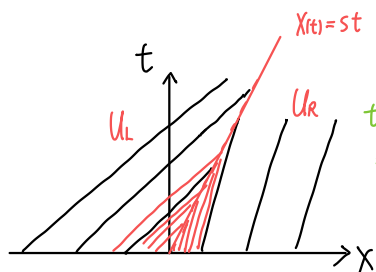
as $\varepsilon \rightarrow 0$

$(**) \Rightarrow (*)$...

Burgers' equation & Riemann problem. $u(x,0) = u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$

case 1. $u_L > u_R > 0$ (shock)

$$s = \frac{u_L + u_R}{2}$$



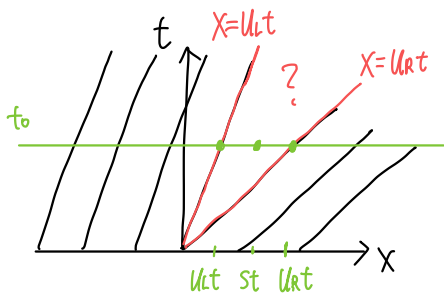
$$u(x,t) = \begin{cases} u_L & x < st \\ u_R & x > st \end{cases}$$

s is the shock speed.

HW. Ex 3.4

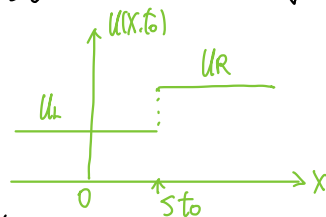
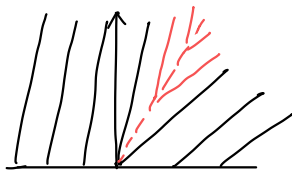
Check the weak solution

Case 2. $u_R > u_L > 0$

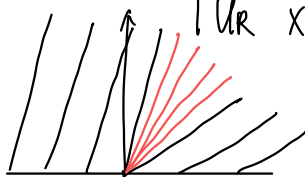


$$\textcircled{1} u(x,t) = \begin{cases} u_L & x < st \\ u_R & x > st \end{cases}$$

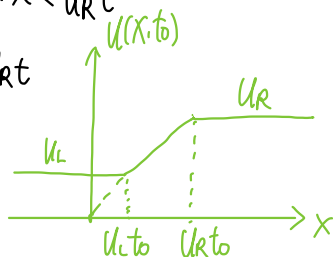
$$s = \frac{1}{2}(u_L + u_R) \quad \checkmark$$



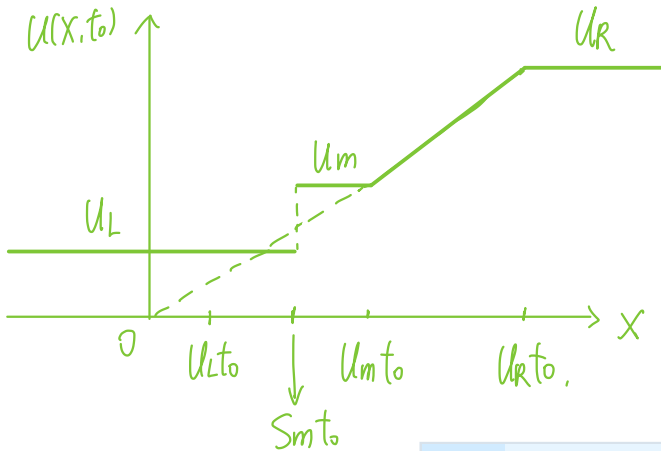
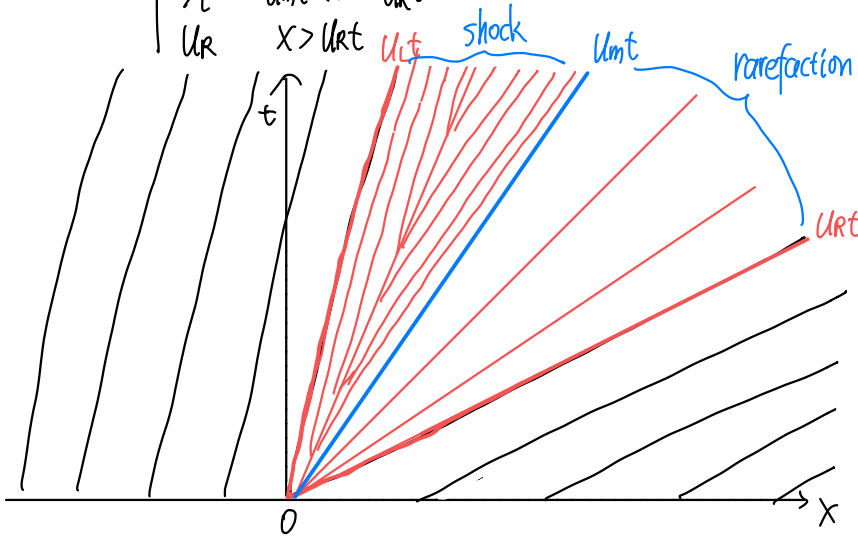
$$\textcircled{2} u(x,t) = \begin{cases} u_L & x < u_L t \\ \frac{x}{t} & u_L t < x < u_R t \\ u_R & x > u_R t \end{cases}$$



(rarefaction wave)



$$\textcircled{3} \quad u(x,t) = \begin{cases} u_L & x < S_m t \\ u_m & S_m t < x \leq u_m t \\ x_f & u_m t \leq x \leq u_R t \\ u_R & x > u_R t \end{cases} \quad \text{for any } u_m \in [u_L, u_R]. \quad S_m = \frac{1}{2}(u_L + u_m)$$

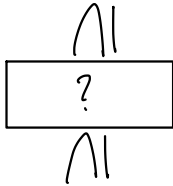


$\Rightarrow$ weak solution is not unique!

Hw P31. Ex3.7

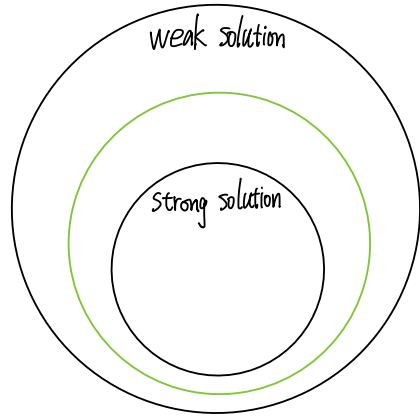
check weak solution

classical solution (strong solution)



weak solution

- (1) vanishing viscosity solution
- (2) entropy solution.



Shock speed:

Rankine - Hugoniot (RH) condition of $u_t + f(u)_x = 0$.

the relation between the shock speed s and the states u_L & u_R :

$$f(u_L) - f(u_R) = s(u_L - u_R)$$

RH jump condition

remark: (1) for scalar equation. $s = \frac{f(u_L) - f(u_R)}{u_L - u_R}$

eg: $u_t + (\frac{1}{2}u^2)_x = 0$ $s = \frac{\frac{1}{2}u_L^2 - \frac{1}{2}u_R^2}{u_L - u_R} = \frac{u_L + u_R}{2}$

(2) for systems. $\vec{u}_L, \vec{f}(\vec{u}_L)$ are vectors. s is a scalar.

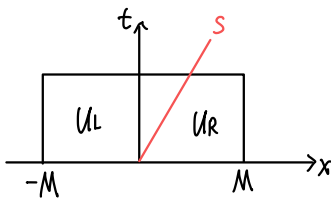
We can not always solve s by RH condition.

eg: $\vec{f}(\vec{u}) = A\vec{u}$. $A\vec{u}_L - A\vec{u}_R = s(\vec{u}_L - \vec{u}_R)$

iff $\begin{cases} \vec{u}_L - \vec{u}_R \text{ is the eigenvector of } A. \\ s \text{ is the eigenvalue of } A. \end{cases}$

Proof of RH condition :

case I. piecewise constant initial data and shock solution with a single discontinuity propagating at constant speed.



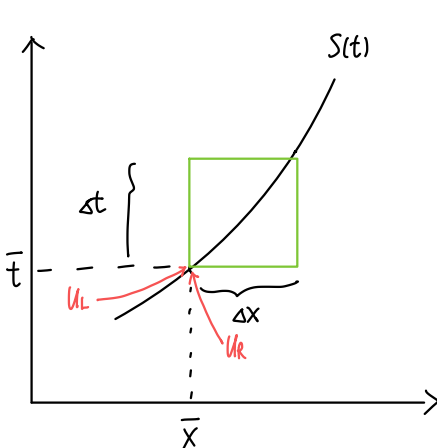
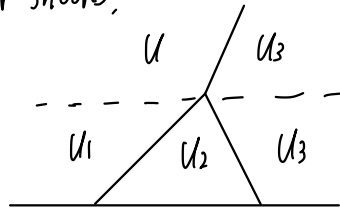
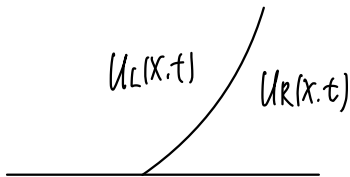
$$\int_{-M}^M u(x,t) dx = \int_{-M}^M u(x,0) dx + \int_0^t f(u(-M,\tau)) d\tau - \int_0^t f(u(M,\tau)) d\tau$$

$$(M+st)u_L + (M-st)u_R = (u_L+u_R)M + t f(u_L) - t f(u_R)$$

$$\Rightarrow f(u_L) - f(u_R) = s(u_L - u_R)$$

case II. Generally, solutions have both smooth regions (PDEs)

the propagating discontinuities whose strength and speed vary as they interact with smooth flux or collide with other shocks,



$$\int_{\bar{x}}^{\bar{x}+\Delta x} u(x, \bar{t}+st) dx = \int_{\bar{x}}^{\bar{x}+\Delta x} u(x, \bar{t}) dx + \int_{\bar{t}}^{\bar{t}+st} f(u(\bar{x}, t)) dt - \int_{\bar{t}}^{\bar{t}+st} f(u(\bar{x}+\Delta x, t)) dt$$

Taylor expand at $(\bar{x}, \bar{t})$... #

$\int u$ conservation

remark: $u_t + (\frac{1}{2}u^2)_x = 0 \Rightarrow S_1 = \frac{u_L + u_R}{2}$

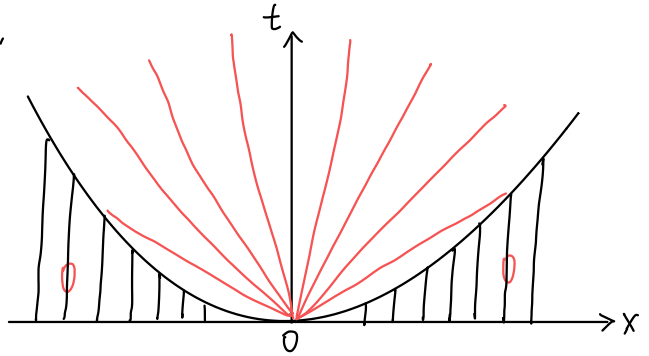
$(\frac{1}{2}u^2)_t + (\frac{1}{3}u^3)_x = 0 \Rightarrow S_2 = \frac{\frac{1}{3}(u_L^3 - u_R^3)}{\frac{1}{2}(u_L^2 - u_R^2)} = \frac{2}{3} \frac{u_L^2 + u_L u_R + u_R^2}{u_L + u_R}$

$\int \frac{1}{2}u^2$ conservation

$S_1 \neq S_2$! different weak solution !

eg: Burgers' equation "N wave"

$$u(x,t) = \begin{cases} x/t & |x| < \sqrt{t} \\ 0 & \text{otherwise} \end{cases}$$



check RH: $u_L = \frac{\sqrt{t}}{t} = \frac{1}{\sqrt{t}}, u_R = 0$

$S = \frac{d}{dt}(\sqrt{t}) = \frac{1}{2\sqrt{t}}$

$f(u_L) - f(u_R) = \frac{1}{2t}$

$S(u_L - u_R) = \frac{1}{2\sqrt{t}} (\frac{1}{\sqrt{t}} - 0) = \frac{1}{2t}. \checkmark$

$\Rightarrow$ weak solution is not unique.

def: entropy and entropy flux

Entropy function: $\eta: \mathbb{R} \rightarrow \mathbb{R}$ if $\eta \in C^2(\mathbb{R})$. $\eta'' \geq 0$.

The corresponding entropy flux: $\xi: \mathbb{R} \rightarrow \mathbb{R}$. if $\xi'(u) = \eta'(u)f'(u)$.

Then, entropy pair (η, ξ) .

remark: The strong solution $u_t + f(u)_x = 0$ satisfies $\eta(u)_t + \xi(u)_x = 0$.

$$u_t + f'(u)u_x = 0$$

$$\Rightarrow \eta'(u)[u_t + f'(u)u_x] = \eta(u)_t + \xi(u)_x = 0$$

def: entropy weak solution

For any entropy pair (η, ξ) , u satisfies $\eta(u)_t + \xi(u)_x \leq 0$ in weak sense
($\forall \phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$, $\phi \geq 0$)

$$-\int_0^{+\infty} \int_{-\infty}^{+\infty} [\phi_t \eta(u) + \phi_x \xi(u)] dx - \int_{-\infty}^{+\infty} \eta(u(x, 0)) \phi(x, 0) dx \leq 0$$

↓
or: $\forall [x_1, x_2] \times [t_1, t_2] \dots$

eg: Burgers' equation $f(u) = \frac{1}{2}u^2$.

$$\eta(u) = \frac{1}{2p} u^{2p} + \frac{\alpha}{2} u^2 \quad p \in \mathbb{N}, p \geq 2, \alpha > 0$$

$$\Rightarrow \eta''(u) = (2p-1)u^{2p-2} + \alpha > 0$$

$$\xi'(u) = \eta'(u)f'(u) = (u^{2p-1} + \alpha u)u = u^{2p} + \alpha u^2$$

$$\Rightarrow \xi(u) = \frac{1}{2p+1} u^{2p+1} + \frac{\alpha}{3} u^3$$

entropy pair (η, ξ)

Thm: The vanishing viscosity solution is an entropy weak solution.

pf: $u(x, t) = \lim_{\varepsilon \rightarrow 0} u^\varepsilon(x, t) \quad u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon$

$$\eta'(u^\varepsilon) [u_t^\varepsilon + f(u^\varepsilon)_x] = \varepsilon \eta'(u^\varepsilon) u_{xx}^\varepsilon$$

$$\Rightarrow \eta(u^\varepsilon)_t + \xi(u^\varepsilon)_x = \varepsilon \left[\eta(u^\varepsilon)_{xx} - \underbrace{\eta''(u^\varepsilon) (u_x^\varepsilon)^2}_{\geq 0} \right] \leq \varepsilon \eta(u^\varepsilon)_{xx}$$

$$\Rightarrow \forall \phi \in C_0'(\mathbb{R} \times \mathbb{R}^+), \phi \geq 0$$

$$\int_0^{+\infty} dt \int_{-\infty}^{+\infty} [\eta(u^\varepsilon)_t + \xi(u^\varepsilon)_x] \phi dx \leq \int_0^{+\infty} dt \int_{-\infty}^{+\infty} \varepsilon \eta(u^\varepsilon)_{xx} \phi dx$$

$$\Rightarrow - \int_0^{+\infty} dt \int_{-\infty}^{+\infty} [\eta(u^\varepsilon)_t \phi + \xi(u^\varepsilon)_x \phi] dx - \int_{-\infty}^{+\infty} \eta(u^\varepsilon(x, 0)) \phi + \xi(u^\varepsilon(x, 0)) \phi dx \leq \int_0^{+\infty} dt \int_{-\infty}^{+\infty} \varepsilon \eta(u^\varepsilon) \phi_{xx} dx \dots$$

$$\varepsilon \rightarrow 0 \Rightarrow u = \lim_{\varepsilon \rightarrow 0} u^\varepsilon \text{ is the entropy solution} \quad \#$$

Thm: (characterization of discontinuous entropy weak solution)

$u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$ and assume there exists a smooth curve $\Gamma: t \rightarrow x(t)$ such that u is piecewise differentiable on both sides of the curve.

Then u is an entropy weak solution iff

① u is a strong solution locally on both sides of Γ .

② $(u_L, u_R, s = x'(t))$ satisfy $f(u_L) - f(u_R) = s(u_L - u_R)$

③ For any entropy pair (ξ, η) , $\xi(u_L) - \xi(u_R) \leq s(\eta(u_L) - \eta(u_R))$

eg: Burgers' equation $f(u) = \frac{1}{2}u^2$

smooth solution: PDE

discontinuity: $S = \frac{\frac{1}{2}u_L^2 - \frac{1}{2}u_R^2}{u_L - u_R} = \frac{u_L + u_R}{2}$

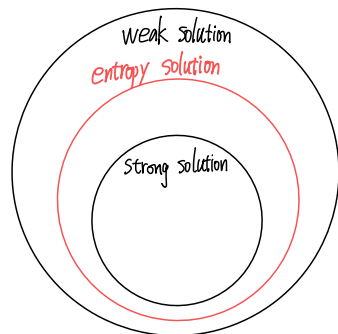
entropy: let $\begin{cases} \eta(u) = u^2 & (\xi(u) = \eta'(u)f(u) = 2uu = 2u^2) \\ \xi(u) = \frac{2}{3}u^3 \end{cases}$

$$\begin{aligned} & \xi(u_L) - \xi(u_R) - S(\eta(u_L) - \eta(u_R)) \\ &= \frac{2}{3}u_L^3 - \frac{2}{3}u_R^3 - \frac{u_L + u_R}{2}(u_L^2 - u_R^2) \\ &= -\frac{1}{6}(u_L - u_R)^3 \leq 0 \quad \Rightarrow \quad u_L > u_R \end{aligned}$$

$\Rightarrow$ if $u_L > u_R$, shock is an entropy solution.

if $u_L < u_R$, shock is a weak solution. but is not an entropy solution!

rarefaction is an entropy solution.



Thm: let u_1, u_2 are piecewise smooth entropy weak solutions to $u_t + f(u)_x = 0$ with $f'' > 0$. and assume that all discontinuities are shocks. Then (maybe different initial value)

$$\frac{d}{dt} \|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^1} \leq 0$$

Cor: The entropy weak solution is unique.

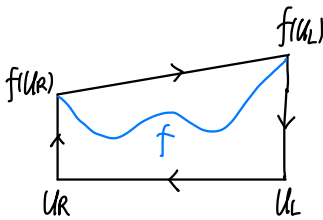
Entropy condition :

(I) let (u_L, u_R, s) be an entropy discontinuity, then (Oleinik)

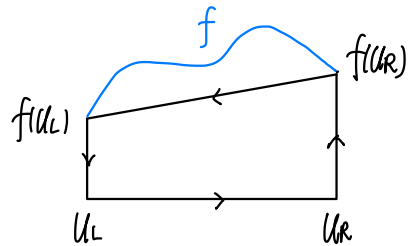
$$\frac{f(u) - f(u_L)}{u - u_L} \geq s \geq \frac{f(u_R) - f(u)}{u_R - u}$$

for all u between u_L and u_R .

remark: 1. $u_L > u_R$



2. $u_L < u_R$



pf: $g(u) = f(u) - f(u_R) - s(u - u_R) \Rightarrow g(u_L) = g(u_R) = 0.$

$H(u) = \xi(u) - s\eta(u) \Rightarrow H(u_R) - H(u_L) \leq 0$ (any entropy pair (ξ, η))

$$\begin{aligned} 0 &\geq H(u_R) - H(u_L) = \int_{u_L}^{u_R} H'(u) du = \int_{u_L}^{u_R} [\xi'(u) - s\eta'(u)] du = \int_{u_L}^{u_R} [\eta'(u)f'(u) - s\eta'(u)] du \\ &= \int_{u_L}^{u_R} \eta'(u)g'(u) du = g(u)\eta'(u) \Big|_{u_L}^{u_R} - \int_{u_L}^{u_R} \eta''(u)g(u) du \\ &= - \int_{u_L}^{u_R} \eta''(u)g(u) du \end{aligned}$$

i.e. $\int_{u_L}^{u_R} \eta''(u)[f(u) - f(u_R) - s(u - u_R)] du \geq 0 \quad \forall \eta''(u) \geq 0$

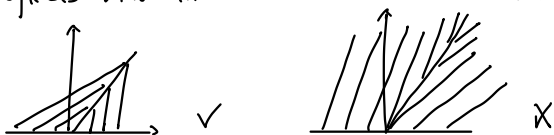
$$\Rightarrow (u_R - u_L) [f(u) - f(u_R) - s(u - u_R)] \geq 0 \Rightarrow \begin{cases} u_L > u_R, & f(u) - f(u_R) - s(u - u_R) \leq 0 \\ u_L < u_R, & f(u) - f(u_R) - s(u - u_R) \geq 0 \end{cases}$$

#

(II) if $f \in C'$, then $f'(u_L) \geq s \geq f'(u_R)$ (Osher)

remark: (1) if $f'' \geq 0$, $\Rightarrow u_L \geq u_R$.

(2) This reflects that the characteristics must run into the shock.



(3) if $f'' > 0$, the entropy solution of a Riemann problem is always either a shock or a rarefaction. self similarity

(i) $u_L > u_R$, $f'(u_L) > f'(u_R)$ $u = \begin{cases} u_L \\ u_R \end{cases}$ $\begin{matrix} x/t < s \\ x/t > s \end{matrix}$ shock solution

(ii) $u_L < u_R$, $f'(u_L) < f'(u_R)$ $u = \begin{cases} u_L \\ (f')^{-1}(x/t) \\ u_R \end{cases}$ $\begin{matrix} x/t < f'(u_L) \\ f'(u_L) \leq x/t \leq f'(u_R) \\ x/t > f'(u_R) \end{matrix}$ rarefaction.

(4) if $f'' = 0$ at some points.

the solution may involve several shocks & rarefactions.

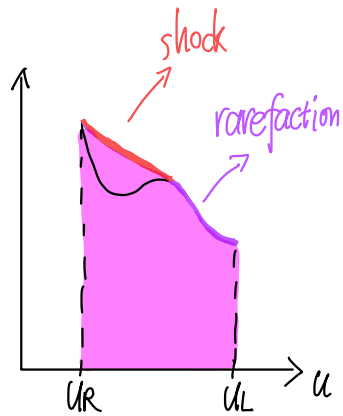
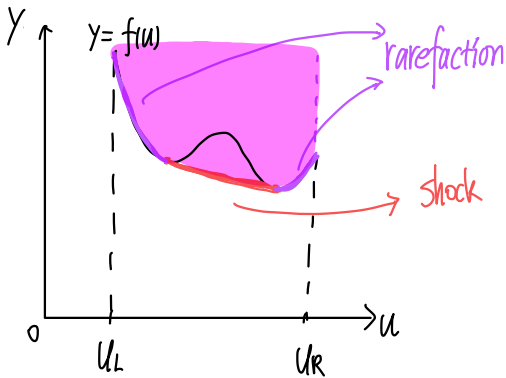


Thm: An entropy weak solution of the Riemann problem can be described as a curve in the plane (u, f)

(i) $u_L > u_R$, the curve is convex envelope/hull of the graph of f .
 $\{y \leq f(u)\}$

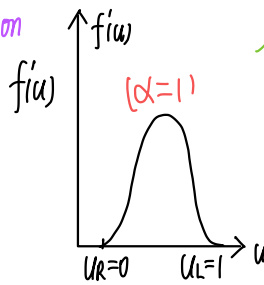
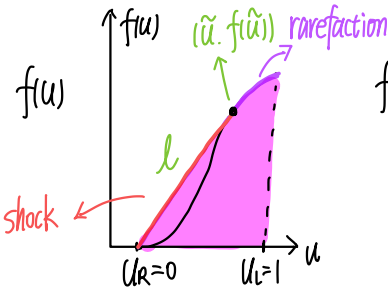
(ii) $u_L < u_R$, the curve is concave envelope/hull of the graph of f .
 $\{y \geq f(u)\}$

Then:



Buckley-leverett equation

Eg: $u_t + f(u)_x = 0$ $f(u) = \frac{u^2}{u^2 + a(1-u)^2}$, $a > 0$, $u \in [0,1]$



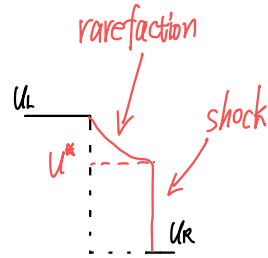
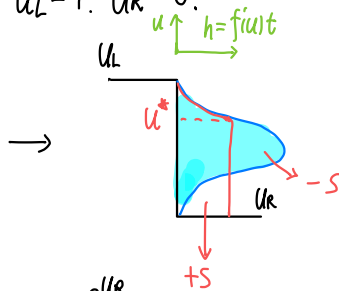
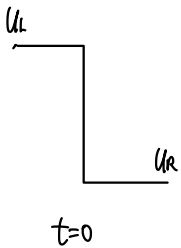
l is the tangent line going through $(\tilde{u}, f(\tilde{u}))$

$$f'(\tilde{u}) = \frac{f(\tilde{u}) - f(u_R)}{\tilde{u} - u_R}$$

$$u_L \longleftrightarrow \tilde{u} \longleftrightarrow u_R$$

rarefaction shock

Riemann problem (I) $u_L = 1$, $u_R = 0$.

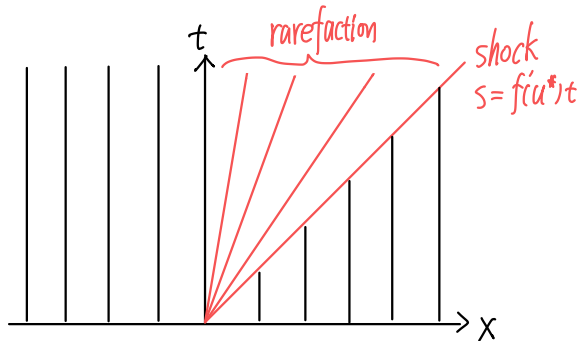
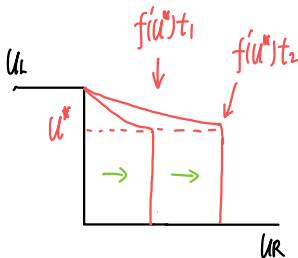


area method = weak form I.

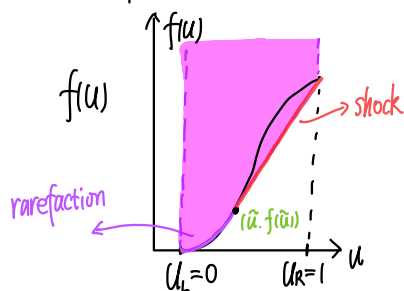
$$\text{area} = f'(u^*)t(u_R - u^*) = \int_{u^*}^{u_R} f'(u)t \, du = t[f(u_R) - f(u^*)]$$

$$\Rightarrow f'(u^*) = \frac{f(u_R) - f(u^*)}{u_R - u^*} = \text{shock speed (constant speed)} \quad (u^* \text{ is constant}) \quad (u^* = \tilde{u})$$

Solution:



Riemann problem (II). $u_L = 0, u_R = 1$



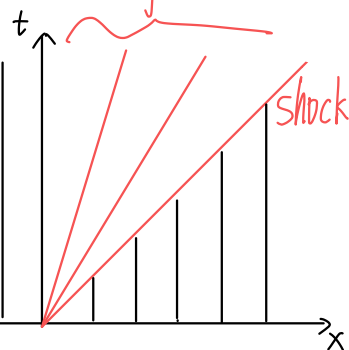
$$f'(\tilde{u}) = \frac{f(\tilde{u}) - f(u_R)}{\tilde{u} - u_R}$$

$$u_L \longleftrightarrow \tilde{u} \longleftrightarrow u_R$$

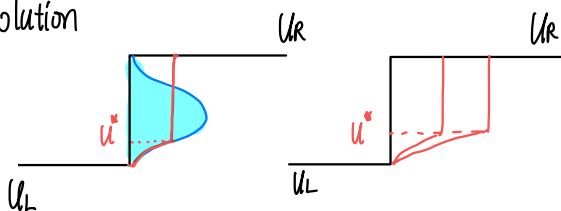
rarefaction

shock

rarefaction



Solution



remark:

$$u_t + f(u)_x = 0 \quad f_\alpha(u) = \frac{u^2}{u^2 + \alpha(1-u)^2}$$

let $w = 1-u$.

$$f_\alpha(u) = \frac{(1-w)^2}{(1-w)^2 + \alpha w^2} = \frac{\frac{1}{\alpha}(1-w)^2}{w^2 + \frac{1}{\alpha}(1-w)^2} = 1 - \frac{w^2}{w^2 + \frac{1}{\alpha}(1-w)^2} = 1 - f_{\frac{1}{\alpha}}(w)$$

$$\begin{cases} f_\alpha(u)_x = -f_{\frac{1}{\alpha}}(w)_x \\ w_t = -u_t \end{cases} \Rightarrow w_t + f_{\frac{1}{\alpha}}(w)_x = 0.$$

HW P34. Ex 3.8

HW. Drive the exact solution of the Riemann problem of
Buckley-leverett equation ($\alpha = \frac{1}{4}$)

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x, 0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases} \end{cases}$$

$$f(u) = \frac{u^2}{u^2 + \frac{1}{4}(1-u)^2} \quad \begin{array}{l} \text{case 1. } u_L = 2, \quad u_R = -2. \\ \text{case 2. } u_L = -3, \quad u_R = 3 \end{array}$$

$$\text{Hw: } \begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & -\pi \leq x \leq \pi \\ u(x, 0) = \alpha + \beta \sin x & (2\pi\text{-periodic boundary}) \end{cases}$$

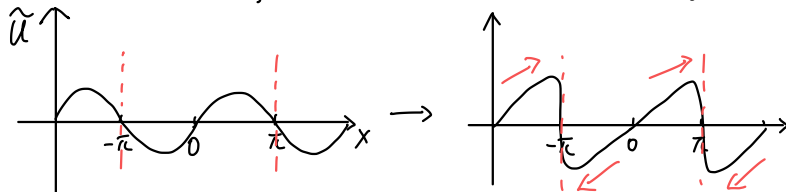
code. ddl 04/04

show the result with $\alpha=1, \beta=2$ at time $t=0.4$ and $t=1$

hint: $u(x, t) = \alpha + \beta \tilde{u}(x - \alpha t, \beta t)$
 $\tilde{u}$ satisfies $\begin{cases} \tilde{u}_t + \left(\frac{\tilde{u}^2}{2}\right)_x = 0 \\ \tilde{u}(x, 0) = \sin x \end{cases}$

$$X = X(t), \quad \begin{cases} \frac{d}{dt} X(t) = f'(u) = \tilde{u}(X(t), t) \Rightarrow \tilde{u}(X(t), t) = \tilde{u}(X_0, 0) \\ X(0) = X_0 \end{cases}$$

Solve: $X = X_0 + f'(u(x_0, 0))t$ ($X \in (0, \pi)$ is enough)



$$\begin{cases} u = \alpha + \beta \tilde{u} \\ u_x = \beta \tilde{u}_x \\ u_t = -\alpha \beta \tilde{u}_x + \beta^2 \tilde{u}_t \end{cases}$$

$$0 = u_t + u u_x = -\alpha \beta \tilde{u}_x + \beta^2 \tilde{u}_t + (\alpha + \beta \tilde{u}) \beta \tilde{u}_x = \beta^2 [\tilde{u}_t + \tilde{u} \tilde{u}_x]$$

Linear hyperbolic system:

$$u_t + A u_x = 0$$

$u: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$ $A \in \mathbb{R}^{m \times m}$ constant, diagonalizable with real eigenvalues

$$A = R \Lambda R^{-1} \text{ with } \Lambda = \text{diag}(\lambda_1 \dots \lambda_n) \quad \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

$$R = (r_1, r_2, \dots, r_n)$$

Remark: strictly hyperbolic, if the eigenvalues are distinct: $\lambda_1 < \lambda_2 < \dots < \lambda_n$

let $v = R^{-1} u \in \mathbb{R}^m$

$$R^{-1} u_t + R^{-1} A u_x = v_t + \Lambda v_x = 0$$

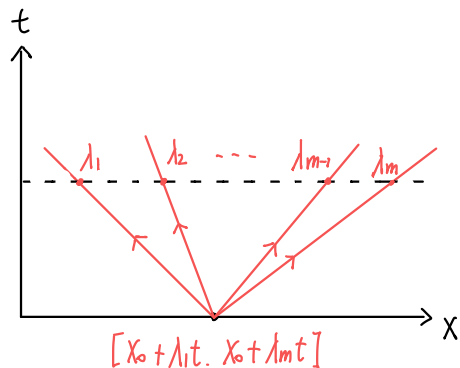
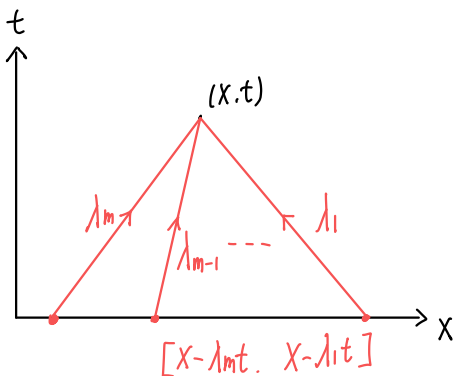
$$(v_p)_t + \lambda_p (v_p)_x = 0 \quad p=1, 2, \dots, m$$

$$\Rightarrow v_p(x, t) = v_p(x - \lambda_p t, 0)$$

$$\Rightarrow u(x, t) = \sum_p v_p(x, t) r_p = \sum_p v_p(x - \lambda_p t) r_p$$

$\left\{ \begin{array}{l} u \text{ conserved variable} \\ v \text{ characteristic variable} \end{array} \right.$

$\rightarrow \ell_p: x = x_0 + \lambda_p t \quad p=1, 2, \dots, m$
"p-characteristic"



$$\text{eg: } \begin{cases} u_{tt} = c^2 u_{xx} & (c > 0) \\ u = u_0, \quad u_t = u_1 \end{cases}$$

$$\text{let } v = u_x, \quad w = u_t \Rightarrow \begin{cases} w_t = c^2 v_x \\ w_x = v_t \\ w = u_1, \quad v = u_0' \end{cases} \quad \begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} -1 & \\ & -c^2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0$$

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix} \quad \lambda_1 = -c, \quad \lambda_2 = c \quad R = \begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix} \quad R^{-1} = \frac{1}{2c} \begin{pmatrix} c & 1 \\ c & -1 \end{pmatrix} \quad AR = R \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

$$\begin{aligned} R^{-1} \begin{pmatrix} v \\ w \end{pmatrix} &= \frac{1}{2c} \begin{pmatrix} c v(x,t) + w(x,t) \\ c v(x,t) - w(x,t) \end{pmatrix} \begin{matrix} \leftarrow \lambda_1 \\ \leftarrow \lambda_2 \end{matrix} = \frac{1}{2c} \begin{pmatrix} c v(x-\lambda_1 t, 0) + w(x-\lambda_1 t, 0) \\ c v(x-\lambda_2 t, 0) - w(x-\lambda_2 t, 0) \end{pmatrix} \\ &= \frac{1}{2c} \begin{pmatrix} c u_0'(x+ct) + u_1(x+ct) \\ c u_0'(x-ct) - u_1(x-ct) \end{pmatrix} \end{aligned}$$

#

$$\text{eg: Riemann problem. } \begin{cases} u_t + A u_x = 0 \\ u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases} \end{cases} \quad u_L = \sum_p \alpha_p r_p \quad u_R = \sum_p \beta_p r_p$$

$$A \text{ is strictly hyperbolic. } \lambda_1 < \lambda_2 < \dots < \lambda_m. \quad A = R \Lambda R^{-1} \quad v = R^{-1} u$$

$$\begin{cases} v_t + \Lambda v_x = 0 \\ v(x,0) = \begin{cases} (\alpha_1 \dots \alpha_m)^T & x < 0 \\ (\beta_1 \dots \beta_m)^T & x > 0 \end{cases} \end{cases}$$

$$\text{i.e. } \begin{cases} (V_p)_t + \lambda_p (V_p)_x = 0 \\ V_p(x, 0) = \begin{cases} \alpha_p & x < 0 \\ \beta_p & x > 0 \end{cases} \end{cases} \Rightarrow V_p(x, t) = \begin{cases} \alpha_p & x - \lambda_p t < 0 \\ \beta_p & x - \lambda_p t > 0 \end{cases} \quad (p = 1, \dots, m)$$

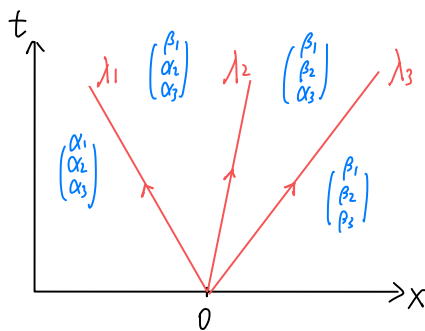
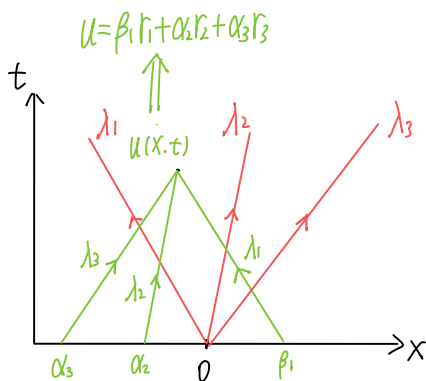
for (x, t) . let $p_m(x, t) \in \{1, \dots, m\}$ be the maximum value of p s.t. $x - \lambda_p t > 0$.

$$V_p(x, t) = (\beta_1 \dots \beta_{p_m}, \alpha_{p_m+1}, \dots, \alpha_m)^T$$

$$\Rightarrow u(x, t) = \sum_p V_p(x, t) r_p = \sum_{p \leq p_m} \beta_p r_p + \sum_{p \geq p_m+1} \alpha_p r_p$$

$$= u_L + \sum_{p \geq p_m+1} (\beta_p - \alpha_p) r_p$$

$$= u_L + \sum_{\lambda_p < \frac{x}{t}} (\beta_p - \alpha_p) r_p \quad \left(= u_R - \sum_{\lambda_p > \frac{x}{t}} (\beta_p - \alpha_p) r_p \right)$$



remark: (1) $u(x,t)$ is constant in each of the wedges.

(2) As we cross the p -th characteristic.

the coefficient of r_p jumps from α_p to β_p . other coefficients remain constant.

$$\text{jump: } [u] = (\beta_p - \alpha_p) r_p$$

$$[f] = [Au] = A[u]$$

$$= A(\beta_p - \alpha_p) r_p = (\beta_p - \alpha_p) \lambda_p r_p = \lambda_p [u]$$

$$\Rightarrow \lambda_p = \frac{[f]}{[u]}. \quad \text{RH jump condition } \checkmark$$

(3) In general case (A is not strictly hyperbolic)

the jump $u_L - u_R$ cannot propagate as a single discontinuity with any speed without violating RH condition. (but more jump, as show below)

$u_L - u_R$ can propagate as a single discontinuity only if $u_L - u_R$ is an eigenvalue of A .

(connected by p -wave directly)

def: (The phase plane $\mathbb{R}^m$)

Hugoniot locus through any point $\hat{u}$. point can be connected to $\hat{u}$ by an elementary

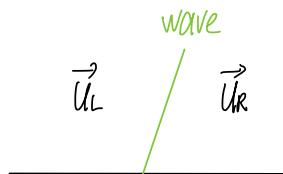
$$p\text{-wave: } \tilde{u}_p(s; \hat{u}) = \hat{u} + s r_p \quad p=1 \dots m. \quad \text{with } f(\tilde{u}_p) - f(\hat{u}) = \lambda_p(\tilde{u}_p - \hat{u})$$

Note that $\tilde{u}_p(0; \hat{u}) = \hat{u}$ for each p .

So through the point $\hat{u}$ in phase space, there are m straight lines.

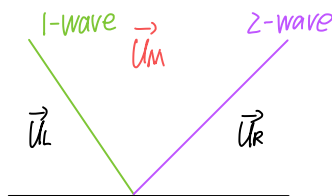
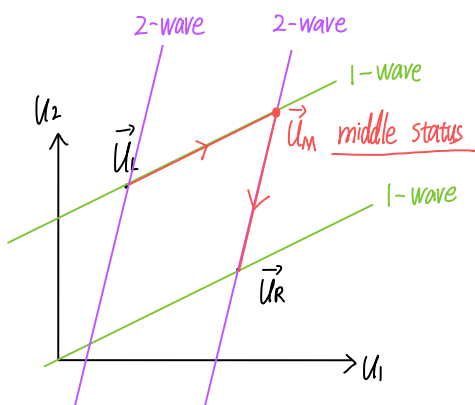
phase plane ($m=2$):

eg 1: $\vec{u}_L - \vec{u}_R$ is eigenvector of A .

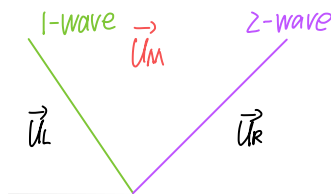
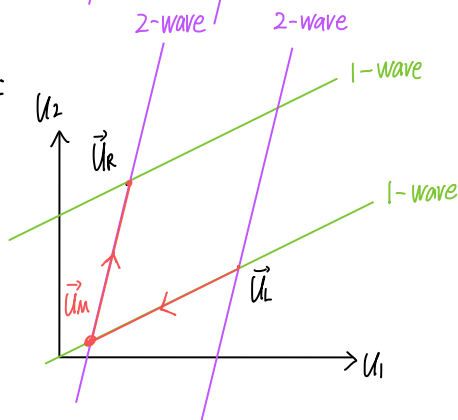


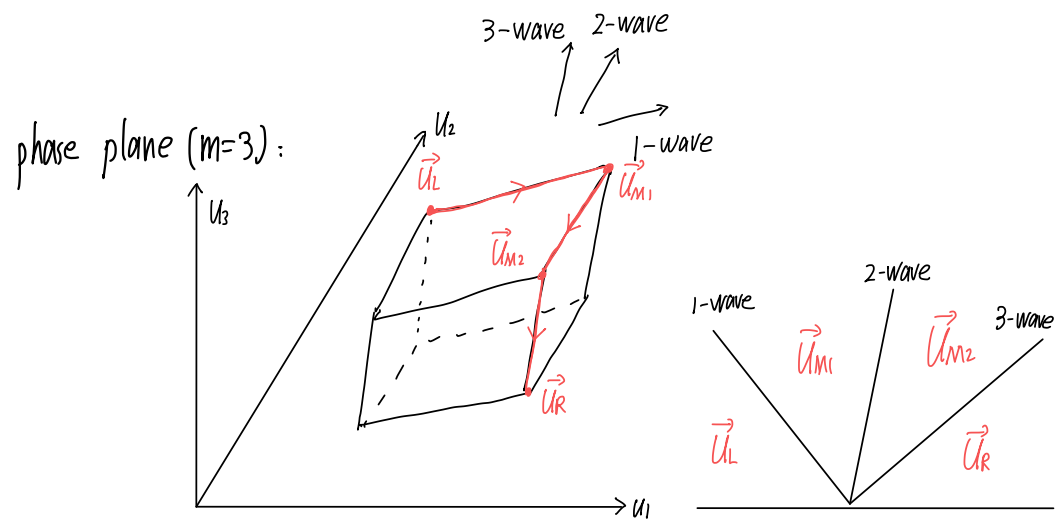
eg 2. otherwise, $\exists \vec{u}_M$ s.t. $\vec{u}_L \xrightarrow{\text{1-wave}} \vec{u}_M \xrightarrow{\text{2-wave}} \vec{u}_R$

eg 2.1



eg 2.2:





in general: Linear system. Riemann problem: only several shocks

A: eigenvectors: $r_1, r_2, \dots, r_m$. $R = (r_1 \dots r_m)$

$$u_L = \sum \alpha_p r_p$$

$$u_L = (\alpha_1, \alpha_2, \alpha_3 \dots \alpha_{m-1}, \alpha_m) R^T \quad \text{1-wave}$$

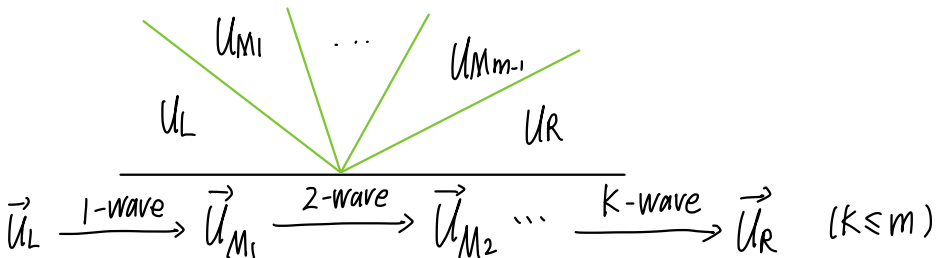
$$u_{M1} = (\beta_1, \alpha_2, \alpha_3 \dots \alpha_{m-1}, \alpha_m) R^T \quad \text{2-wave}$$

$\vdots$

$$u_{Mm-1} = (\beta_1, \beta_2, \beta_3 \dots \beta_{m-1}, \alpha_m) R^T \quad \text{m-wave}$$

$$u_R = \sum \beta_p r_p$$

$$u_R = (\beta_1, \beta_2, \beta_3 \dots \beta_{m-1}, \beta_m) R^T$$



Nonlinear system

$$\vec{u}_t + \vec{f}(\vec{u})_x = 0$$

$$\text{or } \vec{u}_t + A(\vec{u}) \vec{u}_x = 0$$

$$\vec{u}: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m. \quad \vec{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m. \quad A: \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$$

Assume strictly hyperbolic:

$A(u)$ has distinct eigenvalues: $\lambda_1(u) < \lambda_2(u) < \dots < \lambda_m(u)$

and linearly independent eigenvectors: $r_1(u), r_2(u), \dots, r_m(u)$.

$$R(u) = (r_1(u), r_2(u), \dots, r_m(u))$$

$$A(u) = R(u) \Lambda(u) R(u)^{-1}$$

$$\Lambda(u) = \text{diag}(\lambda_1(u), \dots, \lambda_m(u))$$

$$u_t + A(u) u_x = 0 \Leftrightarrow R(u)^{-1} \cdot u_t + \Lambda(u) R(u)^{-1} u_x = 0$$

$$\Leftrightarrow \underline{(R(u)^{-1} u)_t + \Lambda(u) (R(u)^{-1} u)_x = 0}$$

def: There are m characteristic curves through each point (x, t)

$$p\text{-th family } X = X(t) \text{ satisfies } \begin{cases} X'(t) = \lambda_p(u(X(t), t)) \\ X(0) = X_0 \end{cases} \quad (p=1, 2, \dots, m)$$

along the p -th curve $X=X(t)$. $U(X(t), t)$ is not constant.

$$\frac{d}{dt} U(X(t), t) = U_t + U_x X'(t) = U_t + \lambda_p U_x \neq 0$$

Riemann problem in Linear system : shocks



Nonlinear system : shocks and rarefactions.

Shocks :

Consider a discontinuity with speed s . $\hat{u}$ and $\tilde{u}$ are the states.
 $f(\tilde{u}) - f(\hat{u}) = s(\tilde{u} - \hat{u})$

fix the value $\hat{u}$. attempt to determine the set of $\tilde{u}$. $\tilde{u} = \tilde{u}(s; \hat{u})$

$$\begin{cases} f(\tilde{u}(s; \hat{u})) - f(\hat{u}) = S(s; \hat{u}) [\tilde{u}(s; \hat{u}) - \hat{u}] \\ \tilde{u}(0; \hat{u}) = \hat{u} \end{cases}$$

$$\partial_s : f'(\tilde{u}) \tilde{u}'(s) = S'(s) [\tilde{u}(s) - \hat{u}] + S(s) \tilde{u}'(s)$$

$$s=0 : f'(\hat{u}) \tilde{u}'(0) = A(\hat{u}) \tilde{u}'(0) = 0 + S(0) \tilde{u}'(0)$$

Hence, $S(0)$ is the eigenvalue of $A(\hat{u})$, $\tilde{u}'(0)$ is the eigenvector.

$$\begin{cases} S(0) = \lambda_p(\hat{u}) \\ \tilde{u}'(0) = r_p(\hat{u}) \end{cases} \quad p \in \{1, \dots, m\}$$

Linear: $\begin{cases} S = \lambda_p \\ \tilde{u}'(0) = r_p \\ \tilde{u}(\xi) = \hat{u} + \xi r_p \end{cases}$

We can choose m curves through $\hat{u}$. one of each characteristic family $\tilde{u}_p(\xi; \hat{u})$

The curve $\tilde{u}_p(\xi; \hat{u})$ is tangent to $r_p(\hat{u})$ at the point $\hat{u}$.

def: Hugoniot locus

$\tilde{u}_p(\xi; \hat{u})$ is the p -th Hugoniot curve through $\hat{u}$.

$\tilde{u}_p$ and $\hat{u}$ are connected by a p -shock.

eg: the isothermal equation ($m = pV$)

$$\begin{cases} p_t + m x_2 = 0 \\ m_t + \left(\frac{m}{p} + a^2 p \right)_x = 0 \end{cases} \quad a = \text{constant} > 0$$

$$f(U) = \begin{pmatrix} m \\ \frac{m^2}{p} + a^2 p \end{pmatrix}$$

$$U = \begin{pmatrix} p \\ m \end{pmatrix} \quad A(U) = \begin{pmatrix} 0 & 1 \\ \frac{m^2}{p^2} + a^2 & \frac{m}{p} \end{pmatrix}$$

$$\lambda_1 = \frac{m}{p} - a \quad r_1 = \begin{pmatrix} 1 \\ \frac{m}{p} - a \end{pmatrix}$$

$$U_t + A(U)U_x = 0$$

$$\lambda_2 = \frac{m}{p} + a \quad r_2 = \begin{pmatrix} 1 \\ \frac{m}{p} + a \end{pmatrix}$$

$$f(\hat{u}) - f(\hat{u}) = s(\hat{u} - \hat{u}) \quad \text{i.e.} \quad \begin{cases} \hat{m} - \hat{m} = s(\hat{\rho} - \hat{\rho}) \\ \left(\frac{\hat{m}^2}{\hat{\rho}} + a^2 \hat{\rho}\right) - \left(\frac{\hat{m}^2}{\hat{\rho}} + a^2 \hat{\rho}\right) = s(\hat{m} - \hat{m}) \end{cases}$$

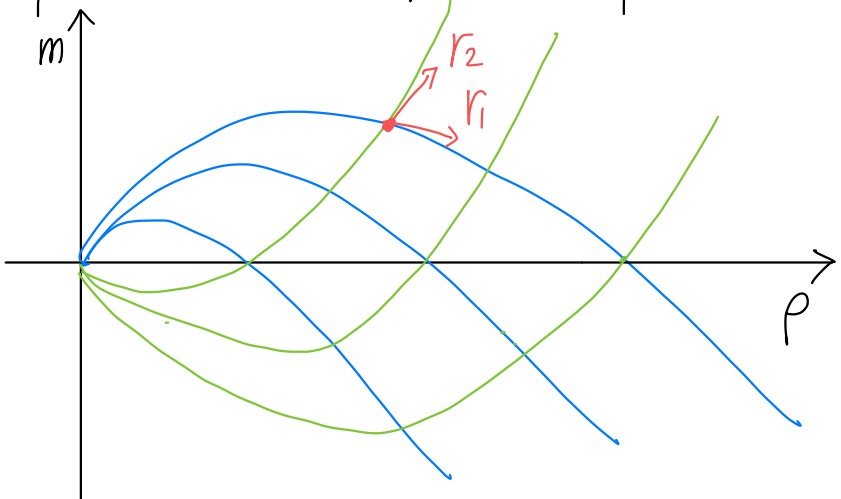
$$\Rightarrow \begin{cases} \tilde{m} = \hat{m} \frac{\tilde{\rho}}{\hat{\rho}} \pm a \sqrt{\frac{\tilde{\rho}}{\hat{\rho}}} (\hat{\rho} - \hat{\rho}) \\ s = \frac{\hat{m}}{\hat{\rho}} \pm a \sqrt{\frac{\tilde{\rho}}{\hat{\rho}}} \end{cases}$$

$$s \geq -1, \tilde{\rho} \geq 0$$

$$(s = -1, \tilde{u}_1 = 0, \tilde{u}_2 = 0)$$

$$\text{Set } \tilde{\rho}(s; \hat{\rho}) = (1+s) \hat{\rho}$$

$$\begin{cases} \tilde{u}_1(s; \hat{u}) = \hat{u} + s \left(\hat{m} - a \sqrt{1+s} \hat{\rho} \right) \\ S_1(s; \hat{u}) = \frac{\hat{m}}{\hat{\rho}} - a \sqrt{1+s} \end{cases} \quad \begin{cases} \tilde{u}_2(s; \hat{u}) = \hat{u} + s \left(\hat{m} + a \sqrt{1+s} \hat{\rho} \right) \\ S_2(s; \hat{u}) = \frac{\hat{m}}{\hat{\rho}} + a \sqrt{1+s} \end{cases}$$



Question : u_1 & u_2 connected by 1-shock $\} \Rightarrow u_1$ & u_3 connected by 1-shock?
 u_2 & u_3 connected by 1-shock $\}$

$$\hat{u}_1(\xi; \hat{u}) = \hat{u} + \xi \begin{pmatrix} \hat{p} \\ \hat{m} - a \hat{p} \sqrt{1+\xi} \end{pmatrix} = (1+\xi) \hat{u} + \begin{pmatrix} 0 \\ -a \hat{p} \sqrt{1+\xi} \xi \end{pmatrix}$$

$$u_2 = (1+\xi_1) u_1 + \begin{pmatrix} 0 \\ -a \rho \xi_1 \sqrt{1+\xi_1} \end{pmatrix}$$

$$u_3 = (1+\xi_2) u_2 + \begin{pmatrix} 0 \\ -a \rho \xi_2 \sqrt{1+\xi_2} \end{pmatrix}$$

$$= (1+\xi_2)(1+\xi_1) u_1 + \begin{pmatrix} 0 \\ -a \rho \xi_1 \sqrt{1+\xi_1} (1+\xi_2) \end{pmatrix} + \begin{pmatrix} 0 \\ -a (1+\xi_1) \rho \xi_2 \sqrt{1+\xi_2} \end{pmatrix}$$

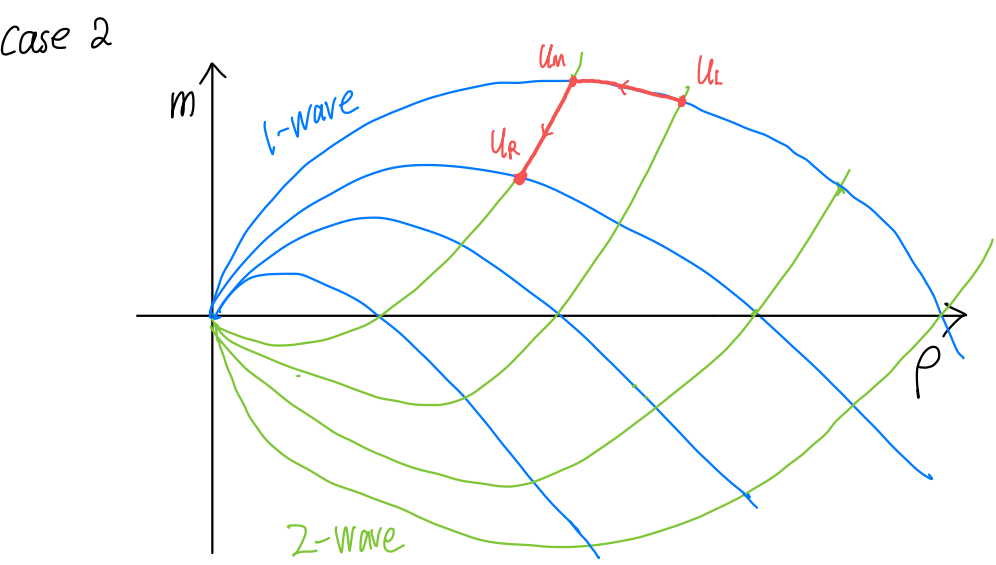
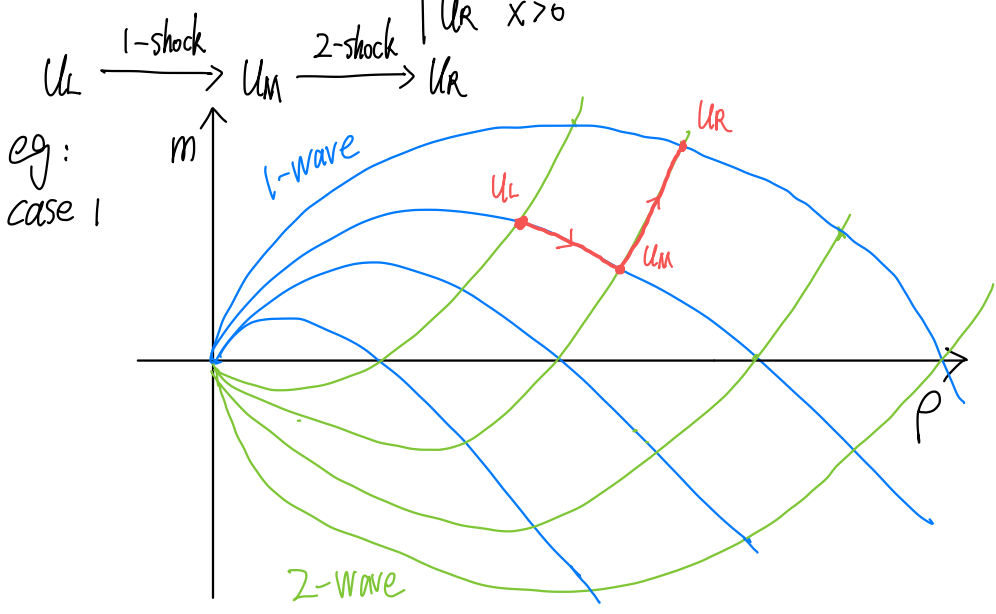
$$= (1+\xi_2)(1+\xi_1) u_1 + \begin{pmatrix} 0 \\ -a \rho ((1+\xi_1)(1+\xi_2) - 1) \sqrt{(1+\xi_1)(1+\xi_2)} \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta \end{pmatrix}$$

$\Delta \neq 0!$ if $\xi_1 \xi_2 \neq 0$

Hw. Ex 7.1 P73

Hugoniot locus (shallow water)

Riemann problem $u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$ for isothermal equation



Algebraically:

$$① \quad u_m = \tilde{u}_1(\xi_1, u_L)$$

$$\begin{pmatrix} p_m \\ m_m \end{pmatrix} = \begin{pmatrix} p_L \\ m_L \end{pmatrix} + \xi_1 \begin{pmatrix} p_L \\ m_L - a \rho_L \sqrt{1 + \xi_1} \end{pmatrix} \Rightarrow m_m = m_L \frac{p_m}{p_L} - a \underbrace{\sqrt{\frac{p_m}{p_L}}}_{\text{1-wave}} (p_m - p_L)$$

$$② \quad u_m = \tilde{u}_2(\xi_2, u_R)$$

$$\Rightarrow m_m = m_R \frac{p_m}{p_R} + a \underbrace{\sqrt{\frac{p_m}{p_R}}}_{\text{2-wave}} (p_m - p_R)$$

$$m_R \frac{p_m}{p_R} + a \sqrt{\frac{p_m}{p_R}} (p_m - p_R) = m_L \frac{p_m}{p_L} - a \sqrt{\frac{p_m}{p_L}} (p_m - p_L) \Rightarrow p_m \Rightarrow u_m = \begin{pmatrix} p_m \\ m_m \end{pmatrix}$$

More generally, for a system of m equations. We can attempt to solve the Riemann problem by finding a sequence of states $u_1 \dots u_{m-1}$

$$u_L \xrightarrow{\text{1-wave}} u_1 \xrightarrow{\text{2-wave}} u_2 \dots \xrightarrow{\text{m-1-wave}} u_{m-1} \xrightarrow{\text{m-wave}} u_m = u_R$$

$$\begin{array}{ccccccc} & & \parallel & & \parallel & & \parallel \\ & & u_1(\xi_1) & & u_2(\xi_1, \xi_2) & & u_m(\xi_1, \dots, \xi_m) \end{array}$$

$$\left. \frac{\partial u_m}{\partial \xi_p} \right|_{\xi_1 = \xi_2 = \dots = \xi_m = 0} \propto f_p(u_L) \quad p=1, 2, \dots, m. \Rightarrow \text{the Jacobian of the mapping is nonsingular.}$$

For any u_R sufficiently close to u_L , there is a unique set of parameter $\{\xi_1, \dots, \xi_m\}$
it is necessary. s.t. $u_R = u_m(\xi_1, \xi_2, \dots, \xi_m)$

def: (Entropy function)

Entropy function: $\eta(u): \mathbb{R}^m \rightarrow \mathbb{R}$

$\eta'' \in \mathbb{R}^{m \times m} > 0$ positive defined

Entropy flux: $\xi(u): \mathbb{R}^m \rightarrow \mathbb{R}$

$$\text{s.t. } \nabla \xi = \underbrace{\nabla \eta \cdot \nabla f}_{\text{matrix}}$$

remark: 1. strong solution u .

$$\eta(u)_t + \xi(u)_x = \nabla \eta \cdot u_t + \nabla \xi \cdot u_x = \nabla \eta (u_t + \nabla f \cdot u_x) = 0$$

2. for scalar case. $\xi'(u) = \eta'(u)f'(u)$

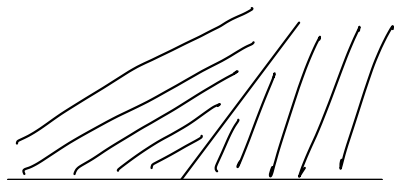
for each $\eta(u)$. we can always find $\xi(u)$. but it's not true for system.

def: (Entropy solution)

u is the entropy solution if $\eta(u)_t + \xi(u)_x \leq 0$ for any weak pair (η, ξ)

(in weak sense)

For scalar equation, in convex case. the simplest case is $f'(u) > s > f'(u_r)$



def: (genuinely nonlinear)

The p -th characteristic field is said to be genuinely nonlinear if

$$\nabla \lambda_p(\vec{u}) \cdot r_p(\vec{u}) \neq 0 \quad \forall \vec{u} \quad (*)$$

$$\text{Here } \nabla \lambda_p(\vec{u}) = \left(\frac{\partial \lambda_p}{\partial u_1} \cdots \frac{\partial \lambda_p}{\partial u_m} \right)^T$$

remark: $m=1$. $\lambda_1(u) = f(u)$, $r_1(u) = 1$, $\nabla \lambda_1(u) = f'(u)$

(*) means $f'(u) \neq 0 \quad \forall u$.

i.e. $f'(u)$ is monotonically increasing or decreasing.

↓
shock

↓
rarefaction

Thm (Lax entropy condition)

For a genuinely nonlinear field, a jump in the p -th field is admissible only if

$$\lambda_p(u_L) > s > \lambda_p(u_R)$$

$$\text{eg: } \lambda_p = \frac{m}{p} \pm a. \quad \nabla \lambda_p = \begin{pmatrix} -\frac{m}{p^2} \\ \frac{1}{p} \end{pmatrix} \quad p=1,2.$$

$$\nabla \lambda_1 \cdot r_1 = -\frac{m}{p^2} + \frac{1}{p} \left(\frac{m}{p} - a \right) = -\frac{a}{p} \neq 0$$

$$\nabla \lambda_2 \cdot r_2 = -\frac{m}{p^2} + \frac{1}{p} \left(\frac{m}{p} + a \right) = \frac{a}{p} \neq 0$$

eg: U_L & U_R are connected by 1-shock

(1) RH jump condition:

$$\text{the shock speed } S_1 = \frac{m_L}{\rho_L} - a \sqrt{\frac{p_R}{p_L}} = \frac{m_R}{\rho_R} - a \sqrt{\frac{p_L}{p_R}}$$

$U_L \rightarrow U_R$

$$\begin{pmatrix} p_R \\ m_R \end{pmatrix} = U_R = \tilde{U}_1(\tilde{s}_1; U_L) = \begin{pmatrix} p_L \\ m_L \end{pmatrix} + \tilde{s}_1 \begin{pmatrix} p_L \\ m_L - a \rho_L \sqrt{p_L \tilde{s}_1} \end{pmatrix} \Rightarrow \tilde{s}_1 = \frac{p_R}{p_L} - 1$$

$$\tilde{s}_1(\tilde{s}_1; U_L) = \frac{m_L}{\rho_L} - a \sqrt{\frac{p_R}{p_L}}$$

or $U_R \rightarrow U_L$

$$\begin{pmatrix} p_L \\ m_L \end{pmatrix} = U_L = \tilde{U}_1(\tilde{s}_2; U_R) \Rightarrow \tilde{s}_2 = \frac{p_L}{p_R} - 1 \Rightarrow \tilde{s}_1(\tilde{s}_2; U_R) = \frac{m_R}{\rho_R} - a \sqrt{\frac{p_L}{p_R}}$$

$$m = \rho v \Rightarrow S < V_L \text{ \& } V_R$$

$\Rightarrow$ 1-shock always slower than the fluid on either side.

so a given fluid particle passes through the shock from left to right.

(2) entropy condition $\frac{m_L}{\rho_L} - a = \lambda_1(U_L) > S > \lambda_1(U_R) = \frac{m_R}{\rho_R} - a$

$$\frac{m_L}{\rho_R} - a \sqrt{\frac{p_R}{p_L}} = \frac{m_R}{\rho_R} - a \sqrt{\frac{p_L}{p_R}} \Rightarrow \underline{p_L < p_R}$$

The density of the gas must increase as it goes through the shock

The gas can only be compressed as the shock passes. (not rarified)

eg 2. U_L & U_R connected by 2-wave

(1) RH jump condition

$$\text{The shock speed } S_2 = \frac{m_L}{\rho_L} + a \sqrt{\frac{\rho_R}{\rho_L}} = \frac{m_R}{\rho_R} + a \sqrt{\frac{\rho_L}{\rho_R}} > V_L \text{ \& } V_R.$$

so 2-shock travel faster than the fluid on either side.

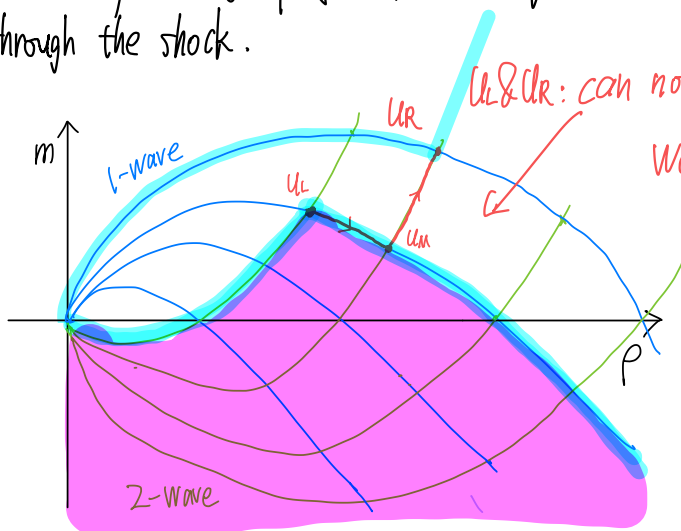
And particles pass through the shock from the right to the left

(2) Entropy condition

$$\frac{m_L}{\rho_L} + a = \lambda_2(U_L) > s > \lambda_2(U_R) = \frac{m_R}{\rho_R} + a \Rightarrow \underline{\rho_R < \rho_L}$$

The density must jump from ρ_R to higher value ρ_L . as the gas go through the shock.

eg:



if U_R in this region, U_L & U_R can be connected by two shocks.

Nonlinear system

$$\vec{u}_t + \vec{f}(\vec{u})_x = 0 \quad \text{or} \quad \vec{u}_t + A(\vec{u})\vec{u}_x = 0$$

$$A(\vec{u}) = R^T(\vec{u}) \Lambda(\vec{u}) R(\vec{u})$$

$$\Lambda(\vec{u}): \lambda_1(\vec{u}) \leq \dots \leq \lambda_m(\vec{u})$$

$$r_1(\vec{u}) \quad \dots \quad r_m(\vec{u})$$

Shocks.

Hugoniot locus (RH)

$\tilde{u}_p(s, \hat{u})$ through $\hat{u}$. $\tilde{u}_p$ and $\hat{u}$ are connected by a p-shock.

$$\begin{cases} f(\tilde{u}_p(s, \hat{u})) - f(\hat{u}) = s_p(s, \hat{u}) [\tilde{u}_p(s, \hat{u}) - \hat{u}] \\ \tilde{u}_p(0, \hat{u}) = \hat{u} \end{cases}$$

$$s_p(0) = \lambda_p(\hat{u}). \quad \tilde{u}_p'(0, \hat{u}) \propto r_p(\hat{u})$$

Entropy condition

Genuinely nonlinear: $\nabla \lambda_p(\vec{u}) \cdot r_p(\vec{u}) \neq 0 \quad \forall \vec{u} \quad (\sim f \text{ is convex})$

(condition)

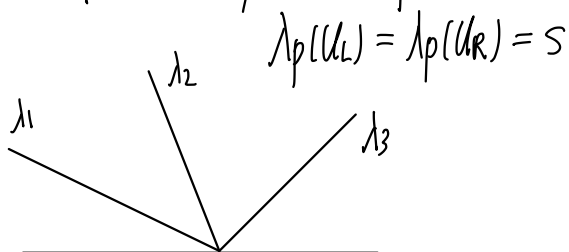
Lax entropy condition: $\lambda_p(\vec{u}_L) > s > \lambda_p(\vec{u}_R)$

(激波. shock wave)

def: (Linear degeneracy 线性退化)

The p -th field is Linear degeneracy if $\nabla \lambda_p(u) \cdot r_p(u) = 0 \quad \forall u$.

remark (1) for linear system, $\nabla \lambda_p = 0 \Rightarrow$ always linear degeneracy



(2) for general systems. if the p -th field is linear degeneracy, and u_L & u_R are connected by a discontinuity in this field. then it can be shown that

$$\lambda_p(u_L) = \lambda_p(u_R) = s$$

The p -characteristic can parallel to the propagating discontinuity on each side
(接触间断 contact discontinuity)

Genuinely nonlinear $\nabla \lambda_p(u) \cdot r_p(u) \neq 0 \quad \forall u$ } (Entropy condition)
or
Linear degeneracy $\nabla \lambda_p(u) \cdot r_p(u) \equiv 0 \quad \forall u$ } $\Rightarrow \lambda_p(u_L) \geq s \geq \lambda_p(u_R)$

↑
(two special case). (general case: too complicated)

Rarefaction waves.

The solution of Riemann problem is a "similarity solution" of PDE. $u(x,t) = v(\frac{x}{t})$

consider the solution:
$$\vec{u}(x,t) = \begin{cases} \vec{u}_L & x \leq s_1 t \\ \vec{w}(\frac{x}{t}) & s_1 t < x < s_2 t \\ \vec{u}_R & x \geq s_2 t \end{cases}$$

$\vec{w}$ is a smooth function, and $\vec{w}(s_1) = \vec{u}_L$, $\vec{w}(s_2) = \vec{u}_R$

$$\vec{u}(x,t) = \vec{w}(\frac{x}{t}) \Rightarrow \vec{u}_t = (\frac{-x}{t^2}) \vec{w}' \quad \vec{u}_x = (\frac{1}{t}) \vec{w}'$$

$$\vec{u}_t + f'(u) \vec{u}_x = 0 \Rightarrow (\frac{-x}{t^2}) \vec{w}' + f'(\vec{w}) (\frac{1}{t}) \vec{w}' = 0 \quad \begin{matrix} \vec{s} = \frac{x}{t} \\ \Rightarrow \end{matrix} \underbrace{f'(\vec{w}(s))}_{\text{matrix}} \cdot \underbrace{\vec{w}'(s)}_{\text{vector}} = \underbrace{s}_{\text{scalar}} \cdot \underbrace{\vec{w}'(s)}_{\text{vector}}$$

Case 1: $\vec{w}'(s) \equiv 0$. $\vec{w}(s) = \text{constant}$ ($s \leq s_1$ or $s \geq s_2$) a jump at s_2 or s_1

Case 2: $\vec{w}'(s) \neq 0$. $s_1 < s < s_2$

$$\begin{cases} \vec{w}'(s) = \alpha(s) \Gamma_p(\vec{w}(s)) \\ s = \lambda_p(\vec{w}(s)) \end{cases}$$

def (integral curves)

An integral curve for $r_p(u)$ is a curve $\vec{u}_p(\eta)$ if the tangent at each point lies in the direction $r_p(u)$

remark: When f is smooth and strictly hyperbolic, $r_p(u)$ is a smooth function.

$\Rightarrow r_p(u)$ is a smooth function.

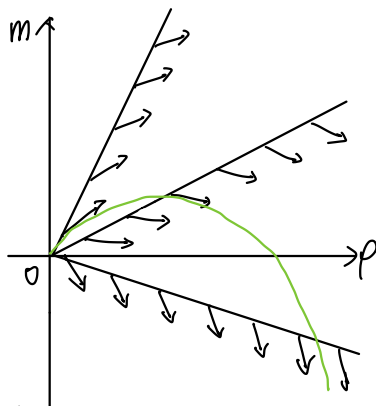
$\Rightarrow$ the integral curve is smooth. $\vec{u}_p(\eta) \quad \eta \in \mathbb{R}$

$$\vec{u}_p'(\eta) = \alpha(\eta) \vec{r}_p(\vec{u}_p(\eta)) \quad \alpha(\eta) \in \mathbb{R}.$$

eg: the isothermal equation ($\alpha=1$)

$$\lambda_1(\vec{u}) = \frac{m}{p} - a \quad \vec{r}_1(\vec{u}) = \begin{pmatrix} m \\ \frac{m}{p} - a \end{pmatrix}$$

$$\vec{u} = \vec{u}_1(\eta) : \vec{u}_1'(\eta) \parallel \vec{r}_1(\vec{u}_1(\eta))$$



$\lambda_1(\vec{u}_1(\eta))$ varies monotonically along any integral curve of $r_1(\vec{u})$

The values $\vec{W}(s)$ all lies along some integral curve of $r_p(\vec{u})$ (rarefaction wave)
start at $\vec{u}_L = \vec{W}(s_1)$. end at $\vec{u}_R = \vec{W}(s_2)$

(necessary condition)

To ensure the function $W\left(\frac{x}{\epsilon}\right)$ is single-valued.

We need $\xi = \lambda_p(W(s))$ to be monotonically increasing as $W(s)$ move from u_L to u_R .

remark:

(1) p -th characteristic field is linearly degeneracy.

$$\nabla \lambda_p(u) \cdot r_p(u) \equiv 0 \quad \forall u$$

$$\nabla \lambda_p(u_p(\eta)) \cdot u_p'(\eta) = \alpha(\eta) \nabla \lambda_p(u_p) \cdot r_p(u_p) = 0$$

$\Rightarrow \lambda_p(u_p(\eta))$ is constant along the integral curve of the vector.

$\Rightarrow$ no rarefaction wave.

(2) p -th characteristic field is genuine nonlinear.

$$\nabla \lambda_p(u) \cdot r_p(u) \neq 0 \quad \forall u.$$

$$\Rightarrow \nabla \lambda_p(u_p(\eta)) \cdot u_p'(\eta) \neq 0$$

$\Rightarrow \lambda_p(u_p(\eta))$ is monotonically increasing or decreasing along the integral curve.

$\Rightarrow u_L$ & u_R can be connected by a rarefaction wave provided they lie on the same integral curve and $\lambda_p(u_L) < \lambda_p(u_R)$

(3) For general case. from a given state u_L . we can move along the integral curve in the direction in which λ_p is increasing.

If $\lambda_p(u_L)$ has a local maximum at u_L in the direction r_p , there is no rarefaction waves with the left state u_L .

To determine the function $W(\xi)$. ($u(x, t) = W(\frac{x}{t})$)

$$\begin{cases} \vec{W}'(\xi) = \alpha(\xi) r_p(\vec{W}(\xi)) \end{cases}$$

$$\begin{cases} \xi = \lambda_p(\vec{W}(\xi)) \end{cases} \xrightarrow{\partial_\xi} 1 = \nabla \lambda_p(\vec{W}(\xi)) \cdot \vec{W}'(\xi) \\ = \alpha(\xi) \underbrace{\nabla \lambda_p(\vec{W}(\xi)) \cdot r_p(\vec{W}(\xi))}_{\neq 0}$$

$$\alpha(\xi) = (\nabla \lambda_p(W(\xi)) \cdot r_p(W(\xi)))^{-1} \quad \# \text{ genuinely nonlinear}$$

$$\Rightarrow \text{ODE: } \begin{cases} \vec{W}'(\xi) = (\nabla \lambda_p(W(\xi)) r_p(W(\xi)))^{-1} r_p(\vec{W}(\xi)) \\ \vec{W}(\xi_1) = \vec{u}_L \quad \vec{W}(\xi_2) = \vec{u}_R \end{cases} \quad \xi_1 = \lambda_p(\vec{u}_L) < \xi_2 = \lambda_p(\vec{u}_R)$$

$$\Rightarrow \text{solve the ODE. } \vec{W} = \vec{W}(\xi) \quad (\xi_1 \leq \xi \leq \xi_2)$$

eg: The isothermal equation

(1) $p=1$. 1-rarefaction

integral curve: $\lambda_1 = \frac{m}{p} - a$. $\nabla \lambda_1 = \begin{pmatrix} -\frac{m}{p^2} \\ \frac{1}{p} \end{pmatrix}$ $r_1 = \begin{pmatrix} 1 \\ \frac{m}{p} - a \end{pmatrix}$

$$\nabla \lambda_1 \cdot r_1 = \frac{-a}{p} \neq 0$$

$$\vec{W}'(s) = \begin{pmatrix} p'(s) \\ m'(s) \end{pmatrix} = \left(\frac{-a}{p}\right)^{-1} r_1 = -\frac{p}{a} \begin{pmatrix} 1 \\ \frac{m}{p} - a \end{pmatrix} = \begin{pmatrix} -\frac{p}{a} \\ p - \frac{m}{a} \end{pmatrix}$$

$$\begin{cases} p'(s) = -\frac{1}{a} p(s) \\ m'(s) = p(s) - \frac{1}{a} m(s) \end{cases}$$

$$\hat{s} = \lambda_1(\hat{u}) = \frac{\hat{m}}{\hat{p}} - a \quad \begin{cases} p(\hat{s}) = \hat{p} \\ m(\hat{s}) = \hat{m} \end{cases}$$

$$\Rightarrow \begin{cases} p(s) = \hat{p} e^{-(s-\hat{s})/a} \\ m(s) = [\hat{p}(s-\hat{s}) + \hat{m}] e^{-(s-\hat{s})/a} \end{cases}$$

$$\text{or } m = m(p) = p \frac{\hat{m}}{\hat{p}} - a p \ln\left(\frac{p}{\hat{p}}\right)$$

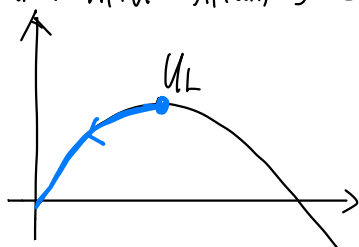
rarefaction wave:

if $\hat{u} = u_L$. $\lambda_1(\hat{u}) = \lambda_1(u_L) = \hat{s} < s$

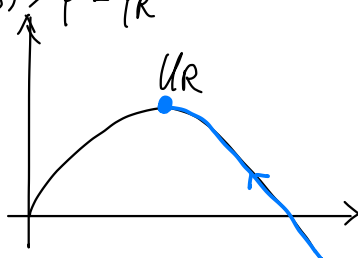
$$\Rightarrow p(s) < p(\hat{s}) = \hat{p} = p_L$$

if $\hat{u} = u_R$. $\lambda_1(\hat{u}) = \lambda_1(u_R) = \hat{s} > s$

$$\Rightarrow p(s) > \hat{p} = p_R$$



1-rarefaction



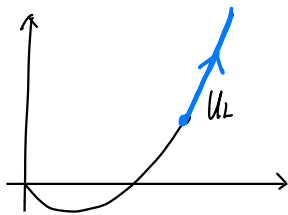
(2) $p=2$. 2-rarefaction
 $\lambda_2 = \frac{m}{p} + a$ $r_2 = \left(\frac{m}{p} + a\right) \Rightarrow \vec{W}'(\xi) = \begin{pmatrix} \frac{p}{a} \\ p + \frac{m}{a} \end{pmatrix}$

$$\Rightarrow \begin{cases} p(\xi) = \hat{p} e^{(\xi - \hat{\xi})/a} \\ m(\xi) = \hat{p}(\xi - a) e^{(\xi - \hat{\xi})/a} \end{cases} \quad \text{or} \quad m(p) = p \frac{\hat{m}}{\hat{p}} + a p \ln \frac{p}{\hat{p}}$$

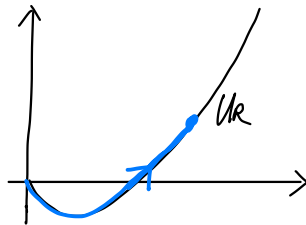
rarefaction wave:

If $\hat{u} = u_L$. $\lambda_2(u_L) = \xi_1 < \xi \Rightarrow p > p(\xi_1) = p_L$

If $\hat{u} = u_R$. $\lambda_2(u_R) = \xi_2 > \xi \Rightarrow p < p(\xi_2) = p_R$



2-rarefaction



Given u_L & u_R . find u_m . s.t. $u_L \xrightarrow{1\text{-rarefaction}} u_m \xrightarrow{2\text{-rarefaction}} u_R$

$$\begin{cases} m_m = m_L \frac{p_m}{p_L} - a p_m \ln \left(\frac{p_m}{p_L} \right) \\ m_m = m_R \frac{p_m}{p_R} + a p_m \ln \left(\frac{p_m}{p_R} \right) \end{cases}$$

$$p_m = \sqrt{p_L p_R} \exp \left[\frac{1}{2a} \left(\frac{m_L}{m_R} - \frac{m_R}{m_L} \right) \right]$$

Step 1: integral curves

Step 2: find intersection

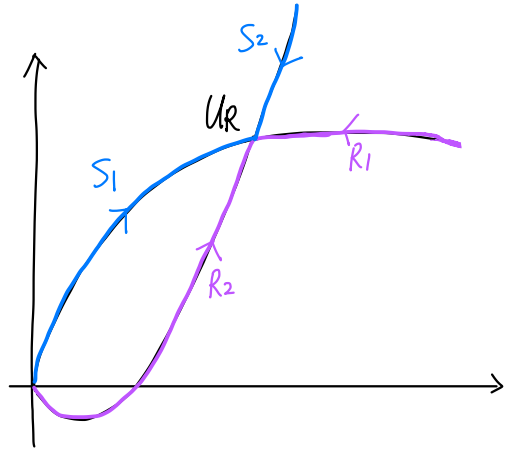
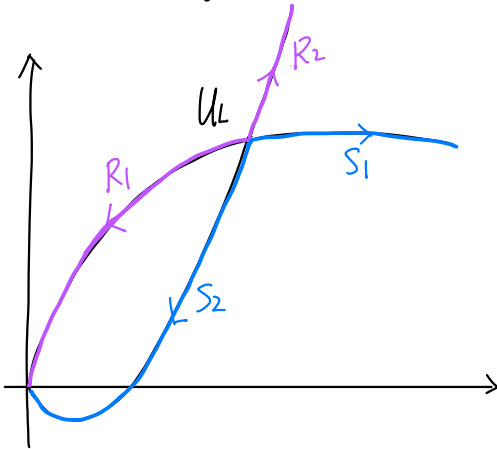
Step 3: entropy condition

General solution of the Riemann problem.

entropy-satisfying waves.

S_p : the locus of states that can be connected by a p-shock
(Hugoniot locus + entropy condition)

R_p : the locus of states that can be connected by a p-rarefaction
(Integral curves + monotonicity)



4 curves from U_L/U_R !

Hugoniot locus: (only at the start point)

$$S_p'(s)|_{s=0} = S_p'(\hat{u}) \propto r_p(\hat{u})$$

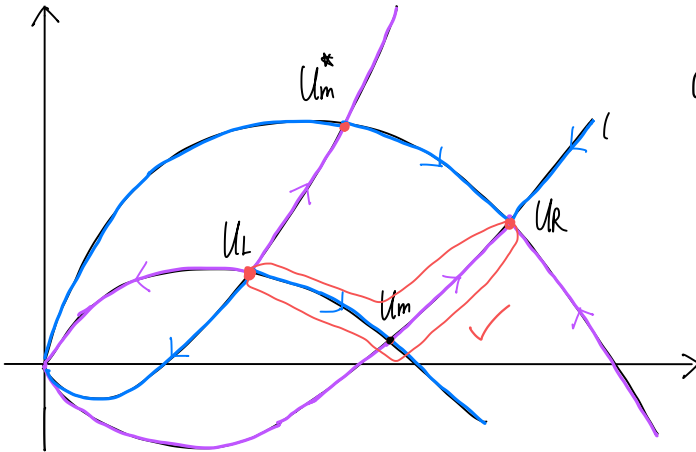
Integral curves:

$$R_p'(s) \parallel r_p(u(R_p(s))) \quad \forall s$$

$$\Rightarrow S_p'(\hat{u}) \parallel R_p'(\hat{u})$$

more over, it can be shown that the curvature of both curve is the same.
therefore, the shock S_p and the rarefaction wave R_p match up smoothly at $\hat{u}$.

eg: $u_L \xrightarrow{\text{1-wave}} u_m \xrightarrow{\text{2-wave}} u_R$. p-wave: shock S_p or rarefaction R_p



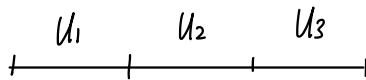
$$(1) u_L \xrightarrow{S_1(u_L)} u_m \xrightarrow{R_2(u_R)} u_R$$

$$(2) u_L \xrightarrow{R_2(u_L)} u_m^* \xrightarrow{S_1(u_R)} u_R$$

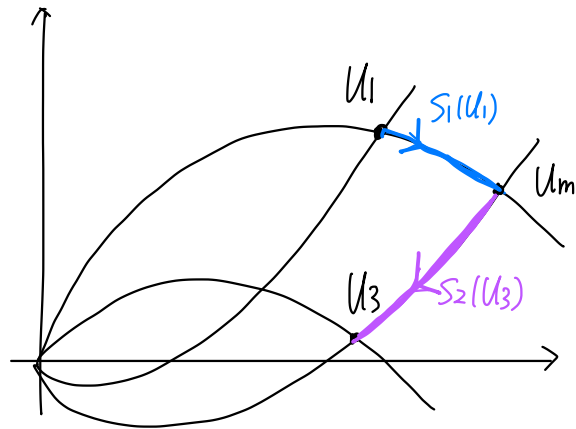
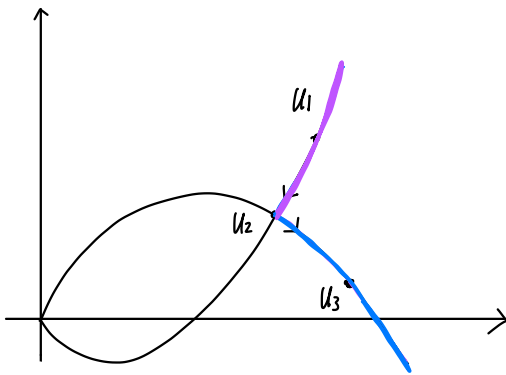
$$u_m: \begin{cases} \text{1-shock: } m_m = m_L \frac{p_m}{p_L} - a \sqrt{\frac{p_m}{p_L}} (p_m - p_L) \\ \text{2-rarewave: } m_m = m_R \frac{p_m}{p_R} + a p_m \ln\left(\frac{p_m}{p_R}\right) \end{cases}$$

eg: shock collisions

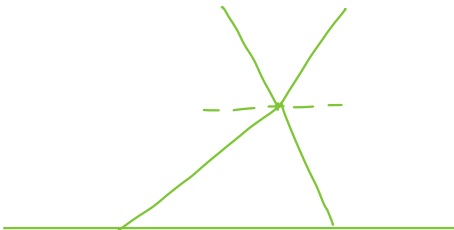
When two shocks collide the two discontinuities merge into one discontinuity that no longer satisfies the R-H condition. $\Rightarrow$ A new Riemann problem.



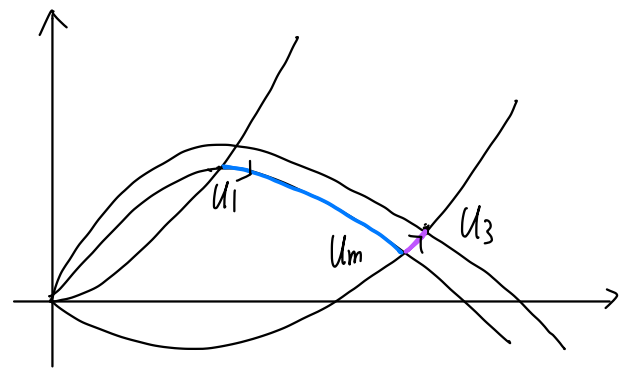
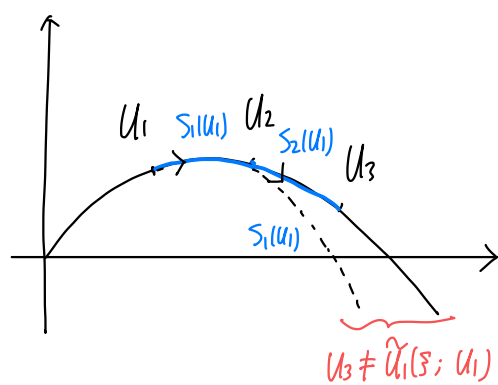
(1) $u_1 \xrightarrow{\text{2-shock}} u_2$ $u_2 \xrightarrow{\text{1-shock}} u_3$



$u_1 \xrightarrow{S_1(u_1)} u_m \xrightarrow{S_2(u_3)} u_3$



$$(2) \quad U_1 \xrightarrow{1\text{-shock}} U_2 \xrightarrow{1\text{-shock}} U_3$$



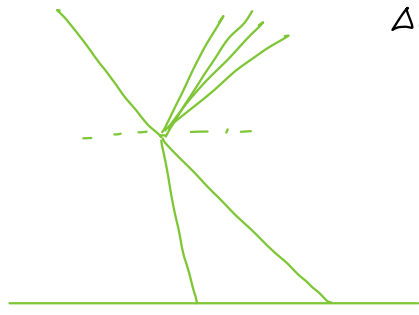
$$U_2 = \tilde{U}_1(s_1; U_1)$$

$$U_3 = \hat{U}_2(s_2; U_2)$$

$$= \tilde{U}_1((1+s_1)(1+s_2)-1; U_1) + \begin{pmatrix} 0 \\ \Delta m \end{pmatrix}$$

$$U_1 \xrightarrow{S_1(U_1)} U_m \xrightarrow{R_2(U_3)} U_3$$

$$\Delta m > 0$$



In particular:

- ① Genuinely nonlinear: $\nabla \lambda_p(u) \cdot \Gamma_p(u) \neq 0 \quad \forall u$
shock + rarefaction
- ② Nonlinear degenerate: $\nabla \lambda_p(u) \cdot \Gamma_p(u) \equiv 0 \quad \forall u$
contact discontinuity

The Riemann problem for the Euler equations

$$\lambda_1(\vec{u}) = V - C \quad \lambda_2(\vec{u}) = V \quad \lambda_3(\vec{u}) = V + C \quad C = \sqrt{\frac{\gamma p}{\rho}}$$

$$r_1(\vec{u}) = \begin{pmatrix} 1 \\ v-c \\ H-vc \end{pmatrix} \quad r_2(\vec{u}) = \begin{pmatrix} 1 \\ v \\ \frac{1}{2}v^2 \end{pmatrix} \quad r_3(\vec{u}) = \begin{pmatrix} 1 \\ v+c \\ H+vc \end{pmatrix} \quad H = \frac{E+p}{\rho}$$

$$\vec{u} = \begin{pmatrix} p \\ \rho v \\ E \end{pmatrix} \quad E = \frac{p}{\gamma-1} + \frac{1}{2} \rho v^2 \quad p = (\gamma-1) \left[E - \frac{1}{2} \rho v^2 \right] = (\gamma-1) \left[E - \frac{1}{2} \frac{(\rho v)^2}{\rho} \right]$$

it's easy to check : $\nabla f_1(\vec{u}) \cdot \vec{r}_1(\vec{u}) \neq 0$ $\nabla f_2(\vec{u}) \cdot \vec{r}_2(\vec{u}) \neq 0$ genuinely nonlinear

$$\nabla \lambda_2(\vec{u}) = \nabla\left(\frac{\rho v}{\rho}\right) = \left(-\frac{\rho v}{\rho^2}, \frac{1}{\rho}, 0\right) = \left(-\frac{v}{\rho}, \frac{1}{\rho}, 0\right)$$

$$\nabla \lambda_2(\vec{u}) \cdot \vec{u} = \left(-\frac{V}{\rho}, \frac{1}{\rho}, 0\right) \cdot \left(1, V, \frac{1}{2}V^2\right) = 0$$

Linearly degenerate

Remark: (1). $\lambda_2 = V$ is constant on the integral curves of r_2 .

$r_2(\vec{u}) = (1, V, \frac{1}{2}V^2)$ is constant along the curves. Hence, the integral curves are straight lines.

$$\text{integral curve: } \left. \begin{aligned} u(s) &= \hat{u} + \alpha(s) r_2(\hat{u}) \\ p(s) &= \hat{p} + \alpha(s) \Rightarrow \alpha(s) = p(s) - \hat{p} \end{aligned} \right\} \Rightarrow u = \hat{u} + (p - \hat{p}) r_2(\hat{u})$$

(2) R/H condition $F(u) = (pV, \frac{1}{2}pV^2 + p \cdot V(E+p))$

$$\begin{cases} pV - \hat{p}\hat{V} = s(p - \hat{p}) \\ (\frac{1}{2}pV^2 + p) - (\frac{1}{2}\hat{p}\hat{V}^2 + \hat{p}) = s(pV - \hat{p}\hat{V}) \\ V(E+p) - \hat{V}(\hat{E} + \hat{p}) = s(E - \hat{E}) \end{cases} \begin{cases} p \neq \hat{p} \\ V = \hat{V} \\ p = \hat{p} \end{cases} \text{ contact discontinuity}$$

if we take $s = V - \hat{V}$, $p = \hat{p}$

$$u - \hat{u} = \begin{pmatrix} p - \hat{p} \\ pV - \hat{p}\hat{V} \\ E - \hat{E} \end{pmatrix} = \begin{pmatrix} p - \hat{p} \\ pV - \hat{p}\hat{V} \\ \frac{p}{\gamma-1} + \frac{1}{2}pV^2 - \frac{\hat{p}}{\gamma-1} - \frac{1}{2}\hat{p}\hat{V}^2 \end{pmatrix} = \begin{pmatrix} p - \hat{p} \\ \hat{V}(p - \hat{p}) \\ \frac{1}{2}\hat{V}^2(p - \hat{p}) \end{pmatrix}$$

$$= (p - \hat{p}) \begin{pmatrix} 1 \\ \hat{V} \\ \frac{1}{2}\hat{V}^2 \end{pmatrix} = (p - \hat{p}) r_2(\hat{u})$$

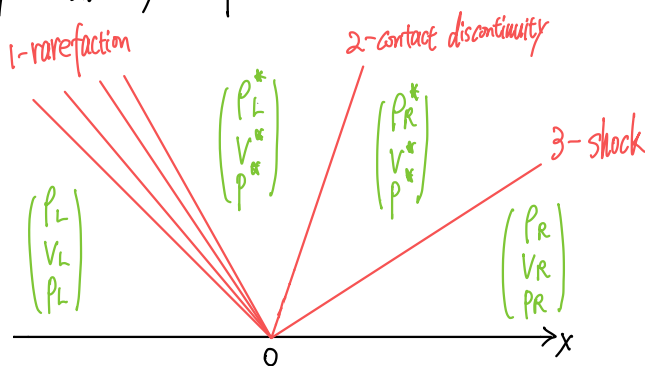
Hugoniot locus: $u = \hat{u} + (p - \hat{p}) r_2(\hat{u})$

In this case, Hugoniot locus $\equiv$ integral curves.

13) Contact discontinuity: $V_L = V_R$, $p_L = p_R$, only p jumps
 $S = V_L = V_R$, the speed of propagation is the same as the fluid velocity v .

1-family: shock / rarefaction
 2-family: contact discontinuity
 3-family: shock / rarefaction

eg:

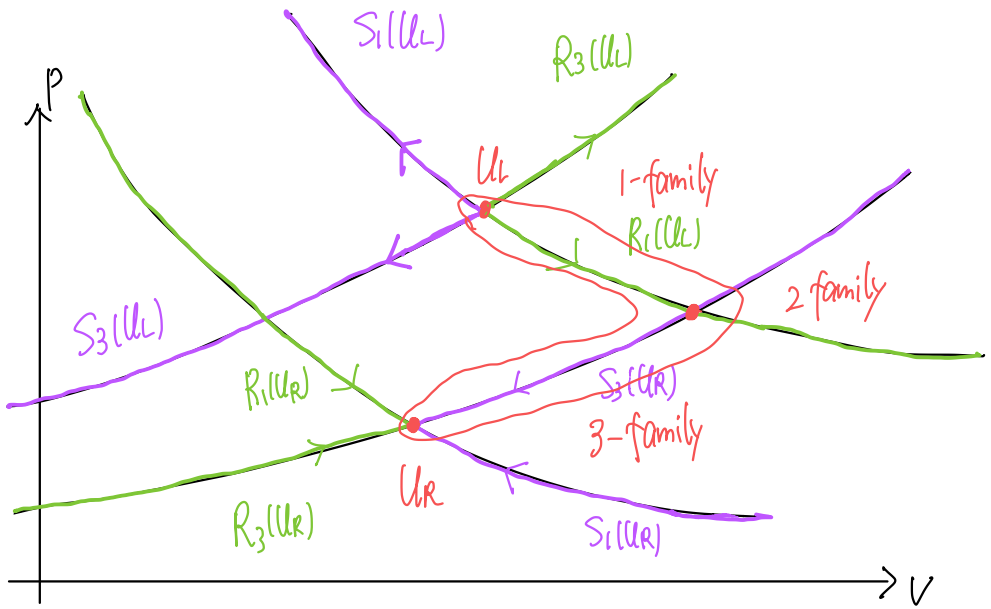


solve: $p_L^* \cdot p_R^* \cdot v^* \cdot p^* = ?$

$$\begin{pmatrix} p_L & v_L & p_L \\ p_L^* & v^* & p^* \\ p_R^* & v^* & p^* \\ p_R & v_R & p_R \end{pmatrix}$$

v - p plane.

2-family is a point in v - p plane
 Use 2-family to connect the same point in v - p plane.



Part II Numerical Method

1D scalar $\begin{cases} u_t + f(u)_x = 0 \\ u(x, 0) = u_0(x) \end{cases}$

space mesh : $h = \Delta x$, $x_j = jh$, $j = 0, \pm 1, \dots$ $x_{j+\frac{1}{2}} = (j+\frac{1}{2})h = \frac{x_j + x_{j+1}}{2}$

time step : $k = \Delta t$, $t^n = nk$, $n = 0, 1, \dots$

finite difference (FD): $u_j^n \approx u(x_j, t^n)$

finite volume (FV): $\bar{u}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n) dx$

finite element (FE): $u_j^n(x) \approx u(x, t^n)$ $x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$

eg: $\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0 \\ u(x, 0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \end{cases}$ entropy solution $u(x, t) = \begin{cases} 1 & x < \frac{t}{2} \\ 0 & x > \frac{t}{2} \end{cases}$

Scheme 1. $\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x} \left[\frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2 \right] = 0$

$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} [(u_j^n)^2 - (u_{j-1}^n)^2]$

$u_j^0 = \begin{cases} 1 & j < 0 \\ 0 & j \geq 0 \end{cases}$

detail:

$u_j^1 = u_j^0 - \frac{\Delta t}{2\Delta x} [(u_j^0)^2 - (u_{j-1}^0)^2] = \begin{cases} j < 0 & 1 - \frac{\Delta t}{2\Delta x} (1 - 1) = 1 \\ j = 0 & 0 - \frac{\Delta t}{2\Delta x} (0 - 1) = \frac{\Delta t}{2\Delta x} \\ j > 0 & 0 - \frac{\Delta t}{2\Delta x} (0 - 0) = 0 \end{cases}$ ✓

Scheme 2.
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + U_j^n \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

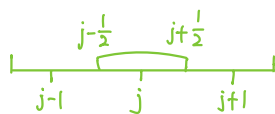
$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} U_j^n (U_j^n - U_{j-1}^n)$$

detail:

$$U_j^1 = U_j^0 - \frac{\Delta t}{\Delta x} U_j^0 (U_j^0 - U_{j-1}^0) = \begin{cases} j < 0 & 1 - \frac{\Delta t}{\Delta x} \times 1 \times (1 - 1) = 1 \\ j = 0 & 0 - \frac{\Delta t}{\Delta x} \times 0 \times (0 - 1) = 0 \\ j > 0 & 0 - \frac{\Delta t}{\Delta x} \times 0 \times (0 - 0) = 0 \end{cases}$$

$$\Rightarrow U_j^1 = U_j^0$$

The converged solution is not a weak solution.



Def: (Conservation scheme)

$$U_t + f(u)_x = 0 \Rightarrow \frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n}{\Delta x} = 0$$

where the numerical flux $\hat{f}$:

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(U_{j-p}, \dots, U_{j+q})$$

satisfies: ① consistency $\hat{f}(u \dots u) = f(u)$

② Locally Lipschitz continuity.

remark: (conservative)

$$0 = \sum_{j=-N}^N \frac{U_j^{n+1} - U_j^n}{\Delta t} + \sum_{j=-N}^N \frac{\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n}{\Delta x}$$

$$= \frac{1}{\Delta t} \frac{1}{\Delta x} \left[\sum_j U_j^{n+1} \Delta x - \sum_j U_j^n \Delta x \right] + \frac{1}{\Delta x} \left[\hat{f}_{N+\frac{1}{2}}^n - \hat{f}_{N-\frac{1}{2}}^n \right]$$

$$\Rightarrow \sum_j U_j^{n+1} \Delta x - \sum_j U_j^n \Delta x = -\Delta t [\hat{f}_R^n - \hat{f}_L^n] \Rightarrow \text{weak solution.}$$

$$\text{weak solution: } \int_{x_L}^{x_R} U(x, t^{n+1}) - U(x, t^n) dx = - \int_{t^n}^{t^{n+1}} \int_{x_L}^{x_R} f(U(x, t)) dx dt$$

A scheme in conservation form is conservative at the discrete level.

$\Rightarrow$ "RH condition" at the discrete level.



Thm: (Lax-Wendroff Theorem)

If the solution to a conservative scheme converges to a total variation bounded solution in L^1 , the converged solution is a weak solution to the conservation law.

$$\|U_j^n - U(x_j, t^n)\|_{L^1} \rightarrow 0$$

$$\text{where } TV(U^n) = \sum_j |U_j^n - U_{j-1}^n| \sim \int \left| \frac{\partial}{\partial x} U(x, t^n) \right| dx$$

eg: $u_t + f(u)_x = 0$ $f(u) = au$ ($a > 0$) $u(x,t) = u_0(x - at, 0)$

1. upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$



conservation scheme ✓

set $\hat{f}_{j+\frac{1}{2}}^n = f(u_j^n) = au_j^n$, $\hat{f}_{j-\frac{1}{2}}^n = au_{j-1}^n$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n}{\Delta x} = 0$$

2. central scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{\Delta x} = 0$$



conservation scheme ✓

set $\hat{f}_{j+\frac{1}{2}}^n = a \frac{u_j^n + u_{j+1}^n}{2} = \frac{1}{2} [f(u_j^n) + f(u_{j+1}^n)]$, $\hat{f}_{j-\frac{1}{2}}^n = a \frac{u_{j-1}^n + u_j^n}{2}$

3. LF scheme

central $u_j^n \rightarrow \frac{u_{j-1}^n + u_{j+1}^n}{2}$

conservation scheme ✓

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n)}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{\Delta x} = 0$$



$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{\frac{1}{2}(au_j^n + au_{j+1}^n) - \frac{1}{2}(au_{j-1}^n + au_j^n)}{\Delta x} + \frac{1}{\Delta t} \left[u_j^n - \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) \right] = 0$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x} \left[\underbrace{\frac{1}{2}(au_j^n + au_{j+1}^n) - \frac{\Delta x}{\Delta t}(u_{j+1}^n - u_j^n)}_{\hat{f}_{j+\frac{1}{2}}^n} - \underbrace{\frac{1}{2}(au_{j-1}^n + au_j^n) + \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)}_{\hat{f}_{j-\frac{1}{2}}^n} \right]$$

4. LW scheme

conservation scheme ✓

Taylor in time: $U(X, t+\Delta t) = U(X, t) + \Delta t \underbrace{U_t(X, t)}_{-aU_x} + \frac{\Delta t^2}{2} \underbrace{U_{tt}(X, t)}_{a^2 U_{xx}} + O(\Delta t^3)$

$$\left. \begin{aligned} U_t + aU_x &= 0 \Rightarrow U_{tt} = -aU_{xt} \\ &\Rightarrow U_{xt} = -aU_{xx} \end{aligned} \right\} \Rightarrow U_{tt} = a^2 U_{xx}$$

$$\Rightarrow U_j^{n+1} = U_j^n + \Delta t \left(-a \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right) + \frac{\Delta t^2}{2} \left(a^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \right)$$

$$\hat{f}_{j+\frac{1}{2}}^n = \frac{1}{2} (aU_{j+1}^n + aU_j^n) - \frac{\Delta t}{2\Delta x} (a^2 U_{j+1}^n - a^2 U_j^n)$$

central $\rightarrow$ upwind.

5. Beam-Warming scheme

$$U_j^{n+1} = U_j^n + \Delta t \left(-a \frac{3U_j^n - 4U_{j-1}^n + U_{j-2}^n}{2\Delta x} \right) + \frac{\Delta t^2}{2} \left(a^2 \frac{U_j^n - U_{j-1}^n + U_{j-2}^n}{\Delta x^2} \right)$$

$$\hat{f}_{j+\frac{1}{2}}^n = \frac{1}{2} (3aU_j^n - aU_{j-1}^n) - \frac{\Delta t}{\Delta x} (a^2 U_j^n - a^2 U_{j-1}^n)$$

eg: $u_t + f(u)_x = 0$ or $u_t + a(u)u_x = 0$

1. a generalization of the upwind scheme.

linear function $\hat{f}_{j+\frac{1}{2}} = \begin{cases} f(u_j) & a \geq 0 \\ f(u_{j+1}) & a < 0 \end{cases}$

nonlinear function $\hat{f}_{j+\frac{1}{2}} = \begin{cases} f(u_j) & \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} \geq 0 \\ f(u_{j+1}) & \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} < 0 \end{cases}$

if $f(u) = au$

2. LW scheme

Taylor in time: $u(x, t+\Delta t) = u(x, t) + \Delta t \underbrace{u_t(x, t)}_{-f(u)_x} + \frac{\Delta t^2}{2} \underbrace{u_{tt}(x, t)}_{(a(u)f(u)_x)_x} + O(\Delta t^3)$

$u_t = -f(u)_x \Rightarrow u_{tt} = -f(u)_{tx} = -(f(u)u_x)_x$
 $= (f(u)f(u)_x)_x = (a(u)f(u)_x)_x$

$\Rightarrow u_j^{n+1} = u_j^n + \Delta t \left(-\frac{f(u_{j+1}^n) - f(u_j^n)}{2\Delta x} \right) + \frac{\Delta t^2}{2} \left[\frac{a_{j+\frac{1}{2}}^n \frac{f(u_{j+1}^n) - f(u_j^n)}{\Delta x} - a_{j-\frac{1}{2}}^n \frac{f(u_j^n) - f(u_{j-1}^n)}{\Delta x} \right]$

$\hat{f}_{j+\frac{1}{2}}^n = \frac{f(u_j^n) + f(u_{j+1}^n)}{2} - \frac{\Delta t}{2\Delta x} a_{j+\frac{1}{2}}^n (f(u_{j+1}^n) - f(u_j^n))$

where $a_{j+\frac{1}{2}}^n = \frac{1}{2}(a(u_j^n) + a(u_{j+1}^n))$

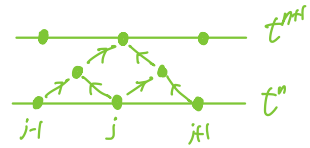
How to avoid $a(u)$?

$f(u) = au$
 LW scheme $\xrightarrow{\text{avoid arcs}}$ $\left\{ \begin{array}{l} \text{Richtmyer LW} \\ \text{MacCormack's scheme} \end{array} \right.$

3. Richtmyer two-step LW scheme

$$\begin{cases}
 u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{u_j^n + u_{j+1}^n}{2} - \frac{\Delta t}{2\Delta x} [f(u_j^n) + f(u_{j+1}^n)] \\
 u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} [f(u_{j+\frac{1}{2}}^{n+\frac{1}{2}}) - f(u_{j-\frac{1}{2}}^{n+\frac{1}{2}})]
 \end{cases}$$

$\Downarrow$
 f



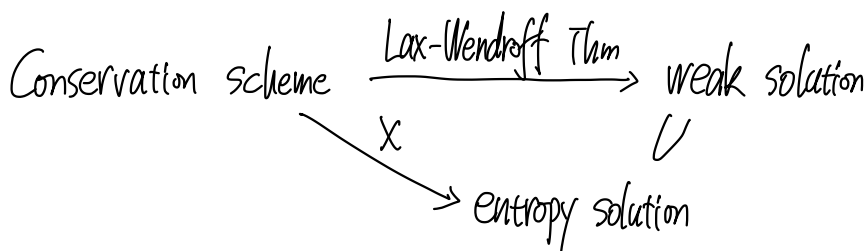
4. MacCormack's scheme

$$\begin{cases}
 u_j^* = u_j^n - \frac{\Delta t}{\Delta x} [f(u_{j+1}^n) - f(u_j^n)] \\
 u_j^{n+1} = \frac{u_j^n + u_j^*}{2} - \frac{\Delta t}{2\Delta x} [f(u_j^*) - f(u_{j-1}^n)]
 \end{cases}$$

$$u_j^n \xrightarrow{\text{right}} u_j^* \xrightarrow{\text{left}} u_j^{n+1}$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} [f(u_{j+1}^n) - f(u_j^n)] - \frac{\Delta t}{2\Delta x} [f(u_j^*) - f(u_{j-1}^n)]$$

$$\hat{f}_{j+\frac{1}{2}}^n = \frac{1}{2} [f(u_j^*) + f(u_{j+1}^n)]$$

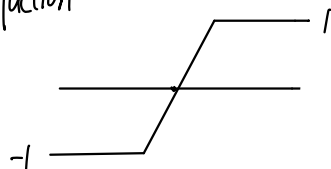


eg: Conservation scheme may not get an entropy solution.

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0 \end{cases}$$

$$u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

entropy solution: rarefaction



use the generalized upwind scheme.

$$\hat{f}_{j+\frac{1}{2}} = \begin{cases} f(u_j) & \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} \geq 0 \\ f(u_{j+1}) & \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} < 0 \end{cases}$$

$$u_j^0 = \begin{cases} -1 & j \leq 0 \\ 1 & j > 0 \end{cases} \quad f(u_j^0) = \frac{1}{2}(u_j^0)^2 = \frac{1}{2} \quad \hat{f}_{j+\frac{1}{2}}^0 = f(u_j^0) = \frac{1}{2}$$

$$\Rightarrow u_j^1 = u_j^0 - \frac{\Delta t}{\Delta x} [\hat{f}_{j+\frac{1}{2}}^0 - \hat{f}_{j-\frac{1}{2}}^0] = u_j^0 \quad \Rightarrow u_j^1 = u_j^0 = \begin{cases} -1 & j \leq 0 \\ 1 & j > 0 \end{cases} \quad \forall n$$

weak solution ✓ entropy solution ✗

def (Entropy Condition)

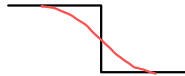
$$\eta(u_j^{n+1}) \leq \eta(u_j^n) - \frac{\Delta t}{\Delta x} [\Phi_{j+\frac{1}{2}}^n - \Phi_{j-\frac{1}{2}}^n]$$

Where $\Phi_{j+\frac{1}{2}}^n = \Phi(u_{j-p}^n, \dots, u_{j+q}^n)$ is the numerical entropy flux function that must be consistent with $f(u)$.

eg:
$$\begin{cases} u_t + u_x = 0 \\ u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \end{cases}$$

Smearred solution:

LF scheme:

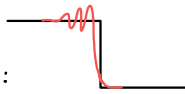


upwind scheme:

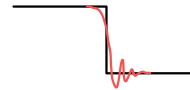


Oscillation:

LW scheme:



Beam-Warming:



Finite volume method

Godunov's method

$$\text{exact} \quad \left\{ \begin{array}{l} \bar{u}_j^n = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n) dx \\ F_{j+\frac{1}{2}}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) dt \end{array} \right.$$

$$u_t + f(u)_x = 0 \quad [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$$

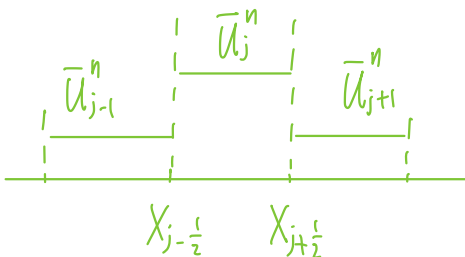
$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^{n+1}) dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n) dx - \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) dt + \int_{t^n}^{t^{n+1}} f(u(x_{j-\frac{1}{2}}, t)) dt$$

$$\Rightarrow \Delta x \bar{u}_j^{n+1} = \Delta x \bar{u}_j^n - \Delta t F_{j+\frac{1}{2}}^n + \Delta t F_{j-\frac{1}{2}}^n$$

$$\frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0$$

set $\hat{f}_{j+\frac{1}{2}}^n \approx F_{j+\frac{1}{2}}^n$, numerical flux.

$$\frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} + \frac{\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n}{\Delta x} = 0 \quad (\text{FV})$$

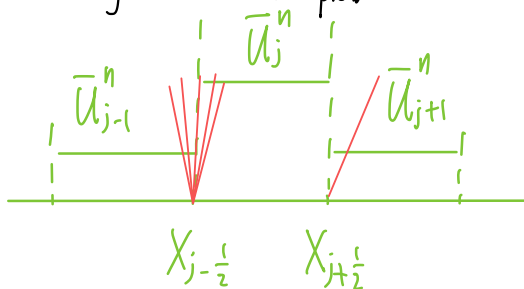


Given $\{\bar{u}_j^n\}$ piecewise constant. Use it as the initial data for the conservation law. We can solve the equation exactly over a short time interval
 $\Rightarrow$ a sequence of Riemann problem.

$u(x_{j+\frac{1}{2}}, t)$: solution of Riemann problem for short time. $\begin{cases} u_L = \bar{u}_j^n \\ u_R = \bar{u}_{j+1}^n \end{cases} \Rightarrow u(x_{j+\frac{1}{2}}, t)$

if $C\Delta t \leq \frac{1}{2}\Delta x$, $C = \max_u |f'(u)|$

the waves from two adjacent Riemann problem have not yet start to interact.



$$\left. \begin{array}{l} u_L = \bar{u}_j^n \\ u_R = \bar{u}_{j+1}^n \end{array} \right\} \Rightarrow u(x_{j+\frac{1}{2}}, t) = u_{j+\frac{1}{2}}^* \quad \hat{f}_{j+\frac{1}{2}}^n = f(u_{j+\frac{1}{2}}^*) = \hat{f}(\bar{u}_j, \bar{u}_{j+1})$$

(self-similarity)

1. consistency: if $u_L = u_R$, $u_{j+\frac{1}{2}}^* = \bar{u}_j = \bar{u}_{j+1}$.

2. Lipschitz continuity follows from the smoothness of f .

Thm: The weak solution obtained by Godunov's method satisfy the entropy condition, provided we use the entropy-satisfying Riemann solution.

Proof: let $u^e(x, t)$ be the piecewise constant solution satisfying the entropy inequality

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(u^e(x, t^{n+1})) dx \leq \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(u^e(x, t^n)) dx - \int_{t^n}^{t^{n+1}} [\xi(u^e(x_{j+\frac{1}{2}}, t)) - \xi(u^e(x_{j-\frac{1}{2}}, t))] dt$$

Since $\eta(u)$ is convex:

$$\eta(\bar{u}_j^{n+1}) = \eta\left(\frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u^e(x, t^{n+1}) dx\right) \leq \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \eta(u^e(x, t^{n+1})) dx$$

$$\eta(\bar{u}_j^{n+1}) \leq \eta(\bar{u}_j^n) - \frac{\Delta t}{\Delta x} [\Psi_{j+\frac{1}{2}}^n - \Psi_{j-\frac{1}{2}}^n] \quad (\{u_j^n\} \rightarrow u^e \xrightarrow{\text{average}} \{u_j^{n+1}\})$$

#

How to solve the Riemann problem in Godunov method?

For scalar case $u_t + f(u)_x = 0$

$$\hat{f}(u_L, u_R) = \begin{cases} \min_{[u_L, u_R]} f(u) & u_L \leq u_R \\ \max_{[u_R, u_L]} f(u) & u_L > u_R \end{cases} \quad (\text{Godunov flux})$$

The numerical solutions converge to the entropy weak solution.

Hw. Godunov's method
Burger's equation

ddl: 4.27 report.

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x = 0 & x \in [-\pi, \pi] \\ u(x, 0) = \frac{1}{2} + \sin x & (T_b = 1) \end{cases}$$

2π -periodic boundary condition. $\alpha^n = \max_u |f(u)|$ $\alpha^n \Delta t = \frac{1}{2} \Delta x$. update Δt^n !

(1) $t = 0.5$ smooth. error table (L^2, L^∞)

$$L^2 \text{ error: } \sqrt{\sum_j |u(x_j, t^n) - \bar{u}_j^n|^2 \Delta x}$$

$$L^\infty \text{ error: } \max_j |u(x_j, t^n) - \bar{u}_j^n| \quad \text{error order: } \frac{\ln \left| \frac{\text{error}(\Delta x)}{\text{error}(\Delta x/2)} \right|}{\ln 2}$$

mesh: $m = 10, 20, 40 \dots 320$.

(2) $t = 1.5$ plot and compare. $\bar{u}_j^n$ Vs $u(x_j, t^n)$ at x_j

mesh: $m = 20, m = 80$



Hw. FV scheme (3rd-order)
(Oscillatory)

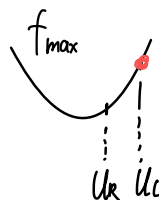
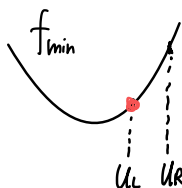
note: time: 3rd Runge-Kutta. ddl 5.11

stencils: $S_j = \{I_{j-1}, I_j, I_{j+1}\}$

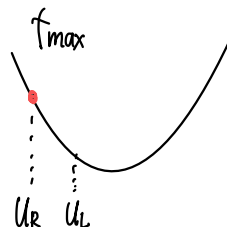
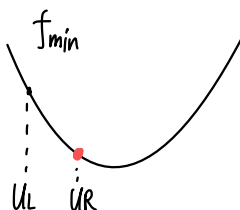
flux: Godunov flux

Remark: In convex case $f'' \geq 0$ $f' \nearrow$

① $f'(u_L) > 0$ $f'(u_R) > 0$ $u^* = u_L$

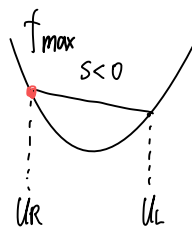
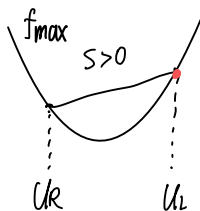


② $f'(u_L) < 0$ $f'(u_R) < 0$ $u^* = u_R$

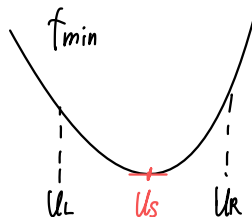


③ $f'(u_L) \geq 0 \geq f'(u_R) \xrightarrow{\text{shock}} u^* = \begin{cases} u_L & s > 0 \\ u_R & s < 0 \end{cases}$
 $(u_R \leq u_L)$

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$



④ $f'(u_R) \geq 0 \geq f'(u_L) \xrightarrow{\text{rarefaction}} u^* = u_s \text{ where } f'(u_s) = 0$
 $(u_L \leq u_R)$



Remark :

(1) for linear case $u_t + a u_x = 0$

$$a > 0: \quad \hat{f}(u_L, u_R) = \begin{cases} a u_L & u_L \leq u_R \\ a u_R & u_L > u_R \end{cases}$$

$$a < 0: \quad \hat{f}(u_L, u_R) = \begin{cases} a u_R & u_L \leq u_R \\ a u_L & u_L > u_R \end{cases}$$

Godunov method $\Rightarrow$ upwind.

(2) for nonlinear case $u_t + f(u)_x = 0$ $f'(u)$ doesn't change sign.

Godunov method $\Rightarrow$ upwind.

For systems. $\vec{u}_t + \vec{f}(\vec{u})_x = 0$

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \vec{u}(x, t^{n+1}) dx = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \vec{u}(x, t^n) dx - \underbrace{\int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j+\frac{1}{2}}, t)) dt + \int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j-\frac{1}{2}}, t)) dt}_{\text{exact Riemann solver}}$$

exact Riemann solver.

Exact Solution :

(1) Linear system : $\vec{f}(\vec{u}) = A\vec{u}$

(2) $\forall u. \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m < 0 \quad \vec{u}(x_{j+\frac{1}{2}}, t) = \vec{u}_{j+1}^n$

(3) $\forall u. 0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m. \quad \vec{u}(x_{j+\frac{1}{2}}, t) = u_j^n$

General nonlinear system ? $\Rightarrow$ Approximate Riemann solver

Approximate Riemann Solver. (system)

(1) Roe flux

$$u_t + f(u)_x = 0 \Rightarrow u_t + A(u)u_x = 0 \quad (A(u) = R(u)\Lambda(u)R(u)^{-1})$$

Linearization.

$$f(u) \rightarrow f^*(u) = A^*u \quad A^* \text{ depends on } u_L \& u_R$$

choice of $A^* \approx A(u) = \nabla f(u)$

condition : (1) A^* is diagonalizable with real eigenvalues.

(2) $A^*(u_L, u_R) \rightarrow A(\bar{u})$ smoothly as $u_L \& u_R \rightarrow \bar{u}$

(3) $A^*(u_R, u_L) = f(u_R) - f(u_L)$

$\downarrow$
Roe matrix

$$(3) \Rightarrow \int_{-M}^M u(x,t) dx = M(u_L + u_R) + [f(u_L) - f(u_R)]t = M(u_L + u_R) + A^*(u_L - u_R)t$$

$u_t + f(u)_x = 0$
 $u_t + A^* u_x = 0$

conservation $\Rightarrow$ In the special case where u_L & u_R are connected by a single shock wave or contact discontinuity, the approximation Riemann solution agrees with the exact Riemann solution.

try: $A^* = \frac{A(u_L) + A(u_R)}{2}$ (1) X

$A^* = A\left(\frac{u_L + u_R}{2}\right)$ (1) (2) $\checkmark$ (3) X

Roe matrix A^* : $A^*(u_L - u_R) = f(u_L) - f(u_R)$

Provided a system of conservation law has a convex entropy.

a Roe matrix with purely real eigenvalues exists.

Roe flux: A^* is diagonalizable: $A^* r_p = \lambda_p r_p$ $u_L - u_R = \sum_{p=1}^m \alpha_p r_p$

denote: $\lambda_p^+ = \max(\lambda_p, 0)$, $\lambda_p^- = \min(\lambda_p, 0)$

$$\begin{cases} u_t + A^* u = 0 \\ u = \begin{cases} u_L \\ u_R \end{cases} \end{cases} \Rightarrow \hat{f}(u_L, u_R) = f(u_L) - \sum_{p=1}^m \lambda_p^+ \alpha_p r_p = f(u_R) + \sum_{p=1}^m \lambda_p^- \alpha_p r_p$$

or $\hat{f}(u_L, u_R) = \frac{1}{2}(f(u_L) + f(u_R)) - \frac{1}{2}|A^*(u_L - u_R)|$

note: $A^* = S^{-1} \Lambda S$ $|\Lambda| = \text{diag}(|\lambda_i|)$ $|A^*| = S^{-1} |\Lambda| S$

Remark: for scalar case: $A^* \in \mathbb{R}$. $A^* = \frac{f(u_L) - f(u_R)}{u_L - u_R} = s$ $|A^*| = |s|$
 $s > 0$: $\hat{f}(u_L, u_R) = \frac{1}{2}(f(u_L) + f(u_R)) - \frac{1}{2}(f(u_R) - f(u_L)) = f(u_L)$ } shock
 $s < 0$: $\hat{f}(u_L, u_R) = f(u_R)$

error when rarefaction wave: $f'(u_L) < 0 < f'(u_R)$, It's also true for a system.

Roe flux: The Riemann problem is dominated by a single shock wave.

Enquist - Osher flux: (Assume the solution is a rarefaction wave)

$$f(u) = f^+(u) + f^-(u)$$

$$\text{Jacobi: } A(u) = A^+(u) + A^-(u) \quad A^+(u) = \nabla f^+(u) \geq 0 \quad \text{i.e. } \lambda_p(A^+) \geq 0$$

$$A^-(u) = \nabla f^-(u) \leq 0 \quad \text{i.e. } \lambda_p(A^-) \leq 0$$

$$f^+(u_R) - f^+(u_L) = \int_{u_L}^{u_R} A^+(u) du \quad \hat{f}(u_L, u_R) = f(u_L) - \int_{u_L}^{u_R} A^+(u) du = f(u_R) + \int_{u_L}^{u_R} A^-(u) du$$

$$\text{or: } \hat{f}(u_L, u_R) = \frac{1}{2}(f(u_L) + f(u_R)) - \frac{1}{2} \int_{u_L}^{u_R} |A(u)| du$$

Remark: for scalar case: $\begin{cases} f^+(u) = \frac{1}{2}f(0) + \int_0^u \max(f'(u), 0) du \\ f^-(u) = \frac{1}{2}f(0) + \int_0^u \min(f'(u), 0) du \end{cases}$

$$\hat{f}(u_L, u_R) = f(0) + \int_0^{u_L} \max(f'(u), 0) du + \int_0^{u_R} \min(f'(u), 0) du$$

if $f'(u)$ does not change sign, it's equivalent with Roe flux and Godunov flux (upwind)

if $f(u_L) < 0 < f(u_R)$, $f'(u_s) = 0 \Rightarrow f(u_s)$. (rarefaction)

$$\text{Osher flux: } \hat{f} = \frac{1}{2}(f(u_L) + f(u_R)) - \frac{1}{2} \left[- \int_{u_L}^{u_s} f'(v) dv + \int_{u_s}^{u_R} f'(v) dv \right] = f(u_s)$$

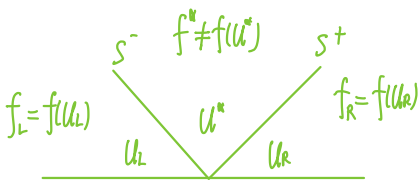
if $f(u_L) > 0 > f(u_R)$. (shock wave)

$$\text{Osher flux: } \hat{f} = \frac{1}{2}(f(u_L) + f(u_R)) - \frac{1}{2} \left[\int_{u_L}^{u_s} f'(v) dv - \int_{u_s}^{u_R} f'(v) dv \right] = f(u_L) + f(u_R) - f(u_s)$$

(consistence $\checkmark$. $u_L, u_R \rightarrow \bar{u} \Rightarrow u_s \rightarrow \bar{u}$)

The flux remains consistent and does not lead to entropy violating solutions.

Harten-Lax-vanleer flux (HLL flux) (Assume the solution contains two shocks)



$$\hat{f}(u_L, u_R) = \begin{cases} f(u_L) & s^- \geq 0 \\ f^* & s^- < 0 < s^+ \\ f(u_R) & s^+ \leq 0 \end{cases}$$

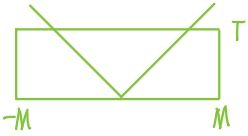
$$f^* - f_L = s^-(u^* - u_L)$$

$$f_R - f^* = s^+(u_R - u^*)$$

$$\Rightarrow f^* = \frac{s^+ f_L - s^- f_R + s^+ s^- (u_R - u_L)}{s^+ - s^-}$$



u^* can be obtained by conservation law on integral form over $[-M, M] \times [0, T]$ $M \geq T \cdot \max |s^\pm|$



$$u_L(s^-T + M) + u^*(s^+T - s^-T) + u_R(M - s^+T) \\ = (u_L + u_R)M + f(u_L)T - f(u_R)T$$

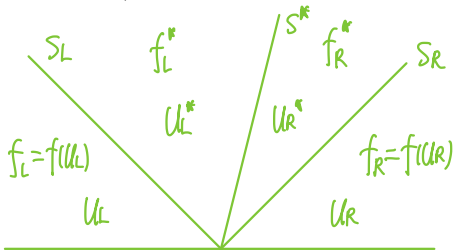
$$\Rightarrow u^* = \frac{s^+u_R - s^-u_L + f(u_L) - f(u_R)}{s^+ - s^-}$$

$\swarrow$
 $\searrow$

$s^- = \min(\lambda_1(u_L), \lambda_1(u^*))$
 $s^+ = \max(\lambda_1(u_R), \lambda_1(u^*))$

so we take $\begin{cases} s^- \leftarrow \min(\lambda_1(u_L), \lambda_1(u_R)) \\ s^+ \leftarrow \max(\lambda_1(u_L), \lambda_1(u_R)) \end{cases}$

HLLC flux (HLL + contact discontinuity) for Euler equation



$$s_L < s^* < s_R$$

$$\hat{f} = \begin{cases} f(u_L) & 0 < s_L \\ f_L^* & s_L < 0 < s^* \\ f_R^* & s^* < 0 < s_R \\ f(u_R) & s_R < 0 \end{cases}$$

Conservation : $\frac{S^* - S_L}{S_R - S_L} u_L^* + \frac{S_R - S^*}{S_R - S_L} u_R^* = U^{HLL}$

jump condition :

$$\begin{aligned} f_L^* - f(u_L) &= S_L(u_L^* - u_L) \\ f_R^* - f_L^* &= S^*(u_R^* - u_L^*) \\ f(u_R) - f_R^* &= S_R(u_R - u_R^*) \end{aligned}$$

contact discontinuity : u_L^*, u_R^*, S^*

Lax-Friedrichs flux (LF).

In HLL flux. if we take $\begin{cases} S^+ = \max\{|\lambda_i(u)|\} \\ S^- = -\max\{|\lambda_i(u)|\} \end{cases} \quad S^+ = -S^- = \alpha$

$\hat{f}(u_L, u_R) = f^* = \frac{f(u_L) + f(u_R)}{2} - \frac{\alpha}{2}(u_R - u_L)$

Remark : $\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{2\Delta x} [f(\bar{u}_{j+1}^n) + f(\bar{u}_{j-1}^n)] + \frac{\alpha \Delta t}{2\Delta x} [\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n]$
 stable if $\alpha \frac{\Delta t}{\Delta x} \leq 1$ $\approx u_{xx}$ viscosity

Remark : global LF : $\alpha = \max_u |\lambda(u)|$

local LF : $\alpha = \max |\lambda(u)|$ u between u_L and u_R

Remark: for scalar case, global LF flux: $\frac{\Delta t}{\Delta x} \max_u |f'(u)| \leq 1$

But in Godunov flux: $\frac{\Delta t}{\Delta x} \max_u |f'(u)| \leq \frac{1}{2}$.

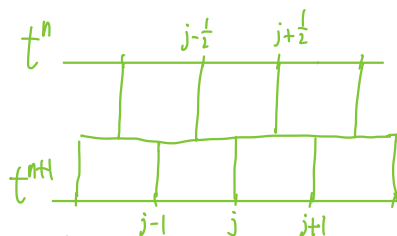
If we take $\frac{\Delta t}{\Delta x} \max_u |f'(u)| \leq 1$, two shocks may combine, but the new shock can't cross $X_{j+\frac{1}{2}}$.

Frame: Godunov's theme + Riemann solver $\left\{ \begin{array}{l} \text{Godunov flux} \\ \dots \\ \text{LF flux} \end{array} \right.$ error: $\left\{ \begin{array}{l} \text{average } \{\bar{u}_j^n\} \\ \text{Riemann solver} \end{array} \right.$

central theme: (avoid Riemann Solve)

t^n cell: $[X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}] \rightarrow \bar{u}_j^n$

$\downarrow$
 t^{n+1} cell: $[X_j, X_{j+1}] \rightarrow \bar{u}_{j+\frac{1}{2}}^{n+1}$



$$\bar{u}_{j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x} \int_{X_j}^{X_{j+1}} u(x, t^{n+1}) dx = \frac{1}{\Delta x} \int_{X_j}^{X_{j+1}} u(x, t^n) dx + \underbrace{\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} [f(u(x_j, t)) - f(u(x_{j+1}, t))] dt}_{\begin{array}{l} \text{average} \\ \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} [f(\bar{u}_j^n) - f(\bar{u}_{j+1}^n)] dt \\ \parallel \\ f(\bar{u}_j^n) - f(\bar{u}_{j+1}^n) \end{array}}$$

Godunov type scheme:

$$(exact) \quad \bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} [F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n] \quad F_{j+\frac{1}{2}}^n = \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t)) dt \approx \hat{f}_{j+\frac{1}{2}}^n$$

$$(numerical) \quad \bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} [\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n]$$

First order: $u_{\Delta x}(x, t^0) = \bar{u}_j^0 = u(x_j, t^0) + O(\Delta x) \quad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \quad \text{step function}$



→ solve the Riemann problem exactly / numerically

high order scheme: $u_{\Delta x}(x, t^0) = u(x, t^0) + O(\Delta x^k) \quad \text{piecewise polynomial} \Rightarrow DG$

→ solve the generalized Riemann problem

Another way: High order FV method

$$u_t + f(u)_x = 0.$$

integral on $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$: $\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx + f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) = 0$



$$\frac{d}{dt} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t) dx + f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) = 0$$



(exact) $\frac{d}{dt} \bar{u}_j(t) + \frac{1}{\Delta x} [f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))] = 0$

Numerical Scheme :

$$\frac{d}{dt} \bar{u}_j(t) + \frac{1}{\Delta x} [\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}] = 0. \quad (\text{semi-discrete})$$

$$\text{numerical flux: } \hat{f}_{j+\frac{1}{2}} = \hat{f}(\bar{u}_{j-p}, \dots, \bar{u}_{j+q}) \approx f(u(x_{j+\frac{1}{2}}, t))$$

$$\Rightarrow \text{ODE system: } \frac{d}{dt} \bar{u}_j(t) = L(\bar{u}(t)) = -\frac{1}{\Delta x} (\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}})$$

Solve the ODE systems (fully discrete)

method of lines
MOL

First order Euler forward :

$$\bar{u}_j^{n+1} = \bar{u}_j^n + \Delta t L(\bar{u}^n) = \bar{u}_j^n - \frac{\Delta t}{\Delta x} (\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n)$$

if we take $\hat{f}_{j+\frac{1}{2}}^n = \hat{f}^{God}(\bar{u}_j^n, \bar{u}_{j+1}^n)$. $\Rightarrow$ Godunov scheme.

Third order SSP Runge-Kutta scheme :

$$\bar{u}_j^{(1)} = \bar{u}_j^n + \Delta t L(\bar{u}_j^n)$$

$$\bar{u}_j^{(2)} = \frac{3}{4} \bar{u}_j^n + \frac{1}{4} (\bar{u}_j^{(1)} + \Delta t L(\bar{u}_j^{(1)}))$$

$$\bar{u}_j^{n+1} = \frac{1}{3} \bar{u}_j^n + \frac{2}{3} (\bar{u}_j^{(2)} + \Delta t L(\bar{u}_j^{(2)}))$$

Frame :

$$\{\bar{u}_j^n\} \xrightarrow{\text{Reconstruction}} u_{\Delta x}(x, t^n) \xrightarrow{\text{flux}} \hat{f}_{j+\frac{1}{2}}^n \longrightarrow \{\bar{u}_j^{n+1}\}$$

$$\parallel$$

$$u(x, t) + O(\Delta x^{k_H})$$

Reconstruction :

$$\{\bar{u}_j\} \longrightarrow \{p_j(x)\}$$

$p_j(x)$ is a polynomial of degree $\leq k$. defined on $I_j = [X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}]$.

Satisfy :

$$(1) \frac{1}{\Delta x} \int_{I_j} p_j(x) dx = \bar{u}_j = \frac{1}{\Delta x} \int_{I_j} u(x, t) dx$$

$$(2) |p_j(x) - u(x, t)| \leq C \Delta x^{k_H} \quad \text{in } I_j$$

choose a stencil $S_j = \{I_{j-p}, \dots, I_{j+q}\}$ $\#S_j = k+1$ (i.e. $p+q=k$, $p, q \geq 0$, not unique)

and require :

($k+1$ possible stencils)

$$(*) \quad \frac{1}{\Delta x} \int_{I_\ell} p_j(x) dx = \bar{u}_\ell \quad , \quad \ell = j-p, \dots, j+q.$$

Remark: if u is smooth in S_j . then $(*) \Rightarrow |p_j(x) - u(x, t)| \leq C \Delta x^{k_H}$

Look at the primitive function $U(x) = \int_{-\infty}^x u(s) ds$

$$U(X_{j+\frac{1}{2}}) = \sum_{\ell=-\infty}^j \int_{X_{\ell-\frac{1}{2}}}^{X_{\ell+\frac{1}{2}}} u(s) ds = \sum_{\ell=-\infty}^j \Delta x \bar{u}_\ell$$

we have the point values of $U(x)$ at the cell boundaries.

suppose $P(x)$ interpolates $U(x)$ at $X_{j-p-\frac{1}{2}}, \dots, X_{j+q+\frac{1}{2}}$. ($k+2$ points) $\deg(P(x)) \leq k+1$.

$$\begin{aligned} \frac{1}{\Delta x} \int_{I_\ell} P'(x) dx &= \frac{1}{\Delta x} [P(X_{\ell+\frac{1}{2}}) - P(X_{\ell-\frac{1}{2}})] \\ &= \frac{1}{\Delta x} [U(X_{\ell+\frac{1}{2}}) - U(X_{\ell-\frac{1}{2}})] = \frac{1}{\Delta x} \int_{X_{\ell-\frac{1}{2}}}^{X_{\ell+\frac{1}{2}}} U(s) ds = \bar{U}_\ell \quad \ell = j-p, \dots, j+q. \end{aligned}$$

take $p_j(x) = P'(x) = [P(x) - U(X_{j-p-\frac{1}{2}})]'$

$$P(x) = \sum_{\ell=-p-1}^q U(X_{j+\ell+\frac{1}{2}}) L_\ell(x) \quad \left(\sum_{\ell} L_\ell(x) = 1 \right)$$

$$P(x) - U(X_{j-p-\frac{1}{2}}) = \sum_{\ell=-p-1}^q [U(X_{j+\ell+\frac{1}{2}}) - U(X_{j-p-\frac{1}{2}})] L_\ell(x) = \sum_{\ell=-p-1}^q \left[\Delta x \sum_{m=j-p}^{j+\ell} \bar{U}_m \right] L_\ell(x)$$

$$p_j(x) = [P(x) - U(X_{j-p-\frac{1}{2}})]' = \sum_{\ell=-p-1}^q \left[\Delta x \sum_{m=j-p}^{j+\ell} \bar{U}_m \right] L_\ell'(x)$$

Then. take $U_{j+\frac{1}{2}} \approx P_j(X_{j+\frac{1}{2}}) = \sum_{\ell=-p}^q a_\ell \bar{U}_{j+\ell}$ $\{a_\ell\} \{b_\ell\}$ constant

$$U_{j-\frac{1}{2}} \approx P_j(X_{j-\frac{1}{2}}) = \sum_{\ell=-p}^q b_\ell \bar{U}_{j+\ell}$$

eg: $k=1$. $p+q=k$.

$$\Rightarrow \begin{cases} p=1, q=0 & S_j = \{I_{j-1}, I_j\} \Rightarrow \begin{cases} U_{j+\frac{1}{2}} = -\frac{1}{2}\bar{U}_{j-1} + \frac{3}{2}\bar{U}_j \\ U_{j-\frac{1}{2}} = \frac{1}{2}\bar{U}_{j-1} + \frac{1}{2}\bar{U}_j \end{cases} \\ p=0, q=1 & S_j = \{I_j, I_{j+1}\} \Rightarrow \begin{cases} U_{j+\frac{1}{2}} = \frac{1}{2}\bar{U}_j + \frac{1}{2}\bar{U}_{j+1} \\ U_{j-\frac{1}{2}} = \frac{3}{2}\bar{U}_j - \frac{1}{2}\bar{U}_{j+1} \end{cases} \end{cases}$$

$k=2$. $p+q=k$.

$$\Rightarrow \begin{cases} p=2, q=0 & S_j = \{I_{j-2}, I_{j-1}, I_j\} \Rightarrow \begin{cases} U_{j+\frac{1}{2}} = \frac{1}{3}\bar{U}_{j-2} - \frac{7}{6}\bar{U}_{j-1} + \frac{11}{6}\bar{U}_j \\ U_{j-\frac{1}{2}} = -\frac{1}{6}\bar{U}_{j-2} + \frac{5}{6}\bar{U}_{j-1} + \frac{1}{3}\bar{U}_j \end{cases} \\ p=1, q=1 & S_j = \{I_{j-1}, I_j, I_{j+1}\} \Rightarrow \begin{cases} U_{j+\frac{1}{2}} = -\frac{1}{6}\bar{U}_{j-1} + \frac{5}{6}\bar{U}_j + \frac{1}{3}\bar{U}_{j+1} \\ U_{j-\frac{1}{2}} = \frac{1}{3}\bar{U}_{j-1} + \frac{5}{6}\bar{U}_j - \frac{1}{6}\bar{U}_{j+1} \end{cases} \\ p=0, q=2 & S_j = \{I_j, I_{j+1}, I_{j+2}\} \Rightarrow \begin{cases} U_{j+\frac{1}{2}} = \frac{1}{3}\bar{U}_j + \frac{5}{6}\bar{U}_{j+1} - \frac{1}{6}\bar{U}_{j+2} \\ U_{j-\frac{1}{2}} = \frac{11}{6}\bar{U}_j - \frac{7}{6}\bar{U}_{j+1} + \frac{1}{3}\bar{U}_{j+2} \end{cases} \end{cases}$$

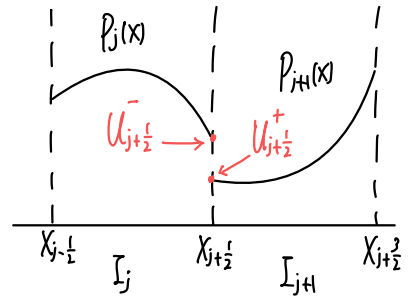
numerical flux $\hat{f}_{j+\frac{1}{2}} \approx f(u(X_{j+\frac{1}{2}}, t))$

eg: Fix $S_j = \{I_{j-1}, I_j, I_{j+1}\} \Rightarrow p_j(x)$

$S_{j+1} = \{I_j, I_{j+1}, I_{j+2}\} \Rightarrow p_{j+1}(x)$

$U_{j+\frac{1}{2}}^- = p_j(X_{j+\frac{1}{2}}) = -\frac{1}{6}\bar{U}_{j-1} + \frac{5}{6}\bar{U}_j + \frac{1}{3}\bar{U}_{j+1}$ } not equal

$U_{j+\frac{1}{2}}^+ = p_{j+1}(X_{j+\frac{1}{2}}) = \frac{1}{3}\bar{U}_j + \frac{5}{6}\bar{U}_{j+1} - \frac{1}{6}\bar{U}_{j+2}$



define $\hat{f}_{j+\frac{1}{2}} = \hat{f}(U_{j+\frac{1}{2}}^-, U_{j+\frac{1}{2}}^+)$

$\hat{f}(\cdot, \cdot)$ satisfies:

- (1) consistency. $\hat{f}(u, u) = f(u)$
- (2) Lipschitz continuity
- (3) monotonicity. $\hat{f}(\uparrow, \downarrow)$ (for stability)

eg: LF flux. Godunov flux. Engquist-Osher flux.

Remark: $\rightarrow$ (different choice of stencil, different choice of flux, time involving)

(1) All these scheme are linear when applied to $u_t + u_x = 0$

i.e.
$$\bar{u}_j^{n+1} = \sum_{l=-p}^q \alpha_l \bar{u}_{j+l}^n \quad (\alpha_l \sim \Delta t \cdot \Delta x, \text{ but not } \bar{u})$$

(2) A linear scheme is either first order or oscillatory.

Artificial viscosity

$$u_t + f(u)_x = \varepsilon u_{xx} \quad \varepsilon > 0, \varepsilon \rightarrow 0.$$

Idea: add an additional "artificial viscosity" term to the hyperbolic equation.

1. This term will vanish as $\Delta t \cdot \Delta x \rightarrow 0$. s.t. it remains consistent with the hyperbolic equation.
2. The term vanishes sufficiently small that the order of the accuracy of the high order method on smooth solutions is unaffected
3. The terms depends on the behavior of the solution, being larger near discontinuity than in smooth regions

Scheme 1:

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} [\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n] + \Delta t Q [\underbrace{\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n}_{\Delta x^2 u_{xx}}] \quad Q \sim \varepsilon$$

if $Q = \text{const}$, the scheme is still a linear scheme

if $Q = Q(\bar{u}_j^n)$, the scheme is not a conservative scheme.

Scheme 2:

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} [\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n] + \Delta t \left\{ Q(\bar{u}_{j+\frac{1}{2}}) (\bar{u}_{j+1}^n - \bar{u}_j^n) - Q(\bar{u}_{j-\frac{1}{2}}) (\bar{u}_j^n - \bar{u}_{j-1}^n) \right\}$$

$$= \bar{u}_j^n - \frac{\Delta t}{\Delta x} [\hat{F}_{j+\frac{1}{2}}^n - \hat{F}_{j-\frac{1}{2}}^n] \quad \hat{F}_{j+\frac{1}{2}}^n = \hat{f}_{j+\frac{1}{2}}^n - \Delta x Q(\bar{u}_{j+\frac{1}{2}}) (\bar{u}_{j+1}^n - \bar{u}_j^n)$$

(conservation scheme)

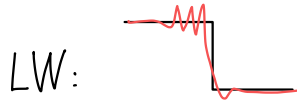
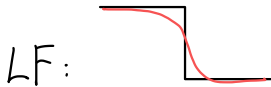
eg: $u_t + (\frac{1}{2}u^2)_x = 0$ $u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases}$

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} \left[\hat{f}(\bar{u}_j^n, \bar{u}_{j+1}^n) - \hat{f}(\bar{u}_{j-1}^n, \bar{u}_j^n) \right]$$

Oscillatory may occur in low order scheme.

$$\hat{f}^{LF}(a, b) = \frac{1}{2}(f(a) + f(b)) - \frac{\max|f'|}{2}(b - a)$$

$$\hat{f}^{LW}(a, b) = \frac{1}{2}(f(a) + f(b)) - \frac{\Delta t}{2\Delta x} f'(\frac{a+b}{2})[f(b) - f(a)]$$



eg: Burgers'. $u(x, 0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

$$\hat{f}^{Roe}(a, b) = \begin{cases} f(a) & s \geq 0 \\ f(b) & s < 0 \end{cases} \quad s = \frac{f(a) - f(b)}{a - b} \quad \Rightarrow \quad \bar{u}_j^n = \bar{u}_j^0 \quad \forall n.$$

A property of the entropy-satisfying weak solution is the following :

If we take two set of initial data u_0 & v_0 . with $u_0(x) \leq v_0(x) \quad \forall x$.
then, we respective entropy solution $u(x,t)$ & $v(x,t)$ satisfy

$$u(x,t) \leq v(x,t) \quad \forall x, \forall t$$

Numerical scheme :

$$\begin{aligned} \bar{u}_j^{n+1} &= \bar{u}_j^n - \frac{\Delta t}{\Delta x} \left[\hat{f}(\bar{u}_{j-p}^n, \dots, \bar{u}_{j+q}^n) - \hat{f}(\bar{u}_{j-p-1}^n, \dots, \bar{u}_{j+q-1}^n) \right] \\ &= H(\bar{u}_{j-p-1}^n \dots \bar{u}_j^n \dots \bar{u}_{j+q}^n) = H(\bar{u}^n; j) \end{aligned}$$

Def: Monotone scheme, if H is monotonically nondecreasing in all arguments.

$$H(\uparrow, \uparrow, \dots, \uparrow)$$

Thm: Good property of monotone scheme.

(1) If $\bar{u}_j^n \leq \bar{v}_j^n$, then $H(\bar{u}^n, j) \leq H(\bar{v}^n, j)$ i.e. $\bar{u}_j^{n+1} \leq \bar{v}_j^{n+1} \quad \forall j$

(2) monotonicity preserving :

$$\bar{u}_j^n \leq \bar{u}_{j+1}^n \quad \underline{\forall j} \Rightarrow \bar{u}_j^{n+1} \leq \bar{u}_{j+1}^{n+1} \quad \forall j$$

(3) maximum principle :

$$m = \min_j \bar{u}_j^n \leq \underbrace{H(\bar{u}^n; j)}_{\bar{u}_j^{n+1}} \leq \max_j \bar{u}_j^n = M \quad \forall j$$

(4) L^1 contractive $\|\bar{u}\|_1 = \sum_j |\bar{u}_j| \Delta x$ (?)

$$\|H(\bar{u}^n; j) - H(\bar{v}^n; j)\|_1 \leq \|\bar{u}_j^n - \bar{v}_j^n\|_1$$

(5) total variation diminishing (TVD) $TV(\bar{u}) = \sum_j |\bar{u}_{j+1} - \bar{u}_j|$

$$TV(\bar{u}^{n+1}) \leq TV(\bar{u}^n)$$

$$\downarrow$$

$$TV(f) = \int |f'| dx$$

Thm: monotone scheme converges to the entropy solution in L^1 -norm.

$$\bar{u}_j^n \rightarrow u(x_j, t^n) \text{ in } L^1 \text{ as } \Delta t, \Delta x \rightarrow 0$$

Remark: (1) monotonicity conserve stability of the scheme and guarantees no oscillation can be created.

(2) If we discretize a single propagating discontinuity with a monotonicity preserving scheme.

(3) True solutions are TVD. it's reasonable to impose the requirement on the numerical solutions as well. yielding a TV-stable and here convergent method.

(4) L^1 contractive scheme $\Rightarrow$ TVD scheme. (choose $\bar{v}_j^n = \bar{u}_{j-1}^n$)

Thm: A TVD scheme is monotonically preserving.

monotone



单调

L^1 contracting



L^1 压缩

TVD



TVD

monotonicity preserving

单调保持

$$\bar{u}_j^{n+1} = H(\bar{u}^n; j). \quad H(\uparrow, \uparrow \dots \uparrow)$$

$$\|H(\bar{u}^n; j) - H(\bar{v}^n; j)\|_1 \leq \|\bar{u}_j^n - \bar{v}_j^n\|_1$$

$$\sum_j |\bar{u}_{j+1}^{n+1} - \bar{u}_j^{n+1}| \leq \sum_j |\bar{u}_{j+1}^n - \bar{u}_j^n|$$

$$\bar{u}_j^n \leq \bar{u}_{j+1}^n \quad \forall j \Rightarrow \bar{u}_j^{n+1} \leq \bar{u}_{j+1}^{n+1} \quad \forall j$$

Thm: A linear monotonically preserving scheme is monotone.

eg: if $\hat{f}_{j+\frac{1}{2}}^n = \hat{f}(\bar{u}_j^n, \bar{u}_{j+1}^n)$. $\bar{u}_j^{n+1} = \bar{u}_j^n - \frac{\Delta t}{\Delta x} [\hat{f}_{j+\frac{1}{2}}^n - \hat{f}_{j-\frac{1}{2}}^n]$
 $= H(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n)$ 3-point.

$$\begin{cases} \frac{\partial H}{\partial \bar{u}_{j-1}} = \frac{\Delta t}{\Delta x} \hat{f}'_1 \geq 0 \\ \frac{\partial H}{\partial \bar{u}_j} = 1 - \frac{\Delta t}{\Delta x} [\hat{f}'_1 - \hat{f}'_2] \geq 0 \\ \frac{\partial H}{\partial \bar{u}_{j+1}} = -\frac{\Delta t}{\Delta x} \hat{f}'_2 \geq 0 \end{cases} \Rightarrow \hat{f}(\uparrow, \downarrow) \quad \text{and CFL: } \frac{\Delta t}{\Delta x} [\hat{f}'_1 - \hat{f}'_2] \leq 1$$

eg:

(1) Roe flux $\times$ not monotone scheme

(2) $\hat{f}(a, b) = \frac{f(a) + f(b)}{2}$ $\times$

(3) LF flux $\checkmark$

$\hat{f}^{LF}(a, b) = \frac{f(a) + f(b)}{2} - \max(|f'|) \frac{b-a}{2}$

(4) Godunov flux $\checkmark$

$\hat{f}^{God}(a, b) = \begin{cases} \min_{a \leq u \leq b} f(u) & a < b \\ \max_{b \leq u \leq a} f(u) & b < a \end{cases}$

(5) Osher flux $\checkmark$

$\hat{f}^{Osher}(a, b) = f(a) + \int_a^b \max(f'(u), 0) du + \int_0^b \min(f'(u), 0) du$

(6) LW flux. unknown

$\hat{f}^{LW}(a, b) = \frac{f(a) + f(b)}{2} - \frac{\Delta t}{2\Delta x} f'(\frac{a+b}{2}) [f(b) - f(a)]$

Thm. monotone schemes are at most first order accurate for smooth region

Remark: A linear scheme is either first order or oscillatory.

Nonlinear limiter :

We will look at the Euler forward time discretization. (same for RK)

suppose Eulerforward $TV(\bar{u}^n + \Delta t L(\bar{u}^n)) \leq TV(\bar{u}^n)$

RK3: (SSP RK. TVD RK)

$$\bar{u}''' = \bar{u}^n + \Delta t L(\bar{u}^n) \Rightarrow TV(\bar{u}''') \leq TV(\bar{u}^n)$$

$$\begin{aligned} \bar{u}^{(2)} = \frac{3}{4}\bar{u}^n + \frac{1}{4}(\bar{u}''' + \Delta t L(\bar{u}''')) &\Rightarrow TV(\bar{u}^{(2)}) \leq \frac{3}{4}TV(\bar{u}^n) + \frac{1}{4}TV(\bar{u}''' + \Delta t L(\bar{u}''')) \\ &\leq \frac{3}{4}TV(\bar{u}^n) + \frac{1}{4}TV(\bar{u}''') \\ &\leq TV(\bar{u}^n) \end{aligned}$$

$$\begin{aligned} \bar{u}^{n+1} = \frac{1}{3}\bar{u}^n + \frac{2}{3}(\bar{u}^{(2)} + \Delta t L(\bar{u}^{(2)})) &\Rightarrow TV(\bar{u}^{n+1}) \leq \frac{1}{3}TV(\bar{u}^n) + \frac{2}{3}TV(\bar{u}^{(2)}) \\ &\leq TV(\bar{u}^n) \end{aligned}$$

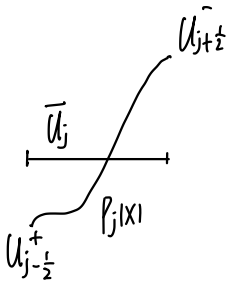
Main idea: $\lambda = \frac{\Delta t}{\Delta x}$

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda [\hat{f}(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) - \hat{f}(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+)]$$

$$\text{1st } \hookrightarrow [\hat{f}(\bar{u}_j, \bar{u}_{j+1}) - \hat{f}(\bar{u}_{j-1}, \bar{u}_j)] \quad \text{monotone.}$$

slope-limiter method:

$$u_{j+\frac{1}{2}}^\pm \xrightarrow{\text{modify}} u_{j+\frac{1}{2}}^{\pm(\text{mod})} \quad \text{or} \quad \hat{f}_{j+\frac{1}{2}}^{n(\text{mod})}$$



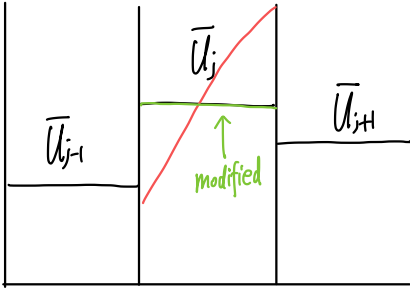
denote $\begin{cases} \tilde{u}_j = u_{j+1/2}^- - \bar{u}_j \longrightarrow \tilde{u}_j^{(mod)} \longrightarrow u_{j+1/2}^- = \bar{u}_j + \tilde{u}_j^{(mod)} \\ \tilde{\tilde{u}}_j = \bar{u}_j - u_{j-1/2}^+ \longrightarrow \tilde{\tilde{u}}_j^{(mod)} \longrightarrow u_{j-1/2}^+ = \bar{u}_j - \tilde{\tilde{u}}_j^{(mod)} \end{cases}$

$$\tilde{u}_j^{(mod)} = m(\tilde{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1})$$

$$\tilde{\tilde{u}}_j^{(mod)} = m(\tilde{\tilde{u}}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1})$$

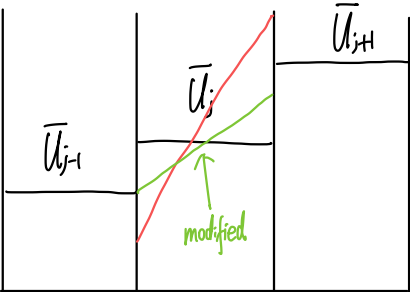
minmod function: $m(a_1, \dots, a_n) = \begin{cases} \text{sign}(a_1) \cdot \min_k |a_k| & \text{if } \text{sign}(a_1) = \text{sign}(a_2) = \dots = \text{sign}(a_n) \\ 0 & \text{otherwise} \end{cases}$

eg 1:



change to 1-st scheme. $\tilde{u}_j^{(mod)} = \tilde{\tilde{u}}_j^{(mod)} = 0$.

eg 2:



Prop: the modified scheme is TVD if $\lambda(\hat{f}_1 + \hat{f}_2) \leq \frac{1}{2}$ ($\lambda = \frac{\sigma t}{\Delta x}$)

$$U_t + f(U)_x = 0.$$

$$\text{MOL: semi-discrete } \frac{d}{dt} \bar{U}_j(t) + \underbrace{\frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta t}}_{\text{high order}} = 0 \xRightarrow[\text{time}]{\text{high order}} \text{fully-discrete}$$

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(U_{j+\frac{1}{2}}^-, U_{j+\frac{1}{2}}^+) \quad \hat{f}(\uparrow, \downarrow)$$

$$\left\{ \begin{array}{l} U_{j+\frac{1}{2}}^- = p_j(X_{j+\frac{1}{2}}) \\ U_{j+\frac{1}{2}}^+ = p_{j+1}(X_{j+\frac{1}{2}}) \end{array} \right.$$

A linear scheme is either first order or oscillatory

↑
nonlinear limiter: TVD, ...

Thm: (Harten lemma)

$$\text{if: } \bar{U}_j^{n+1} = \bar{U}_j^n + C_{j+\frac{1}{2}} [\bar{U}_{j+1}^n - \bar{U}_j^n] - D_{j-\frac{1}{2}} [\bar{U}_j^n - \bar{U}_{j-1}^n]$$

where $C_{j+\frac{1}{2}} \geq 0$, $D_{j+\frac{1}{2}} \geq 0$, $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$, $\forall j, \forall n$. Then the scheme is TVD.

$$\begin{aligned} \text{Proof: } TV(\bar{U}^{n+1}) &= \sum_j |\bar{U}_{j+1}^{n+1} - \bar{U}_j^{n+1}| & (\Delta \bar{U}_j &= \bar{U}_{j+1} - \bar{U}_j) \\ &= \sum_j |\bar{U}_{j+1}^n + C_{j+\frac{1}{2}} \Delta \bar{U}_{j+1}^n - D_{j+\frac{1}{2}} \Delta \bar{U}_j^n - \bar{U}_j^n - C_{j+\frac{1}{2}} \Delta \bar{U}_j^n + D_{j+\frac{1}{2}} \Delta \bar{U}_{j-1}^n| \\ &= \sum_j |(1 - D_{j+\frac{1}{2}} - C_{j+\frac{1}{2}}) \Delta \bar{U}_j^n + C_{j+\frac{1}{2}} \Delta \bar{U}_{j+1}^n + D_{j-\frac{1}{2}} \Delta \bar{U}_{j-1}^n| \\ &\leq \sum_j |1 - D_{j+\frac{1}{2}} - C_{j+\frac{1}{2}}| |\Delta \bar{U}_j^n| + \sum_j |C_{j+\frac{1}{2}}| |\Delta \bar{U}_j^n| + \sum_j |D_{j+\frac{1}{2}}| |\Delta \bar{U}_j^n| \\ &\leq \sum_j |\Delta \bar{U}_j^n| = TV(\bar{U}^n) \quad \# \end{aligned}$$

eg: TVD scheme is TVD.

$$\begin{aligned}
 \bar{u}_j^{n+1} &= \bar{u}_j^n - \lambda \left[\hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j+\frac{1}{2}}^{+(mod)}) - \hat{f}(u_{j-\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)}) \right] \\
 &= \bar{u}_j^n - \lambda \left[\hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j+\frac{1}{2}}^{+(mod)}) - \hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)}) \right. \\
 &\quad \left. + \hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)}) - \hat{f}(u_{j-\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)}) \right] \\
 &= \bar{u}_j^n + (\bar{u}_{j+1}^n - \bar{u}_j^n) \left[-\lambda \frac{\hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j+\frac{1}{2}}^{+(mod)}) - \hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)})}{\bar{u}_j^{n+1} - \bar{u}_j^n} \right] = C_{j+\frac{1}{2}} \\
 &\quad - (\bar{u}_j^n - \bar{u}_{j-1}^n) \left[\lambda \frac{\hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)}) - \hat{f}(u_{j-\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)})}{\bar{u}_j^n - \bar{u}_{j-1}^n} \right] = D_{j-\frac{1}{2}} \\
 &= \bar{u}_j^n + C_{j+\frac{1}{2}} (\bar{u}_{j+1}^n - \bar{u}_j^n) - D_{j-\frac{1}{2}} (\bar{u}_j^n - \bar{u}_{j-1}^n)
 \end{aligned}$$

$$\begin{aligned}
 C_{j+\frac{1}{2}} &= -\lambda \frac{\hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j+\frac{1}{2}}^{+(mod)}) - \hat{f}(u_{j+\frac{1}{2}}^{-(mod)}, u_{j-\frac{1}{2}}^{+(mod)})}{\bar{u}_j^{n+1} - \bar{u}_j^n} = -\lambda \frac{\hat{f}_2(u_{j+\frac{1}{2}}, \xi) (u_{j+\frac{1}{2}}^{+(mod)} - u_{j-\frac{1}{2}}^{+(mod)})}{\bar{u}_j^{n+1} - \bar{u}_j^n} \\
 &= -\lambda \frac{\hat{f}_2 [\bar{u}_{j+1} - \tilde{\alpha}_{j+1} - \bar{u}_j + \tilde{\alpha}_j]^{(mod)}}{\bar{u}_{j+1} - \bar{u}_j} = -\lambda \hat{f}_2 \left[1 - \frac{\tilde{\alpha}_{j+1}^{(mod)}}{\bar{u}_{j+1} - \bar{u}_j} + \frac{\tilde{\alpha}_j^{(mod)}}{\bar{u}_{j+1} - \bar{u}_j} \right] > 0.
 \end{aligned}$$

$\hat{f}(?, \downarrow)$ $\uparrow$ $\uparrow$ $\uparrow$
 $[0.1]$ $[0.1]$ $[0.2]$

$$\Rightarrow 0 < C_{j+\frac{1}{2}} < -2\lambda \min \hat{f}_2$$

① ✓

similar, $0 < D_{j+\frac{1}{2}} < 2\lambda \max \hat{f}_i$
 ② ✓

$$0 < C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} < 2\lambda (\max \hat{f}_1 - \min \hat{f}_2) \Leftrightarrow \underline{\lambda(\hat{f}_1 - \hat{f}_2) \leq \frac{1}{2}}.$$

③ ✓

TVD lose accuracy:

$$U_{j+\frac{1}{2}}^{\pm} = U(X_{j+\frac{1}{2}}) + O(\Delta x^{K_H}) \quad (K \geq 1)$$

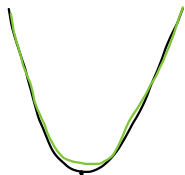
$$\begin{aligned} \bar{U}_j &= \frac{1}{\Delta x} \int_{I_j} [U(X_j) + (X - X_j) U'(X_j) + \frac{1}{2} (X - X_j)^2 U''(\eta)] dx \\ &= U(X_j) + \underline{O(\Delta x^2)} \end{aligned}$$

$$\begin{cases} \tilde{U}_{j+\frac{1}{2}} = U_{j+\frac{1}{2}}^- - \bar{U}_j = U(X_{j+\frac{1}{2}}) - U(X_j) + O(\Delta x^2) = \frac{\Delta x}{2} U'(X_j) + O(\Delta x^2) \\ \bar{U}_{j+1} - \bar{U}_j = U(X_{j+1}) - U(X_j) + O(\Delta x^2) = \Delta x U'(X_j) + O(\Delta x^2) \\ \bar{U}_j - \bar{U}_{j-1} = U(X_j) - U(X_{j-1}) + O(\Delta x^2) = \Delta x U'(X_j) + O(\Delta x^2) \end{cases}$$

if X_j is not near the extrema point: $U'(X_j) = O(1)$.

$$\tilde{U}_{j+\frac{1}{2}}^{(mod)} = \min(\tilde{U}_{j+\frac{1}{2}}^-, \bar{U}_{j+1} - \bar{U}_j, \bar{U}_j - \bar{U}_{j-1}) = \tilde{U}_{j+\frac{1}{2}}^- \quad \text{TVD doesn't destroy the accuracy}$$

if $U'(X_j) \neq O(1)$. near the extrema point. $U_{j+\frac{1}{2}}^- = U_{j-\frac{1}{2}}^+ = \bar{U}_j \Rightarrow$ at most second order.



TVD $\Rightarrow$ TVB

$$m \Rightarrow \bar{m}$$

$$\begin{aligned}\tilde{u}_j^{(mod)} &= \bar{m} (\tilde{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}) \\ &= \begin{cases} \tilde{u}_j & \text{if } |\tilde{u}_j| \leq M \Delta x^2 \\ m(\tilde{u}_j, \bar{u}_{j+1} - \bar{u}_j, \bar{u}_j - \bar{u}_{j-1}) & \text{otherwise} \end{cases}\end{aligned}$$

Hw: 3-rd with TVD/TVB.
Burgers (LF flux)

ddl 05/18

time: $T=0.5$ $T=1.5$ plot & error table

case: ① without limiter

② TVD limiter

③ TVB limiter ($M=0.1, 3, 10$)

remark: $\bar{u}_j^n \xrightarrow{\text{reconstruct}} \hat{p}_j(x) \xrightarrow{\text{limiter}} \bar{u}_{j+\frac{1}{2}}^-, \bar{u}_{j-\frac{1}{2}}^+ \xrightarrow{\sim} \hat{p}_j^{(K)}(x)$

$\uparrow$
 $\hat{p}^K$

$\downarrow$ $\downarrow$ $\downarrow$
 $\hat{f}(\bar{u}_j^n, \bar{u}_{j+1}^n)$ $\hat{f}(\bar{u}_{j-\frac{1}{2}}^+, \bar{u}_{j+\frac{1}{2}}^-)$ $\hat{f}(\bar{u}_{j-\frac{1}{2}}^{+(mod)}, \bar{u}_{j+\frac{1}{2}}^{-(mod)})$

for TVD/TVB limiter. $\tilde{p}_j(x)$ satisfy:

$$(1) \quad \bar{u}_j = \frac{1}{\Delta x} \int_{I_j} \tilde{p}_j(x) dx = \frac{1}{\Delta x} \int_{I_j} \tilde{p}_j(x) dx$$

$$(2) \quad \tilde{p}_j(x_{j+\frac{1}{2}}) = \bar{u}_{j+\frac{1}{2}}^{-(mod)}$$

$$(3) \quad \tilde{p}_j(x_{j-\frac{1}{2}}) = \bar{u}_{j-\frac{1}{2}}^{+(mod)}$$

$K=0$. TVD/TVB limiter do nothing, unique ✓

$K=1$. $\tilde{u} = \tilde{\tilde{u}} \Rightarrow \tilde{u}^{(mod)} = \tilde{\tilde{u}}^{(mod)}$ unique. ✓

$K=2$. unique $\hat{p}_j(x)$ ✓

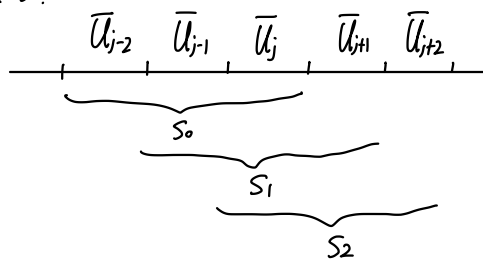
$K \geq 3$. not unique $\tilde{p}_j(x)$

ENO (essentially non-oscillatory) & WENO (weighted ENO)

Given $\{\bar{u}_j\}$ find $p_j(x) \in P^k$ s.t.

$$\bar{u}_\ell = \frac{1}{\Delta x} \int_{I_\ell} p_j(x) dx, \quad \ell = \underbrace{j-p, \dots, j+q}_{k+1} \quad (p+q=k, p, q \geq 0)$$

eg: $k=2$.



} $k+1$ stencils: $S_0, S_1, \dots, S_k$

ENO:

choose different stencil to avoid oscillatory: if discontinuity at $X_{j+\frac{1}{2}}$, choose S_0 .

if discontinuity at $X_{j-\frac{1}{2}}$, choose S_2 .

WENO: use all stencils, but with different weights: if discontinuity at $X_{j+\frac{1}{2}}$, $w_0 \approx 1$, $w_1, w_2 \approx 0$.

if discontinuity at $X_{j-\frac{1}{2}}$, $w_2 \approx 1$, $w_0, w_1 \approx 0$.

Newton's form: ($P' = p$)

$$P(x) = U[X_{j-\frac{1}{2}}] + U[X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}](x - X_{j-\frac{1}{2}}) + U[X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}, X_{j+\frac{3}{2}}](x - X_{j-\frac{1}{2}})(x - X_{j+\frac{1}{2}}) + \dots$$

$$U[X_{j-\frac{1}{2}}] = U(X_{j-\frac{1}{2}}) \quad U[X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}] = \bar{u}_j \quad U[X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}, X_{j+\frac{3}{2}}] = \frac{\bar{u}_{j+1} - \bar{u}_j}{X_{j+\frac{3}{2}} - X_{j-\frac{1}{2}}}$$

$$(*) \quad U[X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}, \dots, X_{j+l-\frac{1}{2}}] = \begin{cases} \frac{U^{(l)}(s)}{l!} & U(x) \text{ is smooth.} \\ \underline{O(\frac{1}{\Delta x^l})} & U(x) \text{ is discontinuous.} \end{cases}$$

ENO: $U(x)$ is "the smoothest" in all stencils.

Way 1: compute $\bar{u}$ in all stencils. (too much cost)

Way 2: (Greedy Algorithm)

step 0, start with one cell $S = \{I_j\}$. 1st order.

step 1. Add $\begin{cases} I_{j-1} \Rightarrow S = \{I_{j-1}, I_j\} \Rightarrow |\bar{u}[X_{j-1}, X_j]| \\ \text{or} \\ I_{j+1} \Rightarrow S = \{I_j, I_{j+1}\} \Rightarrow |\bar{u}[X_j, X_{j+1}]| \end{cases}$

if $|\bar{u}[X_{j-1}, X_j]| < |\bar{u}[X_j, X_{j+1}]|$. choose $S = \{I_{j-1}, I_j\}$ 2nd order

otherwise. choose $S = \{I_j, I_{j+1}\}$

...

Step l . $S = \{I_{j-p}, \dots, I_{j+q}\}$. $p+q=l$.

Add $\begin{cases} I_{j-p-1}, \text{ compare } |\bar{u}[X_{j-p-1}, \dots, X_{j+q}]| \text{ with } |\bar{u}[X_{j-p}, \dots, X_{j+q+1}]|. \\ I_{j+q+1} \end{cases}$ $(l+1)$ th order.

remark: (1) if $u(x)$ is smooth, $p_i(x) = u(x) + O(\Delta x^{k+1})$ in I_j

(2) if $u(x)$ has a discontinuity in I_j . $p_i(x)$ is monotone in I_j .

(3) ENO is TVB

(4) $(k+1)$ candidate stencils in $(2k+1)$ cells. only one stencil is used. $\Rightarrow (k+1)$ th order.

WENO : use all stencils

$$k=1. \quad S_0 = \{I_{j-1}, I_j\} \quad S = \{S_0, S_1\} \quad \text{big stencil: } \{I_{j-1}, I_j, I_{j+1}\} \quad (2k-1)$$

$$S_1 = \{I_j, I_{j+1}\}.$$

$$\begin{aligned} U_{j+\frac{1}{2}}^{(big)} &= -\frac{1}{6} \bar{U}_{j-1} + \frac{5}{6} \bar{U}_j + \frac{1}{3} \bar{U}_{j+1} \\ &= \frac{1}{3} \left(-\frac{1}{2} \bar{U}_{j-1} + \frac{3}{2} \bar{U}_j \right) + \frac{2}{3} \left(\frac{1}{2} \bar{U}_j + \frac{1}{2} \bar{U}_{j+1} \right) \\ &= d_0 U_{j+\frac{1}{2}}^{(0)} + d_1 U_{j+\frac{1}{2}}^{(1)} \end{aligned}$$

S
S₀
S₁

$$k=2 \quad S_0 = \{I_{j-2}, I_{j-1}, I_j\} \quad S = \{S_0, S_1, S_2\}$$

$$S_1 = \{I_{j-1}, I_j, I_{j+1}\}$$

$$S_2 = \{I_j, I_{j+1}, I_{j+2}\}$$

$$\text{big stencil } \{I_{j-2}, \dots, I_{j+2}\}$$

$$\begin{aligned} U_{j+\frac{1}{2}}^{(big)} &= \frac{1}{30} \bar{U}_{j-2} - \frac{13}{60} \bar{U}_{j-1} + \frac{47}{60} \bar{U}_j + \frac{9}{20} \bar{U}_{j+1} - \frac{1}{20} \bar{U}_{j+2} \\ &= \frac{1}{10} \left(\frac{1}{3} \bar{U}_{j-2} - \frac{7}{6} \bar{U}_{j-1} + \frac{11}{6} \bar{U}_j \right) + \frac{3}{5} \left(-\frac{1}{6} \bar{U}_{j-1} + \frac{5}{6} \bar{U}_j + \frac{1}{3} \bar{U}_{j+1} \right) \\ &\quad + \frac{3}{10} \left(\frac{1}{3} \bar{U}_j + \frac{5}{6} \bar{U}_{j+1} - \frac{1}{6} \bar{U}_{j+2} \right) \\ &= d_0 U_{j+\frac{1}{2}}^{(0)} + d_1 U_{j+\frac{1}{2}}^{(1)} + d_2 U_{j+\frac{1}{2}}^{(2)} \end{aligned}$$

$$\text{In general, } S_r = \{I_{j-k+r}, \dots, I_{j+r}\} \quad r=0, 1, \dots, k. \quad S = \{S_0, \dots, S_k\}$$

$$U_{j+\frac{1}{2}}^{(big)} = u(X_{j+\frac{1}{2}}) + O(\Delta x)^{2(K+1)}$$

$$U_{j+\frac{1}{2}}^{(r)} = u(X_{j+\frac{1}{2}}) + O(\Delta x)^{K+1}$$

$$\text{(linear combination; } \{d_r\} \sum_{r=0}^K d_r = 1 \quad (\text{keep } d_r \geq 0. \text{ if } d_r < 0. \text{ split)})$$

$$U_{j+\frac{1}{2}}^{(big)} = \sum_{r=0}^K d_r U_{j+\frac{1}{2}}^{(r)}$$

WENO: change the linear weights $\{d_r\}$ to nonlinear weights $\{w_r\}$ ($\sum w_r = 1$ $w_r \geq 0$)

$$\underline{U_{j+\frac{1}{2}}^{(WENO)} = \sum_{r=0}^K w_r U_{j+\frac{1}{2}}^{(r)}}$$

if we keep $w_r = d_r + O(\Delta x)^K$ in smooth case

$$\begin{aligned} U_{j+\frac{1}{2}}^{(WENO)} &= \sum_{r=0}^K d_r U_{j+\frac{1}{2}}^{(r)} + \sum_{r=0}^K (w_r - d_r) U_{j+\frac{1}{2}}^{(r)} \\ &= U_{j+\frac{1}{2}}^{(big)} + \sum_{r=0}^K (w_r - d_r) [U_{j+\frac{1}{2}}^{(r)} - u(X_{j+\frac{1}{2}})] \\ &= u(X_{j+\frac{1}{2}}) + O(\Delta x)^{2K+1} \end{aligned}$$

$$\sum (w_r - d_r) = 0$$

if $u(x)$ has a discontinuity in one or more stencils. we would hope the corresponding weights w_r to be essentially 0.

Smoothness indicator:

$$\beta_r = \sum_{m=1}^K \int_{X_{j-\frac{r}{2}}}^{X_{j+\frac{r}{2}}} (\Delta x)^{2m-1} \left(\frac{\partial^m p_r(x)}{\partial x^m} \right)^2 dx$$

$$K=1. \quad \beta_0 = (\bar{u}_j - \bar{u}_{j+1})^2 \quad \beta_1 = (\bar{u}_{j+1} - \bar{u}_j)^2$$

$$K=2. \quad \beta_0 = \frac{13}{12} (\bar{u}_{j-2} - 2\bar{u}_{j-1} + \bar{u}_j)^2 + \frac{1}{4} (\bar{u}_{j-2} - 4\bar{u}_{j-1} + 3\bar{u}_j)^2$$

$$\beta_1 = \frac{13}{12} (\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1})^2 + \frac{1}{4} (\bar{u}_{j-1} - \bar{u}_{j+1})^2$$

$$\beta_2 = \frac{13}{12} (\bar{u}_j - 2\bar{u}_{j+1} + \bar{u}_{j+2})^2 + \frac{1}{4} (3\bar{u}_j - 4\bar{u}_{j+1} + \bar{u}_{j+2})^2$$

$$\left. \begin{array}{l} \text{linear weights} \\ \text{indicator} \end{array} \right\} \left\{ \begin{array}{l} \{dr\} \\ \{\beta_r\} \end{array} \right\} \implies \alpha_r = \frac{dr}{(\beta_r + \varepsilon)^2} \xRightarrow{\varepsilon=10^{-6}} W_r = \frac{\alpha_r}{\sum \alpha_r}$$

$$\beta_r = \begin{cases} D(\Delta x)^2 (1 + O(\Delta x)^k) & (\text{smooth}) \\ O(1) & (\text{otherwise}) \end{cases} \quad (D \text{ is same for all } r, \text{ assume } D \neq 0)$$

extreme point. may loss accuracy

$$\alpha_r = \frac{dr}{(\beta_r)^2} = \begin{cases} \frac{dr}{D^2(\Delta x)^4} & (\text{smooth}) \\ O(1) & (\text{otherwise}) \end{cases} \quad W_r = \begin{cases} O(1) & (\text{smooth}) \\ O(\Delta x)^k & (\text{otherwise}) \end{cases}$$

$$\begin{aligned} U_{j+\frac{1}{2}}^{(WENO)} &= \sum_{r=0}^K W_r U_{j+\frac{1}{2}}^{(r)} & \{dr\} &\rightarrow \{\tilde{dr}\} & \tilde{dr} &= \tilde{d}_{K-r} & \text{symmetric skill.} \\ U_{j-\frac{1}{2}}^{(WENO)} &= \sum_{r=0}^K \tilde{W}_r U_{j-\frac{1}{2}}^{(r)} & \{W_r\} &\rightarrow \{\tilde{W}_r\} \end{aligned}$$

WENO:

Step 1. reconstruction on each small stencils $\{U_{j-\frac{1}{2}}^{(r)}\} \{U_{j+\frac{1}{2}}^{(r)}\}$
and the linear weights $dr, \tilde{dr}$ ($\tilde{dr} = d_{K-r}$)

Step 2. Get the smoothness indicator β_r (β_r on I_r . same for $U_{j-\frac{1}{2}}$ and $U_{j+\frac{1}{2}}$)

Step 3. Get the nonlinear weights: $\alpha_r = \frac{dr}{(\beta_r + \varepsilon)^k}$ $W_r = \frac{\alpha_r}{\sum \alpha_r}$
 $\tilde{\alpha}_r = \frac{\tilde{dr}}{(\beta_r + \varepsilon)^2}$ $\tilde{W}_r = \frac{\tilde{\alpha}_r}{\sum \tilde{\alpha}_r}$

Step 4. $U_{j+\frac{1}{2}}^- = \sum_{r=0}^K W_r U_{j+\frac{1}{2}}^{(r)}$ $U_{j-\frac{1}{2}}^+ = \sum_{r=0}^K \tilde{W}_r U_{j-\frac{1}{2}}^{(r)}$

Hw. (K=2) fifth-order WENO

RK3. $\Delta t = CFL \cdot \Delta x^{\frac{5}{3}}$

ddl 05/25

LF flux

Boundary preserving (maximum-principle preserving)

PDE: maximum-principle.

if $M = \max U_0(x)$, $m = \min U_0(x)$. $\Rightarrow U(x,t) \in [m, M] \quad \forall (x,t)$

NPDE: 1st order scheme. $\hat{f}(\uparrow, \downarrow)$ $\lambda = \frac{\Delta t}{\Delta x} \quad \alpha = \max |f'|$

$$\begin{aligned}\bar{u}_j^{n+1} &= \bar{u}_j^n - \frac{\Delta t}{\Delta x} \left[\hat{f}^{LF}(\bar{u}_j^n, \bar{u}_j^{n+1}) - \hat{f}^{LF}(\bar{u}_{j-1}^n, \bar{u}_j^n) \right] \\ &= (1 - \lambda \alpha) \bar{u}_j^n + \frac{1}{2} \lambda [\alpha \bar{u}_{j+1}^n - f(\bar{u}_{j+1}^n)] + \frac{1}{2} \lambda [\alpha \bar{u}_{j+1}^n + f(\bar{u}_{j+1}^n)] \\ &= H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n)\end{aligned}$$

$$\begin{cases} \partial_1 H_\lambda = \frac{1}{2}(\alpha - f') > 0 \\ \partial_2 H_\lambda = (1 - \lambda \alpha) > 0 \\ \partial_3 H_\lambda = \frac{1}{2}(\alpha + f') > 0 \end{cases} \quad \text{it's monotone scheme if } \alpha \lambda \leq 1.$$

Given $\{\bar{u}_j^n\} \subset [m, M]$.

$$m = H_\lambda(m, m, m) \leq \bar{u}_j^{n+1} = H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n) \leq M = H_\lambda(M, M, M) \Rightarrow \{\bar{u}_j^{n+1}\} \subset [m, M]$$

High order scheme

$$\begin{aligned}\bar{u}_j^{n+1} &= \bar{u}_j^n - \lambda \left[\hat{f}^{\text{LF}}(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) - \hat{f}^{\text{LF}}(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+) \right] \\ &= G_\lambda(\bar{u}_j^n, u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) \\ &\quad G_\lambda(\uparrow \uparrow \downarrow \downarrow \uparrow)\end{aligned}$$

Given $\{\bar{u}_j^n\} \subset [m, M]$ & $\{u_{j+\frac{1}{2}}^\pm\} \subset [m, M]$ $\not\Rightarrow \{\bar{u}_j^{n+1}\} \subset [m, M]$.

eg: $u_t + u_x = 0$

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda [u_{j+\frac{1}{2}}^- - u_{j-\frac{1}{2}}^+]$$

choose $\bar{u}_j^n = u_{j-\frac{1}{2}}^+ = u_{j+\frac{1}{2}}^+ = 1$, $u_{j+\frac{1}{2}}^- = 0.99$

$$\bar{u}_j^{n+1} = 1 + 0.01\lambda > 1$$

Way 1: Use limiter to correct $u_{j+\frac{1}{2}}^\pm$ (Xiang xiong Zhang) (same for DG)

$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ Gauss-Lobatto points.

$$x_{j-\frac{1}{2}} = \hat{x}_j^1 < \hat{x}_j^2 < \dots < \hat{x}_j^N = x_{j+\frac{1}{2}} \quad 2N-3 \geq K$$

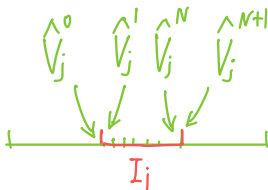
$$\bar{u}_j^n = \frac{1}{\Delta x} \int_{I_j} p_j(x) dx = \sum_{\alpha=1}^N w_\alpha \beta_j(\hat{x}_j^\alpha)$$

$\frac{1}{\Delta x} \hat{V}_j^\alpha$

$$\hat{V}_j^1 = u_{j-\frac{1}{2}}^+$$

$$\hat{V}_j^N = u_{j+\frac{1}{2}}^-$$

let $\hat{V}_j^0 = u_{j-\frac{1}{2}}^-$, $\hat{V}_j^{N+1} = u_{j+\frac{1}{2}}^+$



$$\begin{aligned}
\bar{U}_j^{n+1} &= \sum_{\alpha=1}^N w^\alpha \hat{V}_j^\alpha - \lambda \left[\hat{f}^{LF}(\hat{V}_j^N, \hat{V}_j^{N+1}) - \hat{f}^{LF}(\hat{V}_j^0, \hat{V}_j^1) \right] \\
&= w^1 \left\{ \hat{V}_j^1 - \frac{\lambda}{w^1} \left[\hat{f}^{LF}(\hat{V}_j^1, \hat{V}_j^N) - \hat{f}^{LF}(\hat{V}_j^0, \hat{V}_j^1) \right] \right\} \\
&\quad + w^N \left\{ \hat{V}_j^N - \frac{\lambda}{w^N} \left[\hat{f}^{LF}(\hat{V}_j^N, \hat{V}_j^{N+1}) - \hat{f}^{LF}(\hat{V}_j^1, \hat{V}_j^N) \right] \right\} + \sum_{\alpha=2}^{N-1} w_\alpha \hat{V}_j^\alpha \\
&= w^1 H_{X_{w^1}}(\hat{V}^0, \hat{V}^1, \hat{V}^N) + w^N H_{X_{w^N}}(\hat{V}^1, \hat{V}^N, \hat{V}^{N+1}) + \sum_{\alpha=2}^{N-1} w_\alpha \hat{V}_j^\alpha
\end{aligned}$$

If all the values $\{\hat{V}_j^\alpha\} \subset [m, M]$, then $\bar{U}_j^{n+1} \in [m, M]$, under the CFL condition.

$$\alpha \lambda \leq \min(w_1, \dots, w_N)$$

limiter: modify $p_j(x)$ to $\tilde{p}_j(x)$, choose $U_{j+\frac{1}{2}}^{-(mod)} = \tilde{p}_j(X_{j+\frac{1}{2}})$, $U_{j-\frac{1}{2}}^{+(mod)} = \tilde{p}_j(X_{j-\frac{1}{2}})$

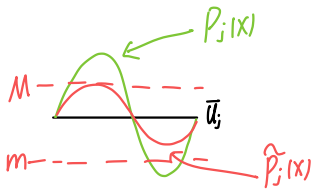
$$\text{s.t. } \tilde{M}_j = \max \tilde{p}_j(\hat{X}_j^\alpha) \quad \tilde{m}_j = \min \tilde{p}_j(\hat{X}_j^\alpha) \Rightarrow \tilde{M}_j \leq M, \quad \tilde{m}_j \geq m.$$

$$\{\bar{U}_j^n\} \subset [m, M] \Rightarrow \underbrace{\{p_j(x)\} \xrightarrow{\text{limiter}} \{\tilde{p}_j(x)\}, \hat{V}_j^\alpha = \tilde{p}_j(\hat{X}_j^\alpha) \in [m, M]} \Rightarrow \{\bar{U}_j^{n+1}\} \subset [m, M]$$

Assume $\bar{U}_j^n \in [m, M]$, $\forall j$.

$$\tilde{p}_j(x) = \bar{U}_j + \theta_j(p_j(x) - \bar{U}_j) = (1-\theta_j)\bar{U}_j + \theta_j p_j(x), \quad \theta_j \in [0, 1]$$

$$\text{Where } \theta_j = \min \left\{ 1, \frac{|M - \bar{U}_j|}{|M - \underline{U}_j|}, \frac{|m - \bar{U}_j|}{|m - \underline{U}_j|} \right\} \quad M_j = \max_\alpha p_j(\hat{X}_j^\alpha), \quad m_j = \min_\alpha p_j(\hat{X}_j^\alpha)$$



remark (1) we only keep $\tilde{M}_j \leq M$. but $\tilde{M}_j = \max_{\alpha} \tilde{p}_j(\hat{X}_j^\alpha) \leq \max_{\mathbb{I}_j} \tilde{p}_j(x)$

so maybe $\tilde{p}_j(x) \notin [m, M]$ at some point.

(2) use limiter in each stage of RK method

(3). $|\tilde{p}_j(x) - p_j(x)| = O(\Delta x^{k+1})$ keep $(k+1)$ th order accuracy.

Way 2. change flux directly. (same for FD) (not in each step of RK)

SSP-RK3:

$$\mathcal{L}(\bar{u}_j) = - \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x}$$

$$\bar{u}_j^{(1)} = \bar{u}_j^n + \Delta t \mathcal{L}(\bar{u}_j^n)$$

$$\bar{u}_j^{(2)} = \frac{3}{4} \bar{u}_j^n + \frac{1}{4} [\bar{u}_j^{(1)} + \Delta t \mathcal{L}(\bar{u}_j^{(1)})]$$

$$\bar{u}_j^{nH} = \frac{1}{3} \bar{u}_j^n + \frac{2}{3} [\bar{u}_j^{(2)} + \Delta t \mathcal{L}(\bar{u}_j^{(2)})]$$

$$\Rightarrow \bar{u}_j^{nH} = \bar{u}_j^n + \Delta t \left[\frac{1}{6} \mathcal{L}(\bar{u}_j^n) + \frac{1}{6} \mathcal{L}(\bar{u}_j^{(1)}) + \frac{4}{6} \mathcal{L}(\bar{u}_j^{(2)}) \right]$$

$$\Rightarrow \bar{u}_j^{nH} = \bar{u}_j^n - \lambda \left[\hat{f}_{j+\frac{1}{2}}^{\text{RK}} - \hat{f}_{j-\frac{1}{2}}^{\text{RK}} \right] \quad (\text{RK3})$$

$$\hat{f}_{j+\frac{1}{2}}^{\text{RK}} = \frac{1}{6} \hat{f}_{j+\frac{1}{2}}^n + \frac{4}{6} \hat{f}_{j+\frac{1}{2}}^{(1)} + \frac{1}{6} \hat{f}_{j+\frac{1}{2}}^{(2)}$$

def flux. $\hat{F}_{j+\frac{1}{2}} = \Theta_{j+\frac{1}{2}}^- \hat{f}_{j+\frac{1}{2}}^{\text{RK}} + (1 - \Theta_{j+\frac{1}{2}}^-) \hat{f}_{j+\frac{1}{2}}^{\text{LF}}(\bar{u}_j^n, \bar{u}_{j+1}^n)$ $\Theta_{j+\frac{1}{2}}^- \in [0, 1]$

$$= \hat{f}_{j+\frac{1}{2}}^{\text{LF}} + \Theta_{j+\frac{1}{2}}^- \left[\hat{f}_{j+\frac{1}{2}}^{\text{RK}} - \hat{f}_{j+\frac{1}{2}}^{\text{LF}} \right] \triangleq \hat{h}_{j+\frac{1}{2}}$$

aim: $\bar{u}_j^{nH} := \bar{u}_j^n - \lambda \left[\hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \right] \in [m, M]$ choose $\Theta_{j+\frac{1}{2}}^-$?

$$\bar{U}_j^n - \lambda [\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}] = \bar{U}_j^n - \lambda [\hat{f}_{j+\frac{1}{2}}^{LF} - \hat{f}_{j-\frac{1}{2}}^{LF}] - \lambda \theta_{j+\frac{1}{2}}^- h_{j+\frac{1}{2}} + \lambda \theta_{j-\frac{1}{2}}^+ h_{j-\frac{1}{2}}$$

$m \leq \downarrow \leq M$ if $\{\bar{U}_j^n\} \subset [m, M]$

$$\text{set } \Gamma_j^M = M - [\bar{U}_j^n - \lambda (\hat{f}_{j+\frac{1}{2}}^{LF} - \hat{f}_{j-\frac{1}{2}}^{LF})] \geq 0$$

$$\Gamma_j^m = m - [\bar{U}_j^n - \lambda (\hat{f}_{j+\frac{1}{2}}^{LF} - \hat{f}_{j-\frac{1}{2}}^{LF})] \leq 0$$

$$\Rightarrow \text{aim: } \begin{cases} -\lambda \theta_{j+\frac{1}{2}}^- h_{j+\frac{1}{2}} + \lambda \theta_{j-\frac{1}{2}}^+ h_{j-\frac{1}{2}} - \Gamma_j^M \leq 0 & (*) \\ -\lambda \theta_{j+\frac{1}{2}}^- h_{j+\frac{1}{2}} + \lambda \theta_{j-\frac{1}{2}}^+ h_{j-\frac{1}{2}} - \Gamma_j^m \geq 0 & (**) \end{cases}$$

$$(*) : \text{choose } \theta_{j-\frac{1}{2}}^{+,M}, \theta_{j+\frac{1}{2}}^{-,M}$$

$$\text{same for } (**) \text{ choose } \theta_{j-\frac{1}{2}}^{+,m}, \theta_{j+\frac{1}{2}}^{-,m}$$

$$1. \text{ If } h_{j+\frac{1}{2}} \geq 0, h_{j-\frac{1}{2}} \leq 0, \text{ any } \theta_{j\pm\frac{1}{2}} \text{ is OK.}$$

$$\text{choose } (\theta_{j-\frac{1}{2}}^{+,M}, \theta_{j+\frac{1}{2}}^{-,M}) = (1, 1)$$

$$2. \text{ If } h_{j+\frac{1}{2}} \leq 0, \underline{h_{j-\frac{1}{2}} \leq 0} \text{ bad}$$

$$\text{choose } \theta_{j-\frac{1}{2}}^{+,M} = 1, \theta_{j+\frac{1}{2}}^{-,M} = \min(1, \frac{\Gamma_j^M}{-\lambda h_{j+\frac{1}{2}}})$$

$$3. \text{ If } h_{j+\frac{1}{2}} \geq 0, \underline{h_{j-\frac{1}{2}} \geq 0} \text{ bad}$$

$$\text{choose } \theta_{j+\frac{1}{2}}^{-,M} = 1, \theta_{j-\frac{1}{2}}^{+,M} = \min(1, \frac{\Gamma_j^M}{\lambda h_{j-\frac{1}{2}}})$$

$$4. \text{ If } \underline{h_{j+\frac{1}{2}} \leq 0}, \underline{h_{j-\frac{1}{2}} \geq 0} \text{ bad}$$

$$\text{choose } \begin{cases} \theta_{j+\frac{1}{2}}^{-,M} = 1, \theta_{j-\frac{1}{2}}^{+,M} = 1 & \text{if small enough} \\ \theta_{j-\frac{1}{2}}^{+,M} = \frac{\Gamma_j^M}{\lambda(h_{j-\frac{1}{2}} - h_{j+\frac{1}{2}})}, \theta_{j+\frac{1}{2}}^{-,M} = \frac{\Gamma_j^M}{\lambda(h_{j-\frac{1}{2}} - h_{j+\frac{1}{2}})} \end{cases}$$

$$\text{Then choose } \theta_{j-\frac{1}{2}}^+ = \min(\theta_{j-\frac{1}{2}}^{+,m}, \theta_{j-\frac{1}{2}}^{+,M}), \theta_{j+\frac{1}{2}}^- = \min(\theta_{j+\frac{1}{2}}^{-,m}, \theta_{j+\frac{1}{2}}^{-,M})$$

FV for 1D system $\vec{u}_t + \vec{f}(\vec{u})_x = \vec{0}$

semi-discrete $\frac{d}{dt} \bar{u}_j(t) + \frac{1}{\Delta x} [\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}] = 0$

flux $\hat{f}_{j+\frac{1}{2}} = \hat{f}(\vec{u}_{j+\frac{1}{2}}^-, \vec{u}_{j+\frac{1}{2}}^+)$

$\hat{f}$: exact / approximate Riemann solver: Roe, HLL, LF ...

$\vec{u}_{j+\frac{1}{2}}^\pm$: reconstruction from $\{\vec{u}_j^n\}$

1. component-wise reconstruction.

for each component of $\bar{u}$, apply the scalar fixed-stencil / ENO / WENO procedure.

not good for high order scheme. may appear oscillation.

2. characteristic-wise reconstruction (better)

Linear system $u_t + Au_x = 0$.

$$\Lambda = \text{diag}(\lambda_1 \dots \lambda_n) \quad R = (r_1 \dots r_n) \quad AR = R\Lambda$$

$$u_t + R\Lambda R^{-1}u_x = 0 \implies v = R^{-1}u \quad v_t + \Lambda v_x = 0.$$

$$\{\bar{u}_j^n\} \longrightarrow \bar{v}_j = R^{-1}\bar{u}_j \xrightarrow{\text{scalar reconstruction. (nonlinear)}} v_{j+\frac{1}{2}}^\pm \longrightarrow u_{j+\frac{1}{2}}^\pm = R^{-1}v_{j+\frac{1}{2}}^\pm$$

nonlinear system $u_t + f(u)_x = 0 \quad u_t + A(u)u_x = 0.$

$$\left. \begin{aligned} \Lambda(u) &= \text{diag}(\lambda_1(u) \dots \lambda_n(u)) \\ R(u) &= (r_1(u) \dots r_n(u)) \end{aligned} \right\} A(u) = R(u) \Lambda(u) R^{-1}(u)$$

$$u_t + R(u) \Lambda(u) R^{-1}(u) u_x = 0. \Rightarrow (R^{-1}(u) u)_t + \Lambda(u) (R^{-1}(u) u)_x = 0.$$

$A(u^*) = R(u^*) \Lambda(u^*) R^{-1}(u^*)$ freeze at some u^* .

$$(R^{-1}(u^*) u)_t + \Lambda(u^*) (R^{-1}(u^*) u)_x = 0$$

$$\Rightarrow V_t + \Lambda(u^*) V_x = 0 \quad \text{with } V = R^{-1}(u^*) u$$

At each fixed $x_{j+\frac{1}{2}} \Rightarrow u_{j+\frac{1}{2}}^{\pm}$

(1) "freeze" $R(u)$, $R^{-1}(u)$ and $\Lambda(u)$ locally at $x_{j+\frac{1}{2}}$.

$$\text{choose } u_{j+\frac{1}{2}} = \frac{1}{2} (\bar{u}_j + \bar{u}_{j+1})$$

$$\text{or } f(\bar{u}_{j+1}) - f(\bar{u}_j) = f'(u_{j+\frac{1}{2}}) (\bar{u}_{j+1} - \bar{u}_j) \quad (\text{Roe average})$$

$$(2) \text{ take } R = R(u_{j+\frac{1}{2}}), \quad R^{-1} = R^{-1}(u_{j+\frac{1}{2}}), \quad \Lambda = \Lambda(u_{j+\frac{1}{2}})$$

(3) transform $\bar{u}$ to the local characteristic field.

$$\bar{V}_\ell = R^{-1} \bar{u}_\ell, \quad \ell = j-p, \dots, j+q$$

(4) component-wise reconstruction. $\{\bar{V}_\ell\} \rightarrow v_{j+\frac{1}{2}}^{\pm}$

(5) transform back into the physical space $u_{j+\frac{1}{2}}^{\pm} = R v_{j+\frac{1}{2}}^{\pm}$

2D problem FV

$$u_t + f(u)_x + g(u)_y = 0.$$

Cartesian grids $I_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$

cell average

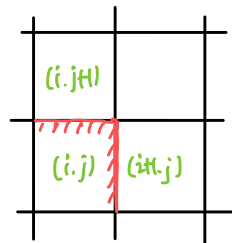
$$\tilde{u}_{ij} = \frac{1}{\Delta x \Delta y} \iint_{I_{ij}} u(x, y) dx dy \quad \left\{ \begin{array}{l} - : \text{average in } x\text{-direction} \\ \sim : \text{average in } y\text{-direction.} \end{array} \right.$$

Fully discrete flux-difference methods: Integrate on $(x, y, t) \in I_{ij} \times [t^n, t^{n+1}]$

$$\tilde{u}_{ij}^{n+1} = \tilde{u}_{ij}^n - \frac{\Delta t}{\Delta x} [F_{i+\frac{1}{2}, j}^n - F_{i-\frac{1}{2}, j}^n] - \frac{\Delta t}{\Delta y} [G_{i, j+\frac{1}{2}}^n - G_{i, j-\frac{1}{2}}^n]$$

$$F_{i+\frac{1}{2}, j}^n = \frac{1}{\Delta t \Delta y} \int_{t^n}^{t^{n+1}} dt \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}}, y, t)) dy$$

$$G_{i, j+\frac{1}{2}}^n = \frac{1}{\Delta t \Delta x} \int_{t^n}^{t^{n+1}} dt \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(u(x, y_{j+\frac{1}{2}}, t)) dx$$



Godunov method: $F_{i+\frac{1}{2}, j}^n \approx \hat{F}_{i+\frac{1}{2}, j}^n = \hat{f}(\tilde{u}_{i, j}, \tilde{u}_{i+\frac{1}{2}, j})$
 $G_{i, j+\frac{1}{2}}^n \approx \hat{G}_{i, j+\frac{1}{2}}^n = \hat{g}(\tilde{u}_{i, j}, \tilde{u}_{i, j+\frac{1}{2}})$

1st order accuracy

semi-discrete methods with RK.

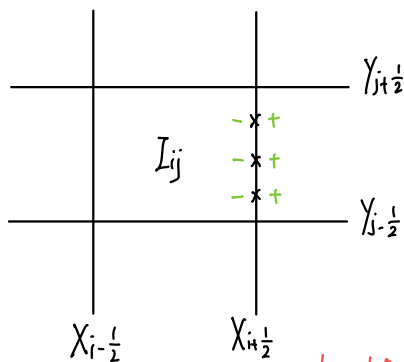
$$\frac{d}{dt} \tilde{u}_j(t) = -\frac{1}{\Delta x \Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}}, y, t)) - f(u(x_{i-\frac{1}{2}}, y, t)) dy$$

$$- \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(u(x, y_{j+\frac{1}{2}}, t)) - f(u(x, y_{j-\frac{1}{2}}, t)) dx$$

$$= -\frac{1}{\Delta x} [F_{i+\frac{1}{2}, j} - F_{i-\frac{1}{2}, j}] - \frac{1}{\Delta y} [G_{i, j+\frac{1}{2}} - G_{i, j-\frac{1}{2}}]$$

$$F_{i+\frac{1}{2}, j} = \frac{1}{\Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f(u(x_{i+\frac{1}{2}}, y, t)) dy = \tilde{f}(u(x_{i+\frac{1}{2}}, y_j, t)) \neq f(\tilde{u}(x_{i+\frac{1}{2}}, y_j, t))$$

(f is nonlinear)



numerical flux:

(Gauss quadrature) $\hat{f}_{i+\frac{1}{2}, j} = \sum_{\alpha} w_{\alpha} h(u(x_{i+\frac{1}{2}}^-, y_j + \beta_{\alpha} \Delta y), u(x_{i+\frac{1}{2}}^+, y_j + \beta_{\alpha} \Delta y))$

$\hat{g}_{i, j+\frac{1}{2}} = \sum_{\alpha} w_{\alpha} h(u(x_i + \beta_{\alpha} \Delta x, y_{j+\frac{1}{2}}^-), u(x_i + \beta_{\alpha} \Delta x, y_{j+\frac{1}{2}}^+))$

← h : 1D Riemann Solver for $u_t + f(u)_x = 0$

← h : 1D Riemann Solver for $u_t + g(u)_y = 0$

2D-reconstruction

Given cell average $\{\tilde{u}_{ij}\}$, find a polynomial $p_{ij}(x,y)$ degree $\leq k$.

s.t. $p_{ij}(x,y) = u(x,y) + \underbrace{O(\Delta)^{k+1}}$

$(x,y) \in I_{ij}, \Delta = \max(\Delta x, \Delta y)$

$\Rightarrow u_{i-\frac{1}{2},j}^+ = p_{ij}(x_{i-\frac{1}{2}}, y_j) \quad u_{i,j-\frac{1}{2}}^+ = p_{ij}(x_i, y_{j-\frac{1}{2}})$

$u_{i+\frac{1}{2},j}^- = p_{ij}(x_{i+\frac{1}{2}}, y_j) \quad u_{i,j+\frac{1}{2}}^- = p_{ij}(x_i, y_{j+\frac{1}{2}})$

$\frac{1}{\Delta x \Delta y} \iint_{I_{ij}} p_{ij}(x,y) dx dy = \tilde{u}_{ij}, \quad \forall I_{ij} \in S_{ij}$

$P^k = \text{span}\{x^i y^j, 0 \leq i,j \leq k, i+j \leq k\}$

$Q^k = \text{span}\{x^i y^j, 0 \leq i,j \leq k\}$

P^3 Q^3

1	x	x ²
y	xy	x ² y
y ²	xy ²	x ² y ²

$\dim P^k = \frac{(k+1)(k+2)}{2}$

$\dim Q^k = (k+1)^2$

remark: 1. Not all candidate stencils can be used to obtain a polynomial $p(x,y)$ of degree at most k satisfying the condition.

eg: $k=1, p(x,y) = \alpha + \beta x + \gamma y \quad S = \{I_{i-1,j}, I_{i,j}, I_{i+1,j}\}$

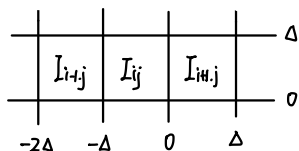
	$I_{i-1,j}$	$I_{i,j}$	$I_{i+1,j}$
-2Δ	$-\Delta$	0	Δ

$\begin{cases} \tilde{u}_{i-1,j} = \alpha - \frac{3}{2}\beta\Delta + \frac{1}{2}\gamma\Delta \\ \tilde{u}_{i,j} = \alpha - \frac{1}{2}\beta\Delta + \frac{1}{2}\gamma\Delta \\ \tilde{u}_{i+1,j} = \alpha + \frac{1}{2}\beta\Delta + \frac{1}{2}\gamma\Delta \end{cases}$

singular systems for (α, β, γ)

2. Even if one obtains a polynomial $p(x,y)$. there is no guarantee that the accuracy conditions will hold.

eg: $k=1$. $p(x,y) = \alpha + \beta x + \gamma y$ $S = \{I_{i-1,j}, I_{i,j}, I_{i+1,j}\}$

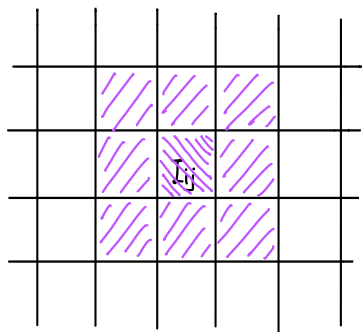


$u(x,y)=0$. $p(x,y) = \Delta - 2y$ (not unique solution, $p(x,y)=0$)
 $u(x,0) - p(x,0) = -\Delta \neq O(\Delta^2)$

choose Q^k . $p(x,y) \in Q^k$

stencil. $S_{rs}(i,j) = \{I_{i+l,j+m}, l=-r \dots -r+k, m=-s \dots -s+k\}$

$U(x,y) = \int_{-\infty}^x \int_{-\infty}^y u(s,\eta) ds d\eta$



$\{\tilde{u}_{ij}\} \rightarrow U(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) \xrightarrow{\text{interpolation}} P_{(x,y)} \in Q^{k+1} \longrightarrow p(x,y) = \frac{\partial^2 P}{\partial x \partial y} \in Q^k$
 $u(x,y) = p(x,y) + O(\Delta^{k+1})$

1D-reconstruction

$$\{\tilde{u}_{ij}\} \xrightarrow{\text{average in x-dir}} \bar{u}_i(y) \xrightarrow{\text{recon in x-dir}} u(x, y)$$

recon in y-dir

or

$$\begin{aligned} \{\tilde{u}_{ij}\} &\xrightarrow{x} \{\tilde{u}_{i+\frac{1}{2},j}^{\pm}\} \xrightarrow{y} \{u_{i+\frac{1}{2},j}^{\pm}\} \rightarrow \{\hat{f}_{i+\frac{1}{2},j}\} \\ \{\tilde{u}_{ij}\} &\xrightarrow{y} \{\bar{u}_{i,j+\frac{1}{2}}^{\pm}\} \xrightarrow{x} \{u_{i,j+\frac{1}{2}}^{\pm}\} \rightarrow \{\hat{g}_{i,j+\frac{1}{2}}\} \end{aligned}$$

remark: 1. $\text{cost}(2D) = 2 \text{cost}(1D)$

for WENO, the weights are different for each direction.

2. use polynomial in $\mathcal{Q}^k$ is waste

Splitting method:

$$u_t + a u_x + b u_y = 0, \quad u(x, y, t) = u(x - at, y - bt, 0)$$

$$\begin{cases} u_t''' + a u_x''' = 0 & u'''(x, y, t) = u'''(x - at, y, 0) \\ u_t'' + b u_y'' = 0 & u''(x, y, t) = u''(x, y - bt, 0) \end{cases}$$

$$\text{let } u'''(x, y, 0) = u'''(x, y, 0) \Rightarrow \underline{u'''(x, y, t)}$$

$$\text{let } u''(x, y, 0) = \underline{u''(x, y, t)} \Rightarrow u''(x, y, t) \quad \text{it's easy to check } u'(x, y, t) = u'(x, y, t)$$

S_t^X, S_t^Y : 1D problems. the solution operator.

S_t : 2D solution operator.

$$U(x, y, t) = S_t U(x, y, 0) = S_t^Y S_t^X U(x, y, 0)$$

$$S_t^X : U''(x, y, t) = U''(x, y, 0) + t \partial_x U''(x, y, 0) + \frac{t^2}{2} \partial_{xx} U''(x, y, 0) + \dots$$

$$= e^{-\alpha t \partial_x} U''(x, y, 0)$$

$$\Rightarrow S_t^X = e^{-\alpha t \partial_x} \quad S_t^Y = e^{-\beta t \partial_y} \quad S_t = e^{-t(\alpha \partial_x + \beta \partial_y)}$$

$$S_t^X \rightarrow H_t^X \quad k\text{-th order accuracy} \quad U''' = H_t U'' = H_t^Y H_t^X U'' \quad (k\text{-th order})$$

$$S_t^Y \rightarrow H_t^Y$$

$$Q: S_t^Y S_t^X = S_t ?$$

$$S_t^Y S_t^X = e^{-t\beta \partial_y} e^{-t\alpha \partial_x}$$

$$= [I - t\beta \partial_y + \frac{t^2}{2} \beta^2 \partial_{yy} + \dots] [I - t\alpha \partial_x + \frac{t^2}{2} \alpha^2 \partial_{xx} + \dots]$$

$$= I - t(\beta \partial_y + \alpha \partial_x) + \frac{t^2}{2} [\beta^2 \partial_{yy} + \alpha^2 \partial_{xx} + \underbrace{2\beta\alpha \partial_y \partial_x}_{\neq (\beta\alpha + \alpha\beta)}] + O(t^3)$$

$$= S_t + \frac{t^2}{2} (\beta\alpha - \alpha\beta) \partial_y \partial_x + O(t^3)$$

$$\text{if } AB = BA, \quad S_t^Y S_t^X = S_t \quad \text{otherwise, } S_t^Y S_t^X = S_t + O(t^2)$$

$$\Rightarrow O(\alpha t^2) \text{ in each time step. } \Rightarrow \text{truncation error } O(\alpha t)$$

1st order method

second order method. strong splitting

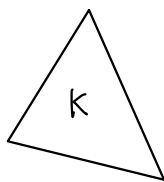
$$e^{-\frac{1}{2}\Delta t A \partial_x} e^{-\Delta t B \partial_y} e^{-\frac{\Delta t}{2} A \partial_x} = e^{-\Delta t (A \partial_x + B \partial_y)} + \underline{O(\Delta t^3)} \Rightarrow \text{2nd order method}$$

$$\begin{aligned} U^n &= e^{-\frac{1}{2}\Delta t A \partial_x} e^{-\Delta t B \partial_y} e^{-\frac{\Delta t}{2} A \partial_x} U^{n-1} \\ &= e^{-\frac{1}{2}\Delta t A \partial_x} e^{-\Delta t B \partial_y} \underbrace{e^{-\frac{\Delta t}{2} A \partial_x} e^{-\frac{1}{2}\Delta t A \partial_x}}_{e^{-\Delta t A \partial_x}} e^{-\Delta t B \partial_y} e^{-\frac{\Delta t}{2} A \partial_x} U^{n-2} \end{aligned}$$

Triangle mesh.

$$U_t + \nabla \cdot F(u) = 0 \quad F = (f(u), g(u))$$

$$\bar{U}_k = \frac{1}{|K|} \int_K u(x, y, t) dx dy$$



$$\begin{aligned} \frac{1}{|K|} \int_K U_t dx dy &= -\frac{1}{|K|} \int_K F dx dy = -\frac{1}{|K|} \int_K \vec{n} \cdot \vec{F} ds \\ &\approx -\frac{1}{|K|} |\partial K| \sum w_j F(u_j) \cdot n \approx -\frac{1}{|K|} |\partial K| \sum w_j \widehat{F_j \cdot n} \end{aligned}$$

LF flux:

$$\widehat{F_j \cdot n} = \frac{1}{2} [F(u^{in}) \cdot n + F(u^{out}) \cdot n] - \frac{C}{2} (u^{out} - u^{in})$$

C: the upper bound of eigen values of Jacobian in $\vec{n}$ direction

$$\underbrace{\vec{n} \cdot (f', g')}_{\text{matrix}} = n_1 f' + n_2 g' \quad (f', g' \text{ are matrix})$$

High order Finite difference method. (FD)

$$\text{FV: } \bar{u}_j(t) = \frac{1}{\Delta t} \int_{t_j}^{t_{j+1}} u(x, t) dx \quad u_t + f(u)_x = 0 \quad \frac{d}{dt} \bar{u}_j(t) + \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} = 0. \quad \hat{f}_{j+\frac{1}{2}} \approx f(u(x_{j+\frac{1}{2}}, t))$$

$$\text{FD: } u_j(t) \approx u(x_j, t). \quad \frac{d}{dt} u_j(t) + f(u_x)|_{x_j} = 0 \quad \{u_j\} \rightarrow p(x) \rightarrow f(u_x)|_{x_j} \approx p'(x_j)$$

This may work for smooth case.

$$\frac{d}{dt} u_j(t) + \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} = 0$$

$$\text{numerical flux: } \hat{f}_{j+\frac{1}{2}} = \hat{f}(u_{j-p}, \dots, u_{j+q})$$

$$\text{s.t. } \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} = f(u_x)|_{x_j} + O(\Delta x^k)$$

$$\text{Q. if } \hat{f}_{j+\frac{1}{2}} = f(u(x_{j+\frac{1}{2}}, t)) + O(\Delta x^k) \quad \text{work? } \times$$

$$\begin{aligned} \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} &= \frac{1}{\Delta x} \left[\cancel{f(u)|_{x_{j+\frac{1}{2}}}} + \frac{\Delta x}{2} f_x + \frac{1}{2} \left(\frac{\Delta x}{2} \right)^2 \cancel{f_{xx}} + \frac{1}{6} \left(\frac{\Delta x}{2} \right)^3 \cancel{f_{xxx}} \dots \right] - \frac{1}{\Delta x} \left[\cancel{f(u)|_{x_{j-\frac{1}{2}}}} - \frac{\Delta x}{2} f_x + \frac{1}{2} \left(\frac{\Delta x}{2} \right)^2 \cancel{f_{xx}} - \frac{1}{6} \left(\frac{\Delta x}{2} \right)^3 \cancel{f_{xxx}} + \dots \right] \\ &= f_x(u(x_j, t)) + O(\Delta x^2) + O(\Delta x^k) \end{aligned}$$

$$\text{But if } \hat{f}_{j+\frac{1}{2}} = f(u(x_{j+\frac{1}{2}}, t)) - \frac{1}{24} \Delta x^2 f_{xx}|_{x_{j+\frac{1}{2}}} + \frac{7}{5760} \Delta x^4 f_{xxxx}|_{x_{j+\frac{1}{2}}} + \dots \quad (\text{uniform mesh})$$

$$\frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} = f_x(u(x_j, t)) + O(\Delta x^k)$$

High order FD

Suppose there is a function $h(x)$. s.t. $f(u(x)) = \frac{1}{\Delta x} \int_{x-\frac{1}{2}\Delta x}^{x+\frac{1}{2}\Delta x} h(s) ds$

$$\Rightarrow f(u(x))_x = \frac{1}{\Delta x} \left[h(x+\frac{1}{2}\Delta x) - h(x-\frac{1}{2}\Delta x) \right] \quad (\text{uniform mesh})$$

We can take $\hat{f}_{j+\frac{1}{2}} = h(x_{j+\frac{1}{2}}) + O(\Delta x^k) \leftarrow O(\Delta x^k)$ is enough to keep k -th order.

$$\text{s.t. } \frac{\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}}{\Delta x} = \frac{h(x_{j+\frac{1}{2}}) - h(x_{j-\frac{1}{2}})}{\Delta x} + O(\Delta x^k) = f(u(x_j))_x + O(\Delta x^k) \quad \text{not } O(\Delta x^{k+1})$$

$$\{u_j\} \longrightarrow f(u_j) = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} h(s) ds = \bar{h}_j \xrightarrow[\text{like FV}]{\text{reconstruction}} \hat{f}_{j+\frac{1}{2}} \approx h_{j+\frac{1}{2}}^{\pm}, h_{j+\frac{1}{2}}^{\pm}$$

① upwind: compute the Roe speed $\bar{a}_{j+\frac{1}{2}} = \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j}$

$$\begin{cases} \bar{a}_{j+\frac{1}{2}} \geq 0, & \hat{f}_{j+\frac{1}{2}} = h_{j+\frac{1}{2}}^- \\ \bar{a}_{j+\frac{1}{2}} < 0, & \hat{f}_{j+\frac{1}{2}} = h_{j+\frac{1}{2}}^+ \end{cases}$$

(entropy violating shock)

② flux splitting: $f(u) = f^+(u) + f^-(u)$. with $\frac{df^+}{du} \geq 0, \frac{df^-}{du} \leq 0$.

and $f^{\pm}(u)$ have as many derivatives as the order of the scheme.

Lax-Friedrichs splitting: $f^{\pm}(u) = \frac{1}{2} [f(u) \pm \alpha u]$ $\alpha = \max(|f'|)$

$$\text{remark: } \hat{f}_{j+\frac{1}{2}}^{\text{LF}}(u^-, u^+) = \frac{1}{2} (f(u^-) + f(u^+)) - \frac{\alpha}{2} (u^+ - u^-) = f^+(u^-) + f^-(u^+)$$

Godunov / Osher flux can not be splitted.

FD splitting flux:

- (1) $f(u) = f^+(u) + f^-(u)$ (stencils like ENO-WENO)
- (2) $f^+(u)$. take $\bar{h}_j = f^+(u_j) \xrightarrow{\text{reconstruction}} h_{j+\frac{1}{2}}^+ \Rightarrow \hat{f}_{j+\frac{1}{2}}^+ = h_{j+\frac{1}{2}}^+$ upwind $\Rightarrow$
- (3) $f^-(u)$. take $\bar{h}_j = f^-(u_j) \xrightarrow{\text{reconstruction}} h_{j+\frac{1}{2}}^- \Rightarrow \hat{f}_{j+\frac{1}{2}}^- = h_{j+\frac{1}{2}}^-$ downwind $\Leftarrow$
- (4) $\hat{f}_{j+\frac{1}{2}} = \hat{f}_{j+\frac{1}{2}}^+ + \hat{f}_{j+\frac{1}{2}}^-$

FD-WENO Framework

- (1) $\{u_j^n\}$ on $\{x_j\}$ $f = f^{(L)} + f^{(R)}$
 - (2) $\bar{h}_j^{(L)} = f^{(L)}(u_j^n)$ $\bar{h}_j^{(R)} = f^{(R)}(u_j^n)$ $f^{(L)} \Rightarrow f^{(R)}$
 $\frac{df^{(L)}}{du} \geq 0$ $\frac{df^{(R)}}{du} \leq 0$
 - (3) $\{\bar{h}_j^{(L)}\} \xrightarrow{\text{ENO-WENO}} h_{j+\frac{1}{2}}^{(L)}$ $\hat{f}_{j+\frac{1}{2}}^{(L)} = h_{j+\frac{1}{2}}^{(L)}$
 $\{\bar{h}_j^{(R)}\} \xrightarrow{\text{ENO-WENO}} h_{j-\frac{1}{2}}^{(R)}$ $\hat{f}_{j-\frac{1}{2}}^{(R)} = h_{j-\frac{1}{2}}^{(R)}$ $\hat{f}_{j+\frac{1}{2}} = \hat{f}_{j+\frac{1}{2}}^{(L)} + \hat{f}_{j+\frac{1}{2}}^{(R)}$
- $\bar{h}_{j-1}^{(L)}$ $\bar{h}_j^{(L)}$ $\bar{h}_{j+1}^{(L)}$ $\bar{h}_{j-\frac{1}{2}}^{(R)}$ $\bar{h}_{j+\frac{1}{2}}^{(R)}$
 x_{j-1} x_j x_{j+1}
 $h_{j-\frac{1}{2}}^{(L)}$ $h_{j+\frac{1}{2}}^{(R)}$
- (4) $u_j^{MH} = u_j^n - \frac{\Delta t}{\Delta x} [\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}]$
- $f^{(L)} = \frac{f(u) + \alpha u}{2}$
 $f^{(R)} = \frac{f(u) - \alpha u}{2}$

1D systems $\vec{u}_t + \vec{f}(\vec{u})_x = \vec{0}$. $\vec{f}(\vec{u}) = \vec{f}^+(\vec{u}) + \vec{f}^-(\vec{u})$

Jacobian $\frac{d\vec{f}}{d\vec{u}}$ is diagonalizable. and have only non-negative eigenvalues, respectively.

LF splitting : $\vec{f}^\pm(\vec{u}) = \frac{1}{2}[\vec{f}(\vec{u}) \pm \alpha \vec{u}]$. $\alpha = \max(|\lambda_i(\frac{d\vec{f}}{d\vec{u}})|)$

FD splitting flux: $(\hat{f}_{j+\frac{1}{2}})$

(1) compute the average state $u_{j+\frac{1}{2}}$.

(2) set $R = R(u_{j+\frac{1}{2}})$. $R^{-1} = R^{-1}(u_{j+\frac{1}{2}})$. $\Lambda = \Lambda(u_{j+\frac{1}{2}})$

(3) transform to the local characteristic fields.

$$\vec{V}_\ell = R^{-1} \vec{u}_\ell \quad \vec{g}_\ell = R^{-1} \vec{f}_\ell \quad \ell = j-p, \dots, j+q$$

(4) flux splitting $g_\ell^\pm = \frac{1}{2}[g_\ell \pm \alpha V_\ell]$

(5) reconstruction on each component of $g^\pm$

$$\left. \begin{array}{l} \bar{h}_\ell = g_\ell^+ \rightarrow \hat{g}_{j+\frac{1}{2}}^+ = h_{j+\frac{1}{2}}^- \\ \bar{h}_\ell = g_\ell^- \rightarrow \hat{g}_{j+\frac{1}{2}}^- = h_{j+\frac{1}{2}}^+ \end{array} \right\} \Rightarrow \hat{g}_{j+\frac{1}{2}} = \hat{g}_{j+\frac{1}{2}}^+ + \hat{g}_{j+\frac{1}{2}}^-$$

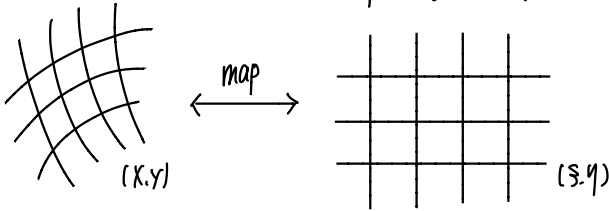
(6) $\hat{f}_{j+\frac{1}{2}} = R \hat{g}_{j+\frac{1}{2}}$

2D : $U_t + U_x + U_y = 0$.

uniform mesh + FD ... ✓

triangle mesh : X FD can not be used on unconstructed mesh !

But we can use map to get triangle mesh. (change the form of equation, too)



Hw : 5th WENO + FD ddl 06/01

		FV	FD
1D	conservative	✓	✓
	accuracy	✓	✓
	cost	✓	✓
	mesh	<u>general mesh</u>	uniform mesh
	monotone flux	<u>all</u> general case	only flux splitting (eg: LF)
2D	conservative	✓	✓
	accuracy	✓	✓
	cost	$\geq 2 \times 1D$ cost	<u>$2 \times 1D$ cost</u> cost less
	mesh	<u>arbitrary</u>	uniform rectangle mesh / smooth curvilinear
	monotone flux	<u>all</u>	only flux splitting (eg: LF)

DG (FE)

$$I_j = [X_{j-\frac{1}{2}}, X_{j+\frac{1}{2}}], \quad X_j = \frac{X_{j-\frac{1}{2}} + X_{j+\frac{1}{2}}}{2}, \quad \Delta X_j = |I_j|, \quad h = \max_j \Delta X_j$$

$$u_t + f(u)_x = 0.$$

$$\Rightarrow 0 = \int_{I_j} u_t v - \int_{I_j} f(u) v_x + f(u) v \Big|_{X_{j+\frac{1}{2}}} - f(u) v \Big|_{X_{j-\frac{1}{2}}} = 0$$

$$\text{DG method: } V_h = \{v : v|_{I_j} \in P^k(I_j)\} \quad (k=0, V_h=1. \Rightarrow \text{FV})$$

$$\text{Find } u_h \in V_h. \text{ s.t. } \forall v_h \in V_h.$$

$$0 = \int_{I_j} (u_h)_t v_h - \int_{I_j} f(u_h) (v_h)_x + \hat{f}_{j+\frac{1}{2}} v_h(X_{j+\frac{1}{2}}^-) - \hat{f}_{j-\frac{1}{2}} v_h(X_{j-\frac{1}{2}}^+) = 0$$

$$\text{where } \hat{f}_{j+\frac{1}{2}} = \hat{f}(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+).$$

$\hat{f}(u^-, u^+)$ satisfies: monotone flux $\hat{f}(\uparrow, \downarrow)$ / Approximate Riemann Solver
LF. Osher. Godunov.

let $V_h=1 \Rightarrow$ conservation

Stability (boundness in L^2 -norm)

$U_h(x,t)$, $W_h(x,t)$ are both solutions of the same scheme with different initial condition.
then :

$$\|U_h(\cdot, t) - W_h(\cdot, t)\|_{L^2} \leq K e^{\alpha t} \|U_h(\cdot, 0) - W_h(\cdot, 0)\|_{L^2}$$

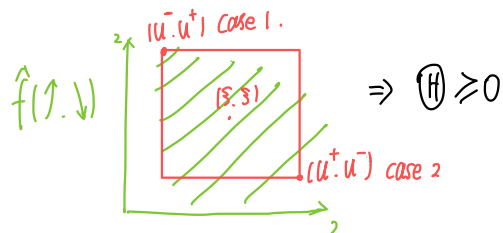
pf: take $V_h = U_h$.

$$0 = \int_{I_j} (U_h)_t U_h - \int_{I_j} f(U_h) U_{hx} + \hat{f}_{j+\frac{1}{2}}(U_h)_{j+\frac{1}{2}}^- - \hat{f}_{j-\frac{1}{2}}(U_h)_{j-\frac{1}{2}}^+ = 0$$

$$\text{let } g(u) = \int f(s) ds, \quad g'(u) = f(u). \quad \int_{I_j} f(U_h) U_{hx} = g(U_h)_{j+\frac{1}{2}}^- - g(U_h)_{j-\frac{1}{2}}^+$$

$$\Rightarrow 0 = \frac{d}{dt} \int_{I_j} \left(\frac{U_h}{2}\right)^2 + \hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}} \quad \begin{cases} \hat{H}_{j+\frac{1}{2}} = -g(U_h)_{j+\frac{1}{2}}^- + \hat{f}_{j+\frac{1}{2}}(U_h)_{j+\frac{1}{2}}^- \\ \hat{H}_{j-\frac{1}{2}} = -g(U_h)_{j-\frac{1}{2}}^- + \hat{f}_{j-\frac{1}{2}}(U_h)_{j-\frac{1}{2}}^- \\ \Theta_{j-\frac{1}{2}} = -g(U_h)_{j-\frac{1}{2}}^- + g(U_h)_{j-\frac{1}{2}}^+ \\ \quad + \hat{f}_{j-\frac{1}{2}}[(U_h)_{j-\frac{1}{2}}^- - (U_h)_{j-\frac{1}{2}}^+] \end{cases}$$

$$\begin{aligned} \Theta &= g(U_h^+) - g(U_h^-) - \hat{f}[U_h^+ - U_h^-] \\ &= [f(s) - \hat{f}(U_h^-, U_h^+)] [U_h^+ - U_h^-] \\ &= [\hat{f}(s, s) - \hat{f}(U_h^-, U_h^+)] [U_h^+ - U_h^-] \\ &= [\hat{f}(s, s) - f(s, U_h^+) + f(s, U_h^+) - f(U_h^-, U_h^+)] [U_h^+ - U_h^-] \end{aligned}$$



$$\Rightarrow \frac{d}{dt} \int_{I_j} \left(\frac{u_h^2}{2} \right) + \hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}} \leq 0$$

sum $j=1 \dots N$

$$\frac{1}{2} \frac{d}{dt} \int_a^b (u_h)^2 + \underbrace{\hat{H}_{N+\frac{1}{2}} - \hat{H}_{\frac{1}{2}}}_{=0} \leq 0$$

$$\Rightarrow \frac{d}{dt} \|u_h(\cdot, t)\|_{L^2} \leq 0 \quad \#$$

=0 for compact supported. or periodic

Remark: entropy pair. $\eta(u) = \frac{1}{2} u^2$. $\eta'(u) = u > 0$

$$\Rightarrow \eta(u)_t + H(u)_x \leq 0$$

$\Rightarrow$ cell entropy inequality

Accuracy $u_t + u_x = 0$ upwind flux for linear case. $\hat{f}(u^-, u^+) = u^-$

$$0 = \int_{I_j} (u_h)_t V_h - \int_{I_j} u_h V_{hx} + (u_h)_{j+\frac{1}{2}}^- (V_h)_{j+\frac{1}{2}}^- - (u_h)_{j-\frac{1}{2}}^- (V_h)_{j-\frac{1}{2}}^+$$

$$0 = \int_{I_j} u_t V_h - \int_{I_j} u V_{hx} + u_{j+\frac{1}{2}} (V_h)_{j+\frac{1}{2}}^- - u_{j-\frac{1}{2}} (V_h)_{j-\frac{1}{2}}^+ \quad (\text{exact solution})$$

denote $e = u - u_h = \underbrace{u - Pu}_{\varepsilon_h} + \underbrace{Pu - u_h}_{e_h \in V_h}$. $P: V \rightarrow V_h$ projection

$$e = \varepsilon_h + e_h$$

$$0 = \int_{I_j} (\varepsilon_h + e_h)_t V_h - \int_{I_j} (\varepsilon_h + e_h) V_{hx} + (\varepsilon_h + e_h)_{j+\frac{1}{2}}^- (V_h)_{j+\frac{1}{2}}^- - (\varepsilon_h + e_h)_{j-\frac{1}{2}}^- (V_h)_{j-\frac{1}{2}}^+$$

$$\Rightarrow \int_{I_j} (e_h)_t V_h - \int_{I_j} e_h V_{hx} + (e_h)_{j+\frac{1}{2}}^- (V_h)_{j+\frac{1}{2}}^- - (e_h)_{j-\frac{1}{2}}^- (V_h)_{j-\frac{1}{2}}^+$$

$$= - \int_{I_j} (\varepsilon_h)_t V_h + \int_{I_j} \varepsilon_h V_{hx} - \underbrace{(\varepsilon_h)_{j+\frac{1}{2}}^- (V_h)_{j+\frac{1}{2}}^-}_{\text{exact}} + \underbrace{(\varepsilon_h)_{j-\frac{1}{2}}^- (V_h)_{j-\frac{1}{2}}^+}_{\text{exact}}$$

take $V_h = \mathcal{E}_h$.

LHS : sum over j (same as stability) $\geq \frac{1}{2} \frac{d}{dt} \int_a^b (\mathcal{E}_h)^2 dx = \frac{1}{2} \frac{d}{dt} \|\mathcal{E}_h\|^2 = \|\mathcal{E}_h\| \frac{d}{dt} \|\mathcal{E}_h\|$

define P : s.t. $Pu \begin{cases} Pu|_{j+\frac{1}{2}} = U_{j+\frac{1}{2}} \\ \int_{I_j} (u - Pu) w_h dx = 0 \quad \forall w_h \in P^{k-1}(I_j) \end{cases}$

$$\Rightarrow \|u - Pu\|_{L^2} \leq Ch^{k+1}$$

Therefore: $\int_{I_j} \mathcal{E}_h (V_h)_x = 0$. $(\mathcal{E}_h)_{j+\frac{1}{2}} = (\mathcal{E}_h)_{j-\frac{1}{2}} = 0$

$$\Rightarrow \text{RHS} = - \int_{I_j} (\mathcal{E}_h)_t \mathcal{E}_h$$

$$\Rightarrow \text{RHS: sum over } j \leq \|(\mathcal{E}_h)_t\| \|\mathcal{E}_h\|$$

$$\Rightarrow \frac{d}{dt} \|\mathcal{E}_h\| \leq \|(\mathcal{E}_h)_t\| \leq Ch^{k+1} \Rightarrow \|\mathcal{E}_h(t)\| \leq \cancel{\|\mathcal{E}_h(0)\|} + Ct h^{k+1}$$

if we take $U_h^0 = Pu(x, 0)$.

$$\Rightarrow \|\mathcal{E}_h(t)\| \leq C(1+t) h^{k+1} \quad \#$$

if we take $U_h^0 = L^2$ projection, then $\|\mathcal{E}_h(0)\| \leq Ch^{k+1}$

limiters: $U_h^{\text{pre}} \xrightarrow{\text{limiter}} U_h \in V_h$.

$$\textcircled{1} \text{ keep } \int_{I_j} U_h = \int_{I_j} U_h^{\text{pre}}$$

$\textcircled{2}$ keep $U_h = U_h^{\text{pre}}$ as much as possible

$\textcircled{3}$ remove or reduce spurious oscillations.

ddl 06/13

Hlw: (3th order) $k=2$ DG (LF flux) repeat the Hlw

error: L^2 Gauss integrate. (and L^∞ error)

$$\left(\sum_j \int_{I_j} |u - u_h|^2 dx \right)^{\frac{1}{2}}$$

plot: midpoint in each cell

06/05 week. gap.

06/12 presentation

limiter : $U_h^{\text{pre}} \longrightarrow U_h \in V_h$.

$$(1) \int_{I_j} U_h = \int_{I_j} U_h^{\text{pre}}$$

(2) keep $U_h = U_h^{\text{pre}}$ as much as possible

(3) remove or reduce spurious oscillations

TVD/TVB limiter.

$$U_h^{\text{pre}}(x)|_{I_j} \longrightarrow U_h|_{j-\frac{1}{2}}^+ \cdot \frac{U_h^{\text{pre}}|_{j+\frac{1}{2}}^-}{U_h^{\text{pre}}|_{I_j}} \xrightarrow{\text{TVD/TVB}} U_h|_{j-\frac{1}{2}}^+ \cdot \frac{U_h|_{j+\frac{1}{2}}^-}{U_h|_{I_j}} \longrightarrow U_h(x)|_{I_j}$$

$k \leq 2$. unique solution

$k \geq 3$. not unique

Moment limiter

$$U_h(x)|_{I_j} = \sum_{l=0}^k U_j^l \mathcal{P}_j^l(x)$$

$$U_j^l = \int_{I_j} U_j(x) \mathcal{P}_j^l(x) dx \quad l=0, \dots, k.$$

$$\mathcal{P}_j^l(x) \text{ legendre on } I_j \text{ and } \int_{I_j} \mathcal{P}_j^m(x) \mathcal{P}_j^n(x) dx = \delta_{mn}$$

$$\text{do: } U_j^k \longrightarrow m(U_j^k, U_{j+1}^{k-1} - U_j^{k-1}, U_j^{k-1} - U_{j-1}^{k-1})$$

$$U_j^{k-1} \longrightarrow m(U_j^{k-1}, U_{j+1}^{k-2} - U_j^{k-2}, U_j^{k-2} - U_{j-1}^{k-2})$$

⋮

$$\downarrow U_j^0 \longrightarrow m(U_j^0, U_{j+1}^0 - U_j^0, U_j^0 - U_{j-1}^0) \quad \sim \text{TVD when } P^0$$

don't change $U_j^0 \sim \overline{U_h}|_{I_j}$

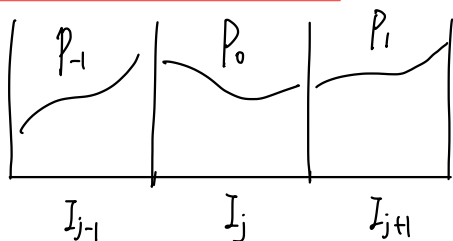
note: FV+WENO on each cell. at first.

DG+WENO: identify trouble cells

only use WENO on trouble cells

$U_h^{pre}(x) = P_0(x)$ in I_j

WENO-based limiter



Aim: $U_h(x) = W_{-1} \tilde{P}_{-1}(x) + W_0 \tilde{P}_0(x) + W_1 \tilde{P}_1(x)$ in I_j

($W_{-1} + W_0 + W_1 = 1$)

(keep $\frac{1}{\Delta x_j} \int_{I_j} U_h(x) dx = \frac{1}{\Delta x_j} \int_{I_j} U_h^{pre}(x) dx = \frac{1}{\Delta x_j} \int_{I_j} P_0(x) dx$)

$$\begin{aligned} \tilde{P}_1(x) &= P_1(x) - \frac{1}{\Delta x_j} \int_{I_j} (P_1(x) - P_0(x)) dx \\ &= P_1(x) + C_1 \end{aligned}$$

$$\text{(keep } \frac{1}{\Delta x_j} \int_{I_j} \tilde{P}_1(x) dx = \frac{1}{\Delta x_j} \int_{I_j} P_0(x) dx)$$

$$\begin{aligned} \tilde{P}_{-1}(x) &= P_{-1}(x) - \frac{1}{\Delta x_j} \int_{I_j} (P_{-1}(x) - P_0(x)) dx \\ &= P_{-1}(x) + C_{-1} \end{aligned}$$

linear weights : $d_{-1} + d_0 + d_1 = 1$. $d_i \geq 0$

$$U_h(x) = d_{-1} \hat{p}_{-1}(x) + d_0 \hat{p}_0(x) + d_1 \hat{p}_1(x) \quad \text{choose } d_0 = 1 - \frac{2}{10^n}, \quad d_1 = d_{-1} = \frac{1}{10^n}$$

Maximum-principle-preserving limiter:

supposed $U_h^n(x) \in [m, M]$. $x \in S_G$ (Gauss-Lebatto points)

$$\xrightarrow{\text{Euler forward}} U_h^{n+1, \text{pre}}(x) \begin{cases} m \leq \bar{U}_j^{n+1, \text{pre}} \leq M \\ U_j^{n+1, \text{pre}}(x) - u(x, t^{n+1}) = O(h^{k_H}) \end{cases}$$

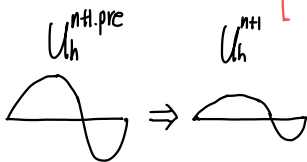
$$\xrightarrow{\text{limiter}} U_h^{n+1}(x) \begin{cases} \bar{U}_j^{n+1} = \bar{U}_j^{n+1, \text{pre}} \quad \checkmark \\ U_j^{n+1}(x) - u(x, t^{n+1}) = O(h^{k_H}) \\ \underline{U}_h^{n+1}(x) \in [m, M] \quad \forall x \in S_G \end{cases}$$

$\bar{U}_j^{n+1} = \theta_j U_h^{n+1, \text{pre}}(x) + (1 - \theta_j) \bar{U}_j^{n+1}$

set θ_j

$$\theta_j = \min \left[1, \frac{|M - \bar{U}_j^{n+1}|}{|M_j - \bar{U}_j^{n+1}|}, \frac{|m - \bar{U}_j^{n+1}|}{|m_j - \bar{U}_j^{n+1}|} \right]$$

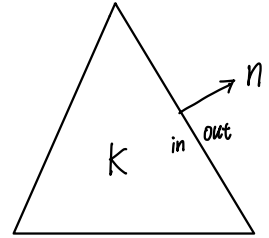
$$M_j = \max_{x \in S_G} U_h^{n+1, \text{pre}}(x) \quad m_j = \min_{x \in S_G} U_h^{n+1, \text{pre}}(x)$$



DG in $\mathbb{R}^n$

$$U_t + \nabla \cdot F(U) = 0.$$

$$\int_K (U_h)_t V_h - \int_K F(U) \cdot \nabla V_h + \int_{\partial K} \hat{F} \cdot \hat{n} \cdot V_h^{in} = 0$$



$$\hat{F} \cdot \hat{n} = \frac{1}{2} [F(u^{in}) \cdot n + F(u^{out}) \cdot n] - \frac{\alpha}{2} [u^{out} - u^{in}]$$

FD

$$u_j^n \approx u(x_j, t^n)$$

FV

$$\bar{u}_j^n \approx \frac{1}{\Delta x_j} \int_{I_j} u(x, t^n) dx$$

DG

$$u_j^n(x, t) \approx u(x, t^n)|_{I_j}$$

compact but more stage

Conservative

✓

✓

✓

accuracy

✓

✓

✓

mesh

uniform / smooth.

general

general

monotone flux

flux splitting

all

all

Compare freedoms : eg : P^k N cells DG. $\sim$ $(k+1)N$ cells FV

Multi-Dimension Cost : cost(FD) < cost(FV)

cost(DG) < cost(FV)

Final project: scheme: RK3 + FD/FV/DG. order ≥ 3 .

1. 2D Burgers' equation (periodic boundary)

$$\begin{cases} u_t + (\frac{1}{2}u^2)_x + (\frac{1}{2}u^2)_y = 0 & (x,y) \in [-2,2]^2 \quad (1) \text{ exact solution.} \\ u(x,y,0) = \frac{1}{2} + \sin(\frac{\pi}{2}(x+y)) \end{cases}$$

(2) $T = 0.5/\pi$. error table.

$T = 1.5/\pi$. plot u & u_h
 (+ limiter) $\begin{cases} \text{surface (3D)} \\ \text{cut at } x=y \text{ (2D)} \end{cases}$

2. 1D Euler equations

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0 \quad E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2 \quad \gamma = 1.4$$

$$(1) \begin{cases} \rho(x,0) = 1 + 0.2 \sin x & [-\pi, \pi] \text{ periodic} \\ u(x,0) = 1 \\ p(x,0) = 1 \end{cases}$$

初值 $\begin{cases} \rho = \alpha + \beta \sin(Wx + \varphi) \\ u = u_0 \\ p = p_0 \end{cases} \Downarrow \begin{cases} \rho = \alpha + \beta \sin(W(x - ut) + \varphi) \end{cases}$

$T = 0.6$. show the error table of ρ .

$$(2) \quad x < 0 \quad \begin{pmatrix} \rho \\ v \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad x > 0 \quad \begin{pmatrix} \rho \\ v \\ p \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0 \\ 0.1 \end{pmatrix} \quad (+ \text{limiter})$$

$x \in [-2, 2]$. plot ρ, v, p at $T = 0.6$
 (exact solution by 1st order method)

ddl: 06/20.