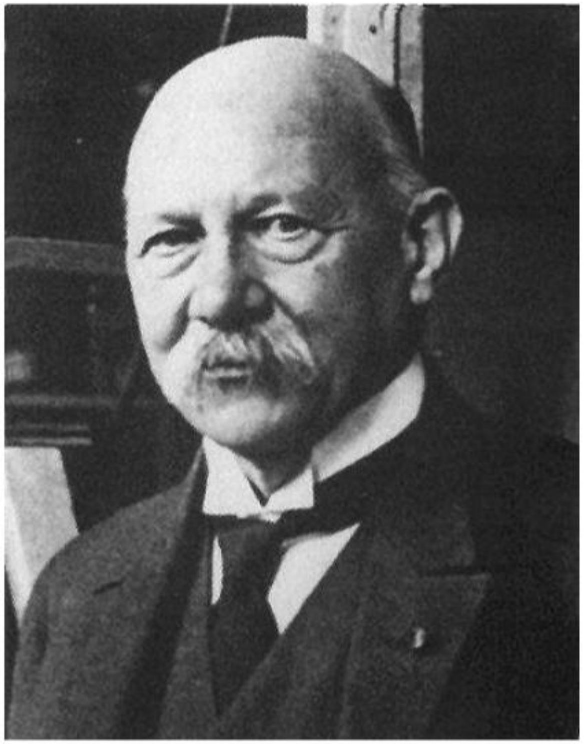
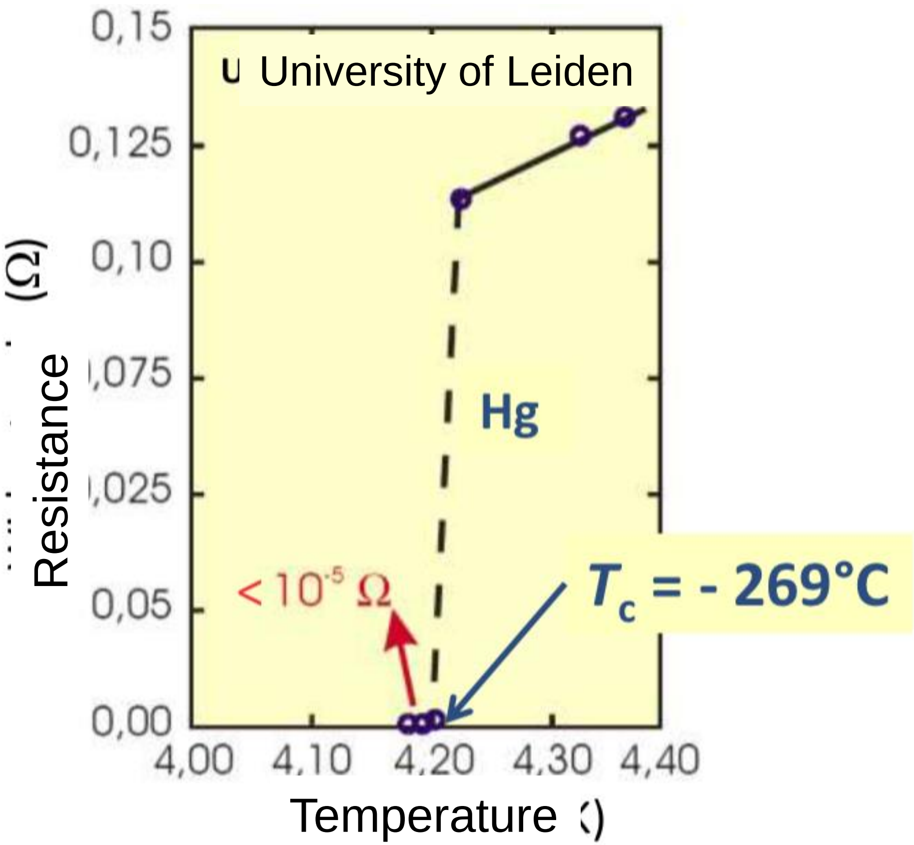
Lecture 9
Superconductors

- Superconductivity
- Semiclassical theory
- Berry curvature effects
- Summary

Discovery of Superconductivity

Heike Kammerlingh Onnes (1853-1926)



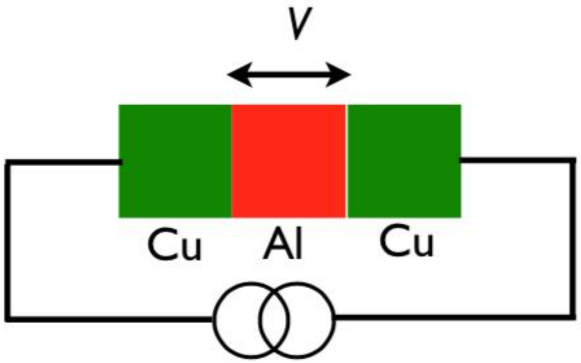


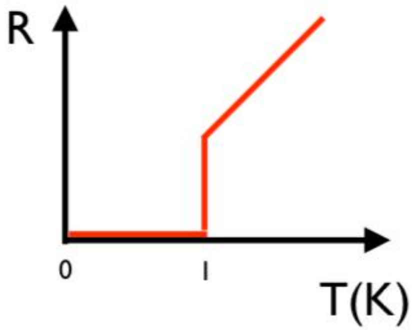
H. K. Onnes, Comm. Leiden 120b, 122b, 124c (1911)

Nobel Price in Physics 1913

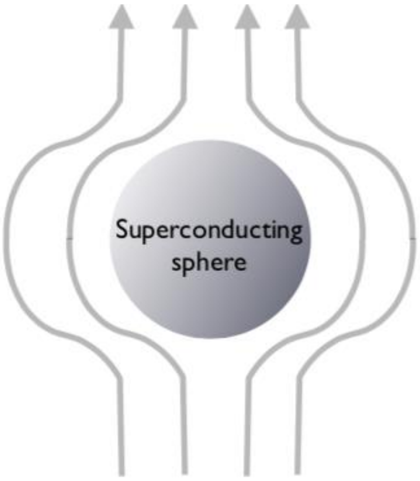
"for his investigations on the properties of matter at low temperatures which led, inter alia to the production of liquid helium"

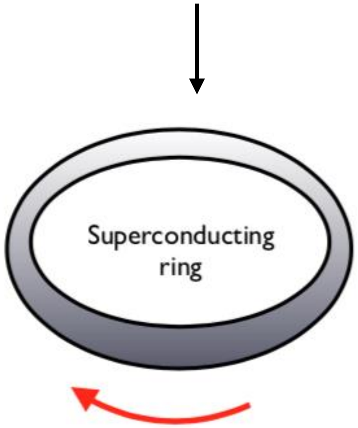
Characters of Superconductivity





Zero resistance

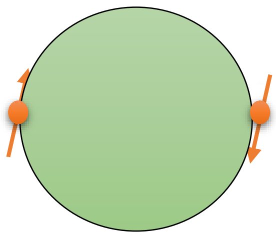




Meissner effect
(perfect diamagnetism)
equilibrium effect

Persistent currents
metastable effect

Microscopic theory



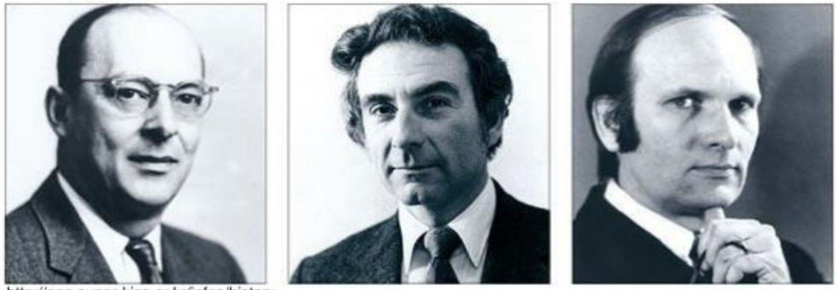
Cooper pair:

$$\phi\,(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = (\downarrow\uparrow - \downarrow\uparrow)\phi(\boldsymbol{R}, \boldsymbol{r})$$

spin-singlet $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ S or D wave

Energy can be lowered below 2 x fermi energy, if interaction is attractive, which can be mediated by phonons, etc.

Bardeen -Cooper -Schrieffer (1957)



http://eng.super-kics.or.kr/infos/history

$$\Psi = \prod_k (\mu_k + \nu_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow})|vac\rangle$$

Bogoliubov de-Gennes theory

- Interaction Hamiltonian

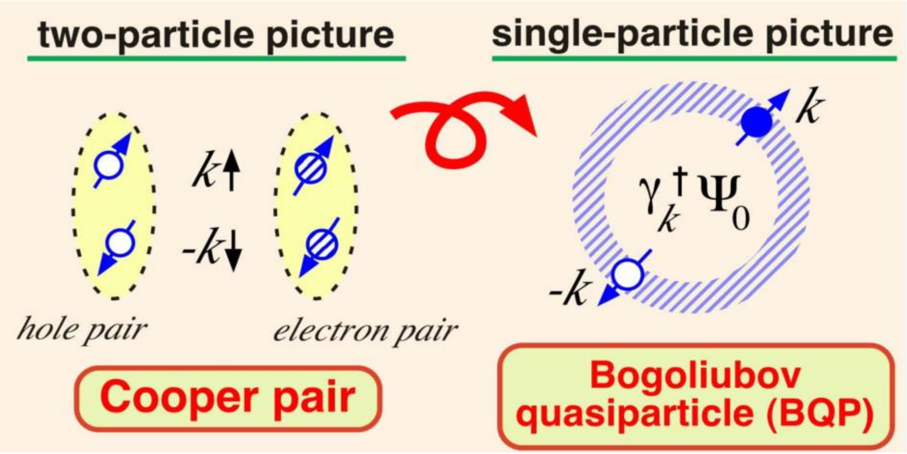
$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}$$

- Mean-field approximation

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} + \Delta^{*}_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right) + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \left\langle c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} \right\rangle$$

- Bogoliubov transformation

$$\begin{aligned} c_{\mathbf{k}\uparrow} &= u^{*}_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \gamma^{\dagger}_{-\mathbf{k}\downarrow} \\ c^{\dagger}_{-\mathbf{k}\downarrow} &= u_{\mathbf{k}} \gamma^{\dagger}_{-\mathbf{k}\downarrow} - v^{*}_{\mathbf{k}} \gamma_{\mathbf{k}\uparrow} \end{aligned}$$



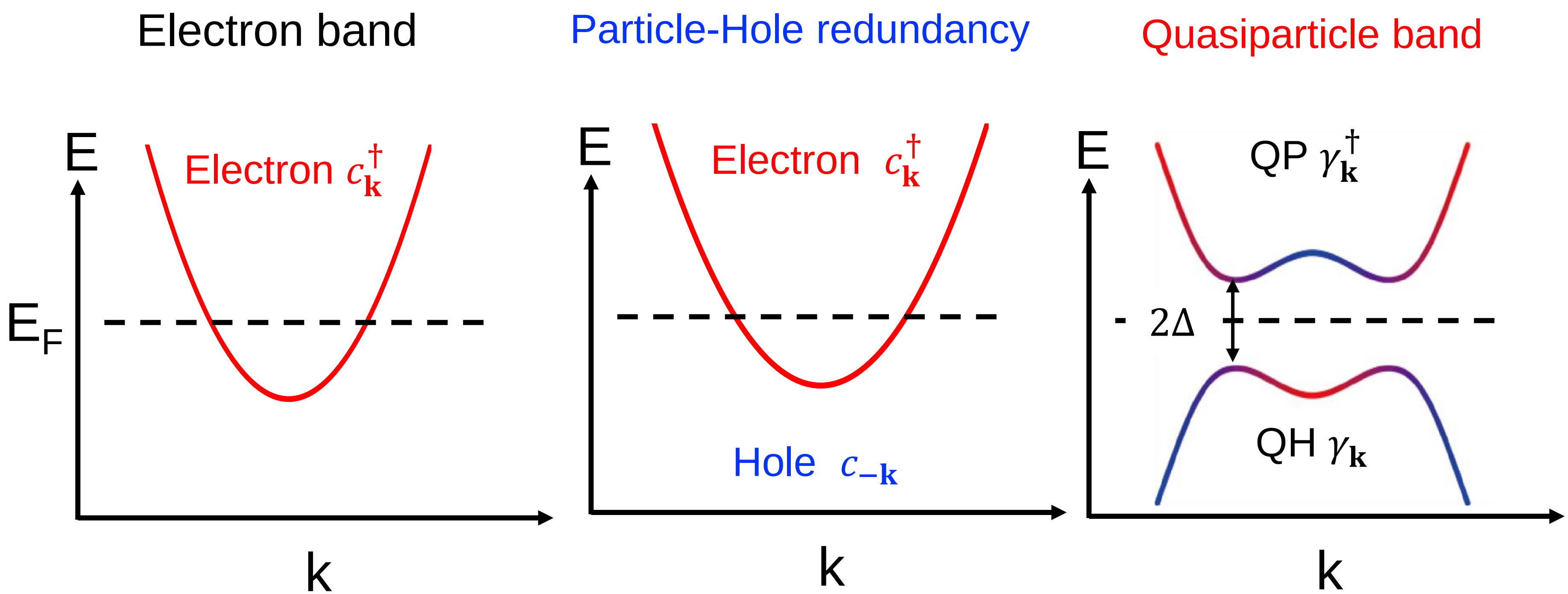
- $\gamma_{\mathbf{k}}$ obey fermionic commutation

$$\{\gamma^{\dagger}_{\mathbf{k}}, \gamma_{\mathbf{k}'}\} = 1 \quad \{\gamma^{\dagger}_{\mathbf{k}}, \gamma^{\dagger}_{\mathbf{k}}\} = 0$$

- Diagonalization

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}} (\gamma^{\dagger}_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + \gamma^{\dagger}_{\mathbf{k}\downarrow} \gamma_{\mathbf{k}\downarrow}) + \mathrm{E}_g$$

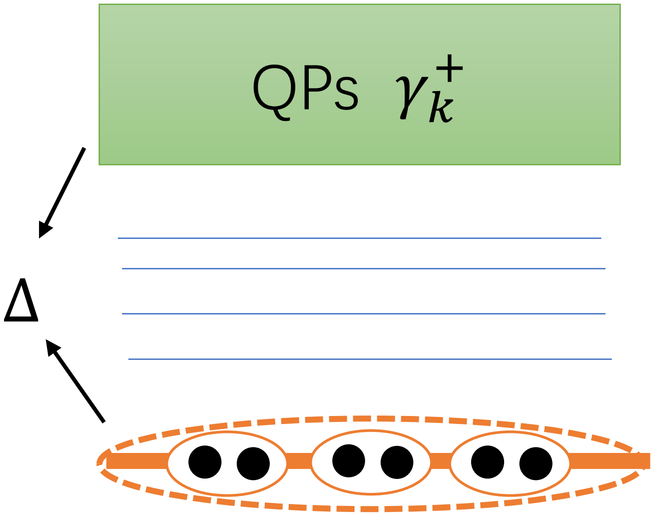
Nanbu-Dirac picture



- BCS ground state is given by filling the quasi-hole band:

$$H = \sum_{\mathbf{k}} E_{\mathbf{k}} \gamma_{\mathbf{k}\sigma}^{\dagger} \gamma_{\mathbf{k}\sigma} + E_g \longrightarrow \gamma_{\mathbf{k}\sigma} |\Psi\rangle = 0 \longrightarrow |\mathrm{BCS}\rangle = \mathcal{N} \prod_{\sigma \boldsymbol{k}} \gamma_{\sigma \boldsymbol{k}} |0\rangle$$

Topological and Geometrical effects



Cooper Pair condensation

Quasiparticles: Berry curvatures

Edge/Bound states: Majorana etc

Condensate topology: p+ip, d+id, etc

Expected applications in:

- Topological superconductors
- Non-centrosymmetric superconductors
- Triplet superconductors
- Multiband superconductors

Semiclassical quasiparticle dynamics

Construct a quasiparticle wave packet

$$|\Psi_\sigma(\mathbf{r}_c)\rangle = \int [d\mathbf{k}]\, \alpha(\mathbf{k},t)\gamma^{\dagger}_{\sigma\mathbf{k};\mathbf{r}_c}|G\rangle$$

$\boldsymbol{k}_c$ $\boldsymbol{r}_c$ Center of WP in real and momentum space

Semiclassical equations of motion and Berry curvatures:

$$\dot{\boldsymbol{r}} = \nabla_{\boldsymbol{k}}E + \dot{\boldsymbol{k}} \times \underbrace{(\nabla_k\rho \times \nabla_k\theta)}_{\boldsymbol{\Omega}_k} + \underbrace{\nabla_{\boldsymbol{k}}\left(\rho\boldsymbol{v}^s - \boldsymbol{B} \times \tilde{\boldsymbol{d}}\right) \cdot \dot{\boldsymbol{r}} - \dot{\boldsymbol{r}} \cdot \nabla_r(\rho\nabla_k\theta)}_{\boldsymbol{\Omega}_{rk}}$$

$$\dot{\boldsymbol{k}} = -\nabla_r E + \dot{\boldsymbol{r}} \times \underbrace{\left(e\rho\boldsymbol{B} - \nabla_r\rho \times \boldsymbol{v}^s + \nabla_r \times \left(\boldsymbol{B} \times \tilde{\boldsymbol{d}}\right)\right)}_{\boldsymbol{\Omega}_r} - \underbrace{\nabla_r(\rho\nabla_k\theta) \cdot \dot{\boldsymbol{k}} + \dot{\boldsymbol{k}} \cdot \nabla_{\boldsymbol{k}}\left(\rho\boldsymbol{v}^s - \boldsymbol{B} \times \tilde{\boldsymbol{d}}\right)}_{\boldsymbol{\Omega}_{kr}}$$

Z. Wang, L. Dong, C. Xiao and Q. Niu, PRL 126, 187001 (2021)

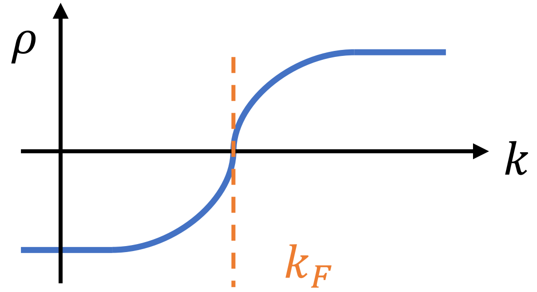
Momentum-space Berry Curvature

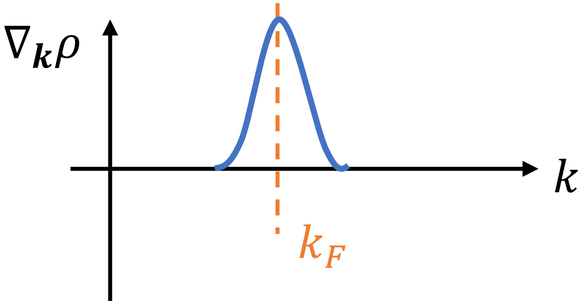
The Berry curvature in the momentum space

$$\boldsymbol{\Omega}_{\boldsymbol{k}} = -\nabla_{\boldsymbol{k}}\rho \times \nabla_{\boldsymbol{k}}\theta$$

effective charge $\rho = \dfrac{\xi_k}{E_{\boldsymbol{k}}}$ SC phase $\theta = \dfrac{1}{2}\arg\Delta$

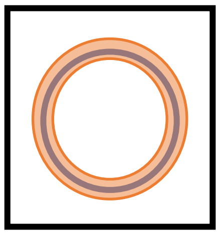
- The formula is simplified by neglecting band geometry
- Effective charge changes sharply across Fermi surface





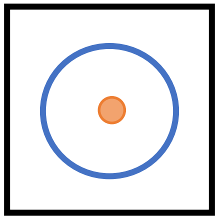
Geometry and topology

- Berry curvature $\boldsymbol{\Omega}_{\boldsymbol{k}}$ concentrates around the Fermi surface



Contrast: $\boldsymbol{B}_{\boldsymbol{k}} = -\nabla_{\boldsymbol{k}} \times \nabla_{\boldsymbol{k}} \theta$

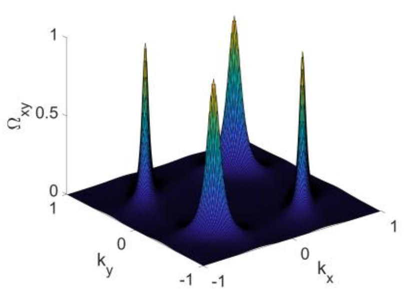
S. Murakami et al., PRL 90, 057002 (2003)



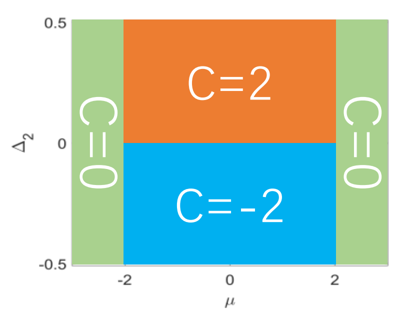
- Example: Square lattice with d+id Pairing

Yang et al PRB 98, 104515 2018

$$\hat{H} = \sum_{\sigma,\mathbf{k}} \xi_{\mathbf{k}}\, c^{\dagger}_{\sigma\mathbf{k}}\, c_{\sigma\mathbf{k}} + \sum_{\boldsymbol{k}} [(\Delta_{\mathbf{k}} + i\Delta'_{\mathbf{k}}) c^{\dagger}_{\uparrow\mathbf{k}} c^{\dagger}_{\downarrow-\mathbf{k}} + h.c.]$$



Berry curvature $\boldsymbol{\Omega}_{\boldsymbol{k}}$



Chern number $\boldsymbol{C}$

Real-space Berry Curvature

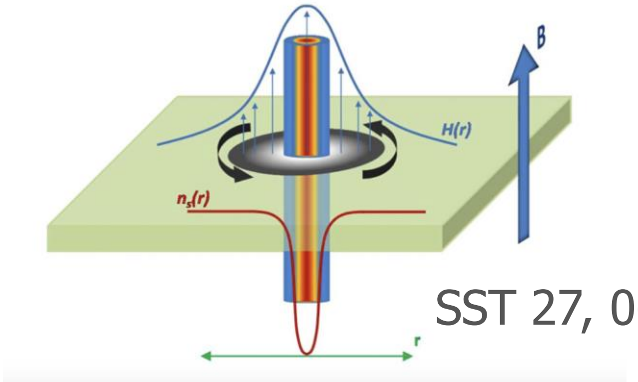
The Berry curvature in the real space:

$$\boldsymbol{\Omega}_r = e\rho\boldsymbol{B} - \nabla_r\rho \times \boldsymbol{v}^s + \nabla_r \times \left(\boldsymbol{B} \times \tilde{\boldsymbol{d}}\right)$$

Supercurrent   Charge dipole

Supercurrent velocity

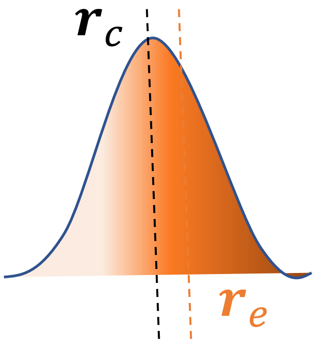
$$\boldsymbol{v}^s = \nabla_{\boldsymbol{r}}\theta - e\boldsymbol{A}$$



SST 27, 063001 (2014)

Dipole moment from the charge distribution of the wave packet

$$\tilde{\boldsymbol{d}} = \frac{1}{2}(\rho^2 - 1)\nabla_{\boldsymbol{k}}\theta$$



Quasiparticle properties

Anomalous thermal Hall	$\kappa_{xy} = \frac{1}{T}\int [d\boldsymbol{k}]\, \Omega_{\boldsymbol{k}} \int_{E_k}^{\infty} d\eta \eta^2 f'(\eta, T)$ f: Fermi distribution function
Anomalous Nernst	$\alpha_{xy}^{e} = -\int [d\boldsymbol{k}] \frac{\partial g}{\partial T}(\rho \Omega_{\boldsymbol{k}} + \partial_{k_x} d_y)$ $g = T \ln(1 - f(\boldsymbol{k}, T))$
Anomalous spin Nernst	$\boldsymbol{\alpha}_{xy}^{s} = -\int [d\boldsymbol{k}] \frac{\partial g}{\partial T}(\boldsymbol{s} \Omega_{\boldsymbol{k}} + \partial_{k_x} \boldsymbol{d}_y^{s})$
Phase-space measure	$\mathcal{D}(\boldsymbol{r}, \boldsymbol{k}) = 1 + \mathrm{Tr}\, \boldsymbol{\Omega}_{\boldsymbol{kr}} - \boldsymbol{\Omega}_{\boldsymbol{r}} \cdot \boldsymbol{\Omega}_{\boldsymbol{k}}$
Density of states	$n(\boldsymbol{r}, \omega) = \int [d\boldsymbol{k}]\, \mathcal{D}(\boldsymbol{r}, \boldsymbol{k})[\lvert\mu\rvert^2 \delta(\omega - E) + \lvert\nu\rvert^2 \delta(\omega + E)]$
Momentum space	$n(\boldsymbol{k}) = \iint d\,\omega d\boldsymbol{r} \mathcal{D}(\boldsymbol{r}, \boldsymbol{k})[\lvert\mu\rvert^2 \delta(\omega - E) + \lvert\nu\rvert^2 \delta(\omega + E)]$

Z. Wang, L. Dong, C. Xiao and Q. Niu, PRL 126, 187001 (2021)
C. Xiao and Q. Niu, PRB 104, L241411 (2021)

Example: Anomalous Thermal Hall Effect

Semiclassical energy current:

$$\mathbf{j}^{\mathrm{E}}(\mathbf{r}) = \int_{\mathbf{k}_c} Df(E)\, E\dot{\mathbf{r}}_c|_{\mathbf{r}_c=\mathbf{r}}$$

Intrinsic thermal Hall current:

$$\mathbf{j}^{\mathrm{E}} = -\nabla T \times \frac{\partial}{\partial T}\int_{\mathbf{k}} h\boldsymbol{\Omega}_{\mathbf{k}}$$

$$h(E,T) = -\int_{E}^{\infty} d\eta f(\eta)\,\eta$$

Thermal Hall conductivity:

$$\kappa_{xy}^{q} = \frac{2}{T}\int [d\boldsymbol{k}]\,\Omega_{\boldsymbol{k}} \int_{E_k}^{\infty} d\eta \eta^2 f'(\eta, T)$$

C. Xiao and Q. Niu, PRB 104, L241411 (2021)

Quantized thermal Hall conductivity

Our formula κ_{xy}^{q} accounts for the quasiparticles beyond the condensate, it is reasonable to make the connection of

$$\kappa_{xy}^{q} + \kappa_{xy}^{0} = \kappa_{xy}^{BdG}$$

quasiparticle — Condensate — BdG picture

Comparing with the previous BdG result obtained with Green function method

$$\kappa_{xy}^{BdG} = \frac{1}{T}\int [d\boldsymbol{k}]\,\Omega_{\boldsymbol{k}} \left(\int_{E_k}^{\infty} - \int_{-E_k}^{\infty}\right) \eta^2 f'(\eta, T)$$

Sumiyoshi et al., JPSJ 82, 023602 (2013)

We can obtain the condensate result of

$$\kappa_{xy}^{0} = -\frac{1}{T}\int [d\boldsymbol{k}]\,\Omega_{\boldsymbol{k}} \int_{-\infty}^{\infty} d\eta\,\eta^2 f'(\eta, T) = \frac{\pi C k_B T}{6\hbar}$$

Supercurrent

We can regard the supercurrent as the charge current carried by the filled quasi-hole band:

$$\mathbf{j} = -e \int d\mathbf{k} \, (\rho \dot{\mathbf{r}}_c)$$

Doppler shift from the supercurrent velocity: $$E_{\mathbf{k}} = E_{0\mathbf{k}} + \mathbf{v}_s \cdot \frac{\partial \xi_{\mathbf{k}}}{\partial \mathbf{k}}$$

$$\begin{aligned} \dot{\mathbf{k}}_c &= 0 \\ \dot{\mathbf{r}}_c &= \frac{\partial E_{0\mathbf{k}}}{\partial \mathbf{k}} + \frac{\partial}{\partial \mathbf{k}}\left(\mathbf{v}_s \cdot \frac{\partial \xi_{\mathbf{k}}}{\partial \mathbf{k}}\right) \end{aligned}$$

→ $\mathbf{j} = e\tilde{n}_s \cdot \mathbf{v}_s$

↓

SC density tensor $$\int d\mathbf{k} \frac{\rho \partial^2 \xi}{\partial k_\alpha \partial k_\beta}$$

For an inhomogeneous system with $\partial E / \partial \boldsymbol{r}_c \neq 0$,

$$\begin{aligned} \dot{\mathbf{k}}_c &= -\frac{\partial E}{\partial \mathbf{r_c}} \\ \dot{\mathbf{r}}_c &= \frac{\partial E}{\partial \mathbf{k_c}} - \dot{\mathbf{k}}_c \times \mathbf{\Omega}_{\mathbf{k}} = \frac{\partial E_{0\mathbf{k}}}{\partial \mathbf{k}} + \frac{\partial E}{\partial \mathbf{r_c}} \times \mathbf{\Omega}_{\mathbf{k}} \end{aligned}$$

→ $$\mathbf{j}_{sH} = -e \int d\mathbf{k} \sum_{\sigma} \rho_k \left[\frac{\partial E}{\partial \mathbf{r_c}} \times \mathbf{\Omega}_{\mathbf{k}} \right]$$

Hall-type supercurrent

Toy Models

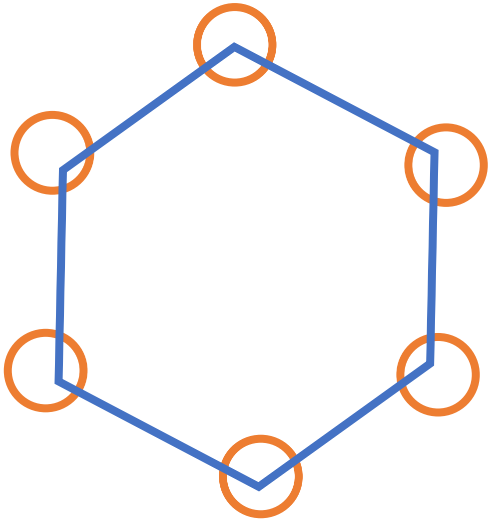
Honeycomb lattice with d+id pairing

$$\hat{H} = \sum_{\sigma,\mathbf{k}} \xi_{\mathbf{k}}\, c_{\sigma\mathbf{k}}^{\dagger}\, c_{\sigma\mathbf{k}} + \sum_{\boldsymbol{k}} [\Delta_{\mathbf{k}} c_{\uparrow\mathbf{k}}^{\dagger} c_{\downarrow-\mathbf{k}}^{\dagger} + h.c.]$$

$$\Delta(\boldsymbol{k}) = \sum_{i=1}^{3} \Delta_i \cos(\boldsymbol{k}\cdot\boldsymbol{R}_i - \varphi_{\boldsymbol{k}})$$

$$(\Delta_1, \Delta_2, \Delta_3) \equiv \left(2\Delta, -\Delta + i\sqrt{3}\Delta', -\Delta - i\sqrt{3}\Delta'\right)$$

Jiang et al, PRB 77, 235420 (2008)



TBG tight-binding model

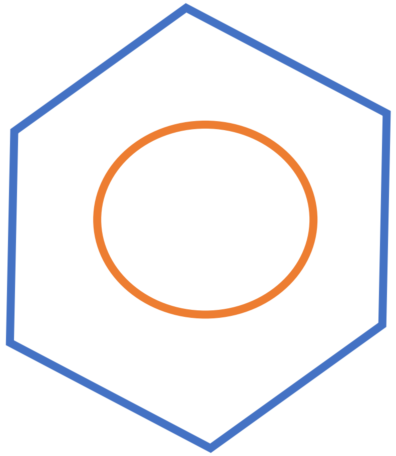
$$H = -\mu \sum_{i} \tilde{c}_i^{\dagger}\tilde{c}_i + t_1 \sum_{\langle i,j\rangle} \tilde{c}_i^{\dagger}\tilde{c}_j$$

$$+ \sum_{[i,j]} \tilde{c}_i^{\dagger}\left[\left(t_2\sigma_0 + it_3\sigma_y\right)\otimes \sigma_0\right]\tilde{c}_j + h.c.$$

$$\Delta(\boldsymbol{k}) = \sum_{i=1}^{3} \Delta_i \cos(\boldsymbol{k}\cdot\boldsymbol{R}_i - \varphi_{\boldsymbol{k}})$$

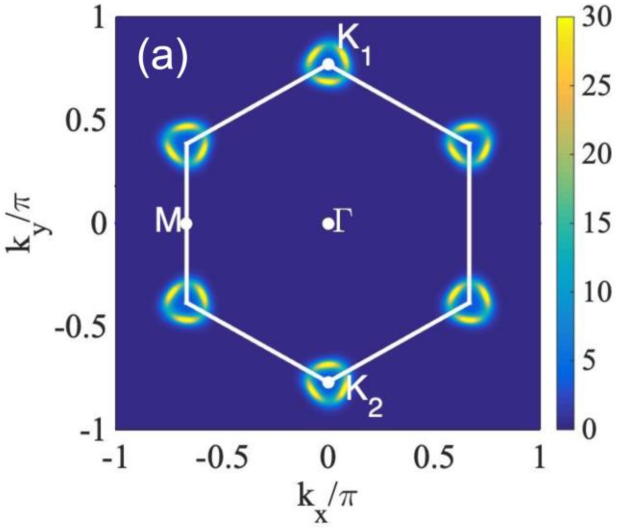
Liu et al, PRL 121, 217001 (2018)

Yuan et al., PRB 98, 045103 (2018)

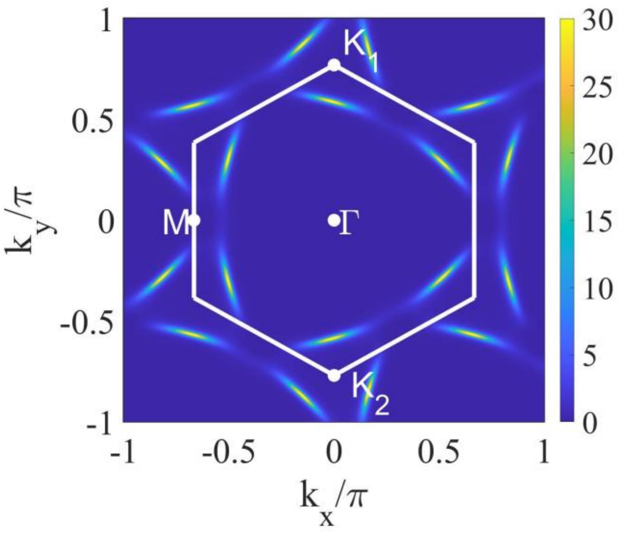


Berry Curvature distribution

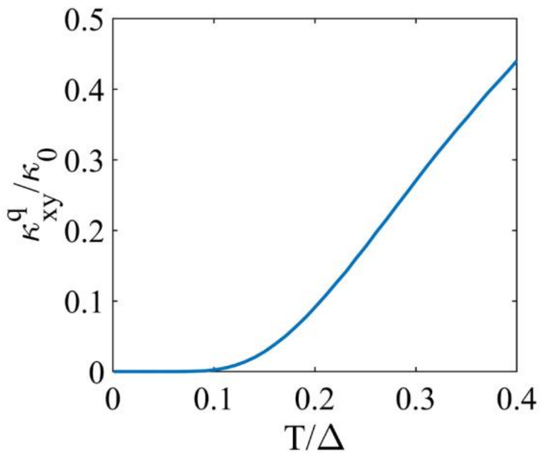
Honeycomb lattice



TBG



Anomalous Thermal Hall



Summary

- Equation of motion for SC quasiparticles
- Berry curvatures in real, momentum, and phase space
- Berry curvature effects in transport properties