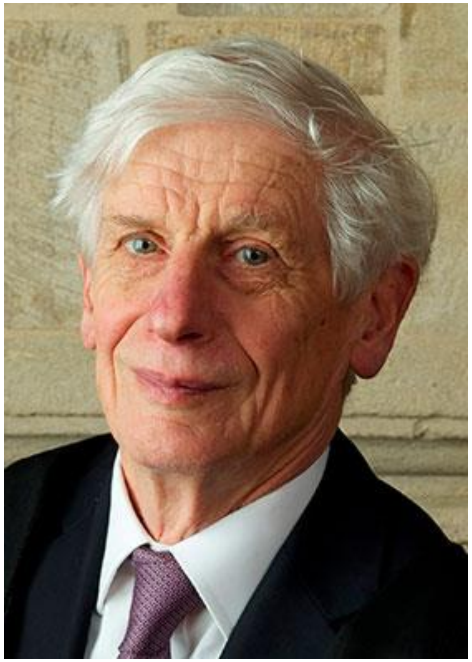
Lecture 1
Chern Number and Berry Curvature

- 2016 Nobel Physics Prize
- Working with Thouless
- Geometric phase and Berry curvature
- Understand Hofstadter butterfly

2016 Nobel Physics Winners

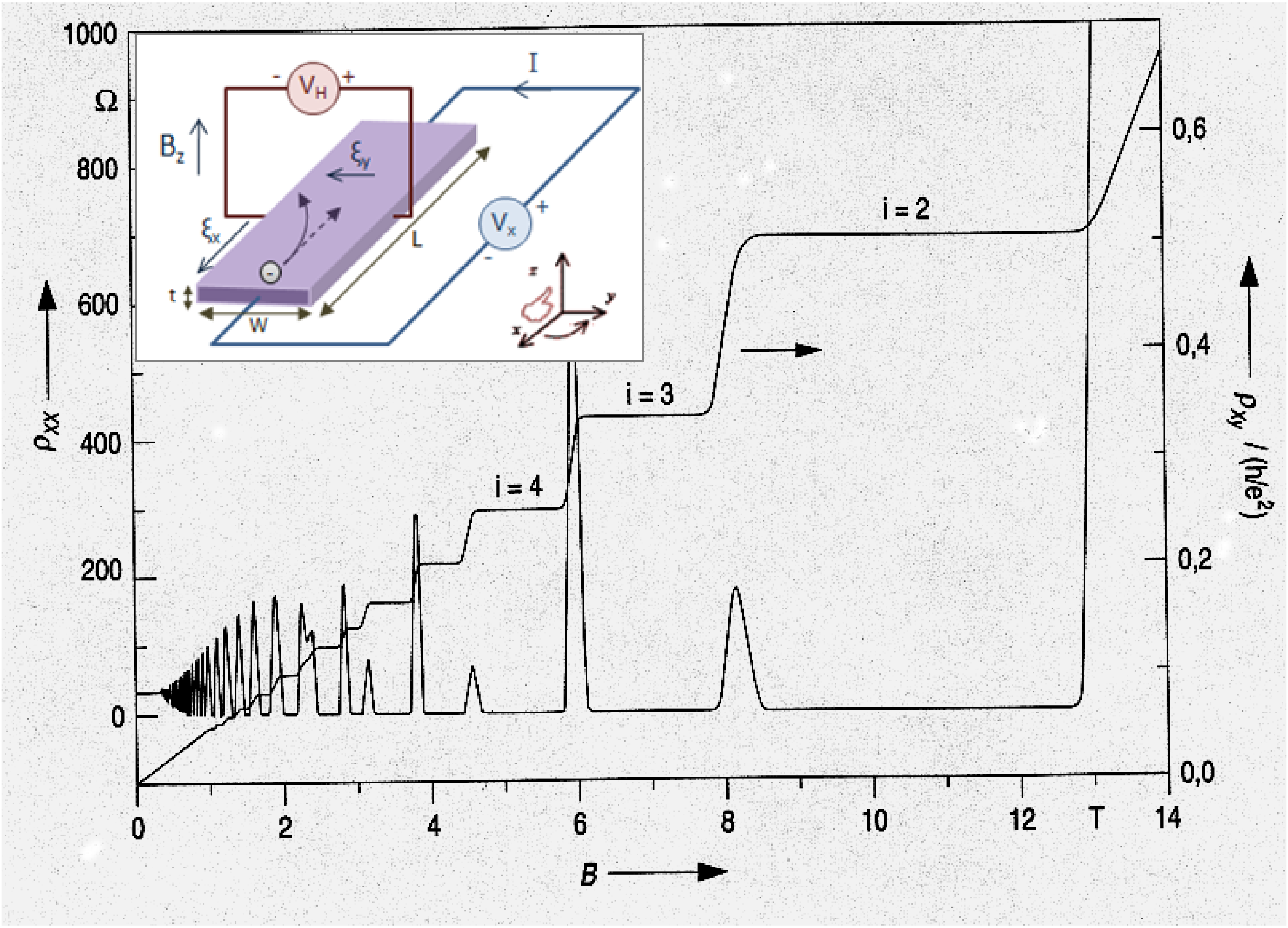


David J. Thouless
U of Washington

F. Duncan M. Haldane
Princeton U

J. Michael Kosterlitz
Brown U

for theoretical discoveries of topological phase transitions and topological phases of matter



Quantum Hall effect; von Klitzing, Nobel ‘85

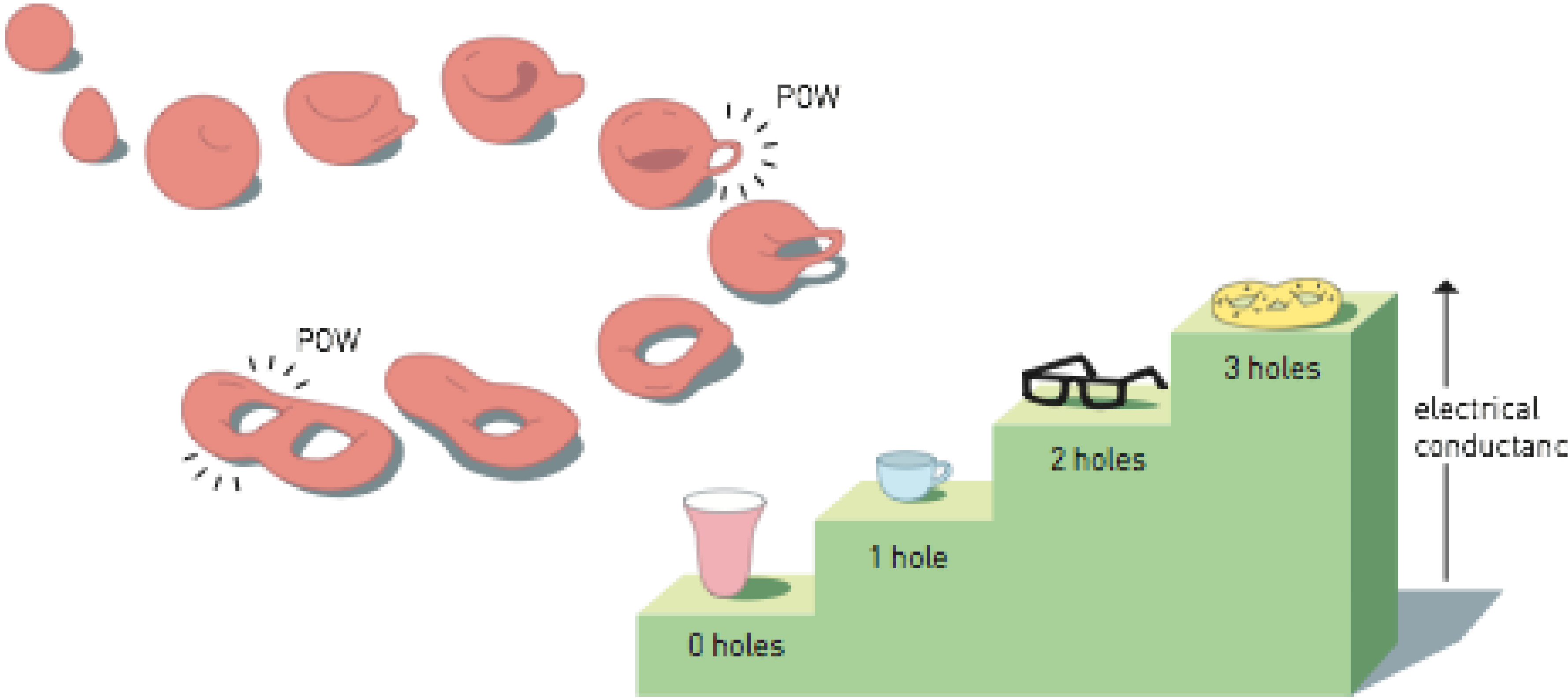


Illustration: ©Johan Jarnestad/The Royal Swedish Academy of Sciences

Topological quantum phase transition

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

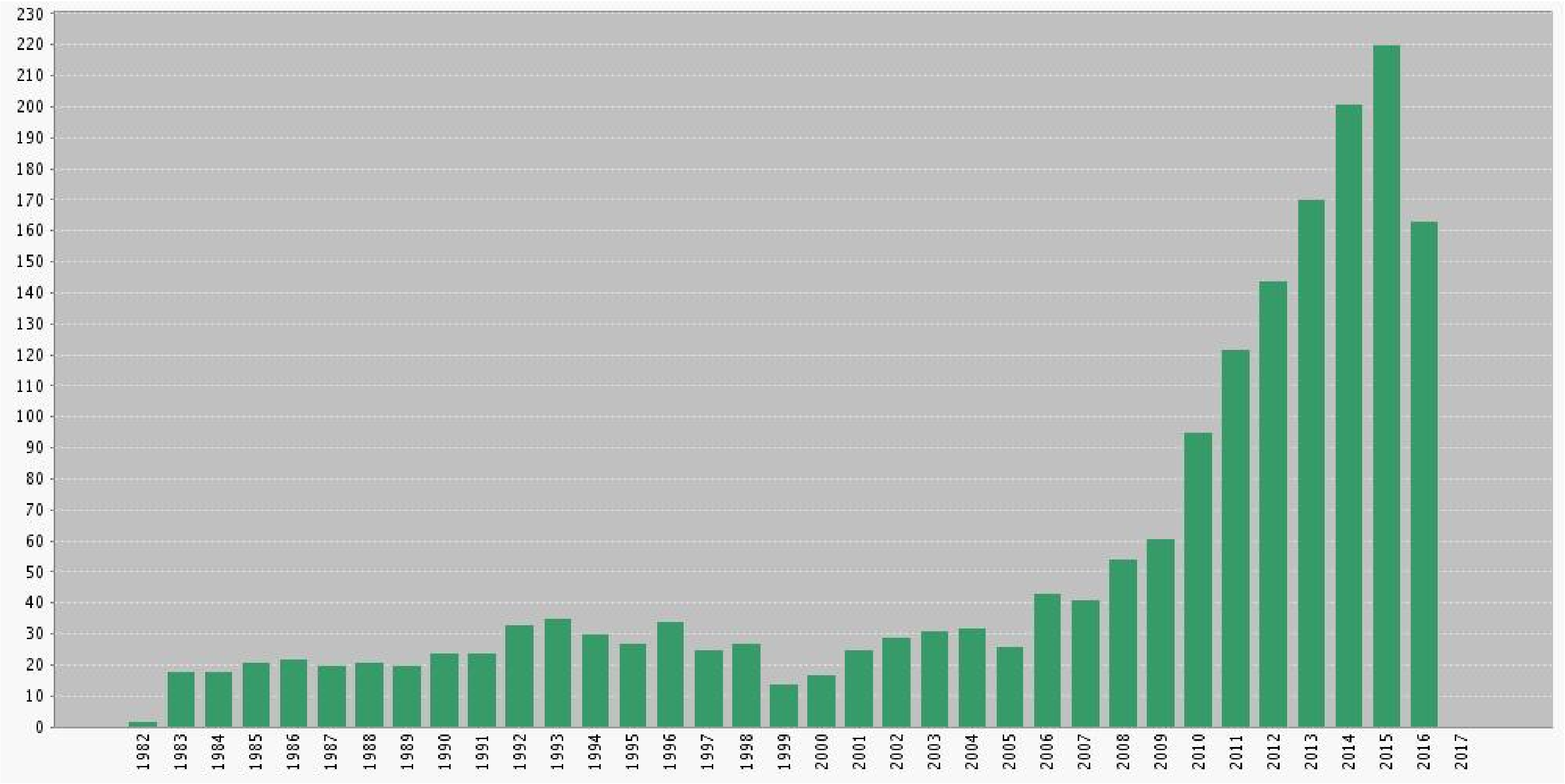
PACS numbers: 72.15.Gd, 72.20.My, 73.90.+b

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

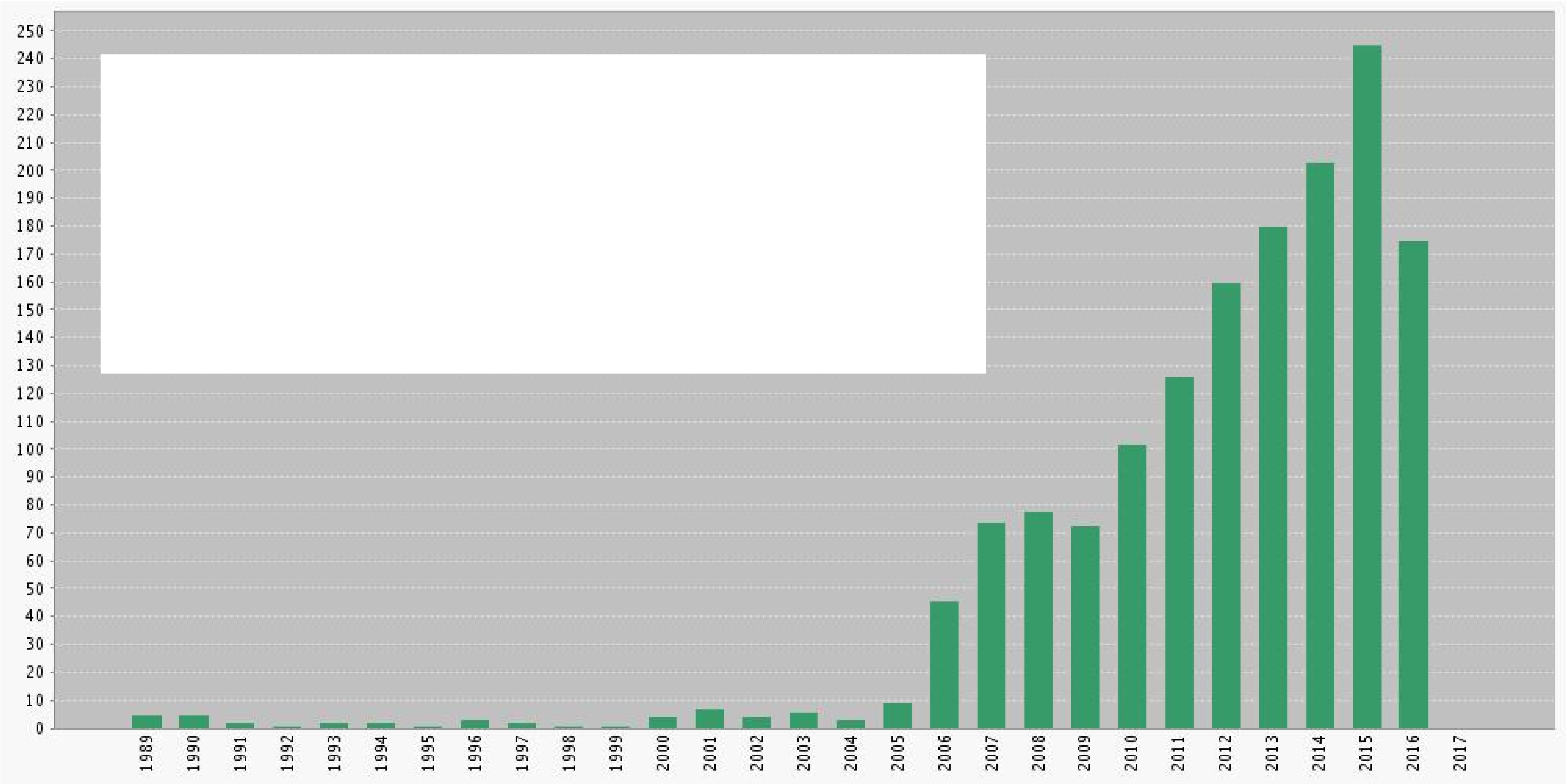
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane
Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 16 September 1987)

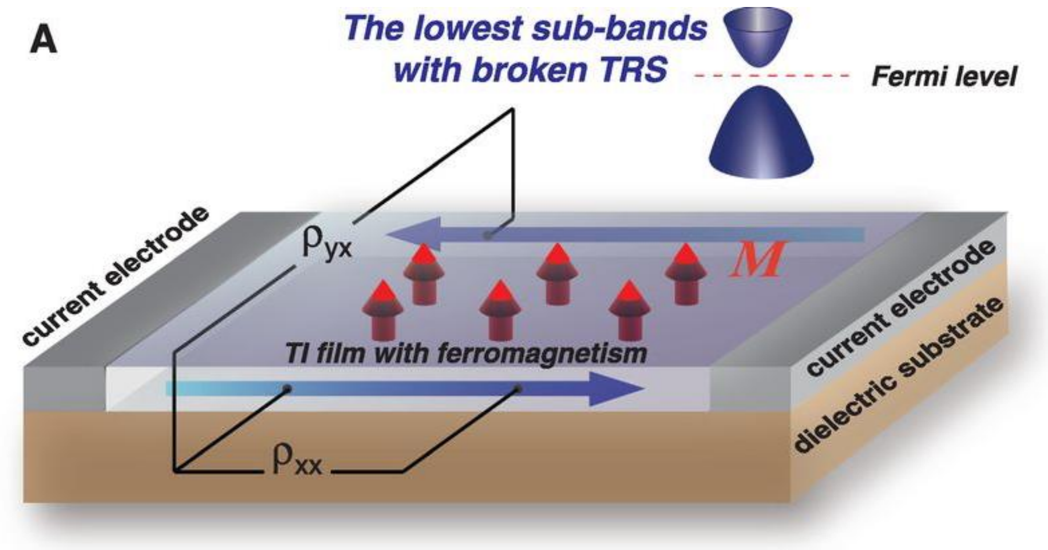
A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.



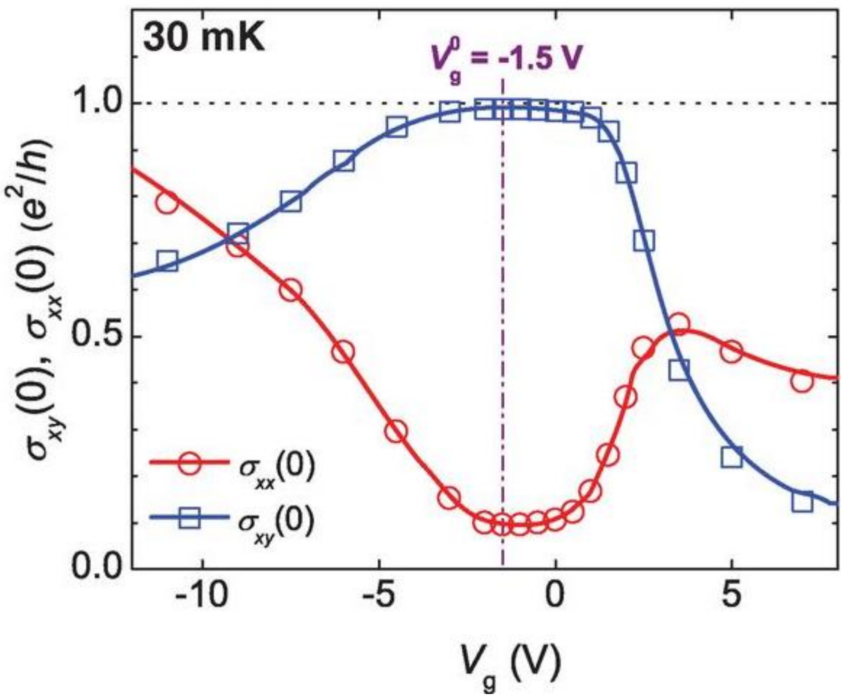
Citation to TKNN paper



Citation to Haldane’s paper



Q.K. Xue’s group
Science (2013)



How I met David Thouless

- Entered UW in 1981 through TD Lee's CUSPEA program
  - the same year David moved from Yale to UW
- Joined David in 1982:
  - His work with Kosterlitz had already become a classic
  - He was an authority on Anderson localization
  - He just finished with his collaborators the TKNN paper
- Thouless and Niu (1983):
  - Wavefunction scaling in a quasiperiodic potential
    derived from the Hofstadter problem

Hofstadter Butterfly

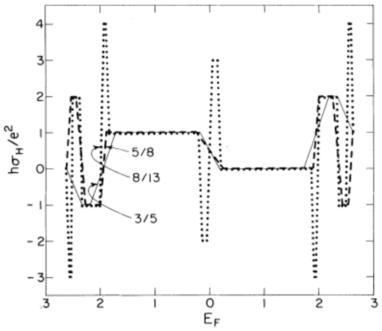
Hofstadter (1976): 2d Bloch band in a magnetic field

The band is split into q sub-bands, if the magnetic flux per unit cell is a fraction of h/e

TKNN (1982): Kubo formula for the Hall conductance

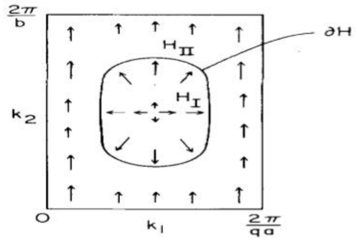
$$\sigma_{\mathrm{H}} = \frac{ie^2}{2\pi h}\sum\int d^2k\int d^2r\left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1}\right)$$

$$= \frac{ie^2}{4\pi h}\sum\oint dk_j\int d^2r\left(u^*\frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j}u\right),$$

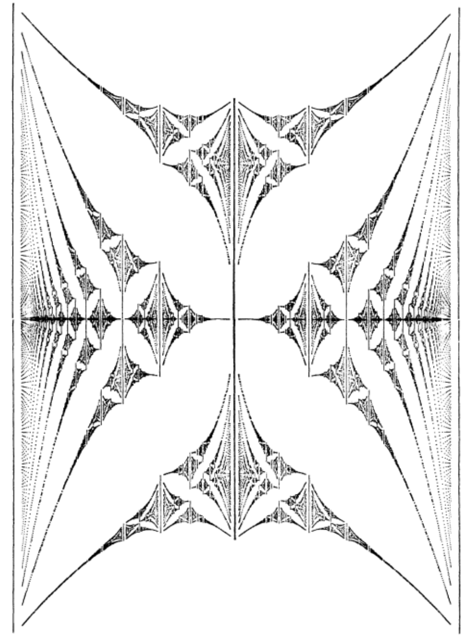


Chern number

Kohmoto (1985):



energy



Magnetic flux

Adiabatic Quantum Pump

Thouless (1983):

In a Bloch insulator, charge transfer

induced by a time periodic change is quantized.

$$C = \frac{i}{2\pi} \sum_{\lambda} f_{\lambda} \int_0^T dt \int_0^{2\pi/L} dk \left[\left\langle \frac{\partial \psi_{\lambda k}}{\partial t} \middle| \frac{\partial \psi_{\lambda k}}{\partial k} \right\rangle - \left\langle \frac{\partial \psi_{\lambda k}}{\partial k} \middle| \frac{\partial \psi_{\lambda k}}{\partial t} \right\rangle \right]$$

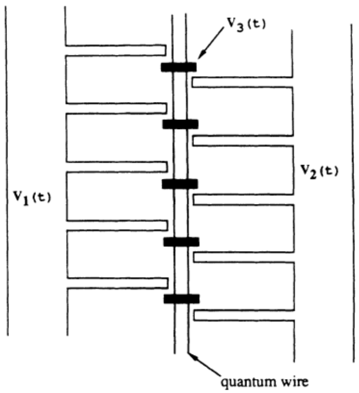


FIG. 2. The quantum wire and the voltage leads.

Niu and Thouless (1984): generalize.

The pumped current can be written as a Berry curvature
in time and an imaginary flux.

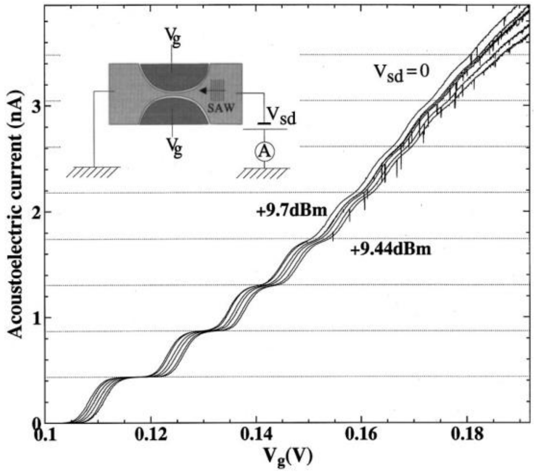
The charge transfer is an integral over a time cycle,
which becomes a Chern number after averaging over the flux.

The average is justified when the Fermi energy lies in a gap
or mobility gap.

Quantum Pump: implications

– Quantum pump for a current standard:

M. Pepper's group (2000): quantized within 60 ppm

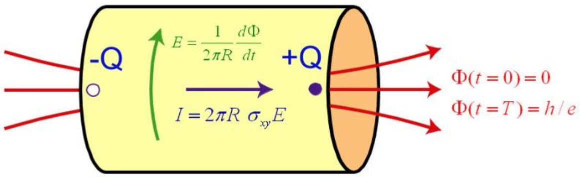


– Polarization as a geometric phase

•King-Smith and Vanderbilt (1993)

•Ortitz and Martin (1994)

– Laughlin on integer quantum Hall effect

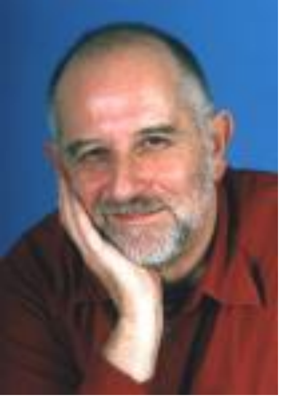


$$\Delta Q = \int_0^T \sigma_{xy}\frac{d\Phi}{dt}dt = \sigma_{xy}\frac{h}{e}$$

Quantization of charge transfer needs justification

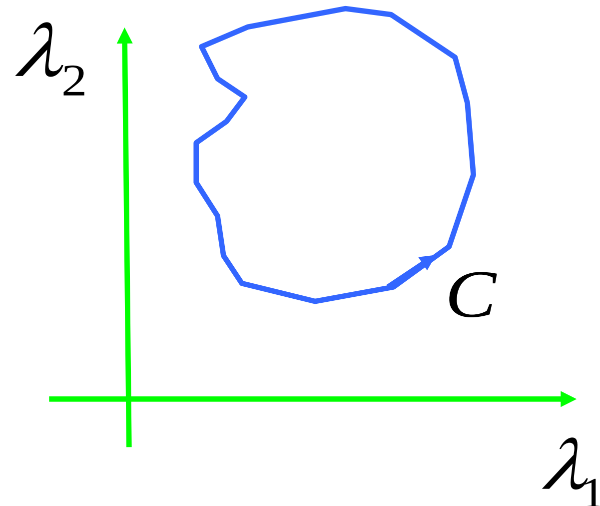
Quantum Hall conductance as a topological invariant

- Niu, Thouless, and Wu (1985):
  - Disordered and/or interacting 2d electrons in a torus geometry.
  - The Hall conductance can still be expressed as a Berry curvature of the many body ground state as a function of magnetic fluxes through the holes.
  - It becomes a Chern number after average, which can be justified if the Fermi energy lies in a gap or mobility gap.
  - To have fractional quantum Hall state, one must have ground state degeneracy.
- Tao and Wu (1984): gauge argument for the fractional case.
- Wen and Niu (1990): degeneracy depends on surface topology.



Berry's Geometric Phase (1984)

$$\gamma_n = \oint_C d\lambda \, \langle \psi_n | i \frac{\partial}{\partial \lambda} | \psi_n \rangle$$



Stokes theorem

$$\gamma_n = \iint d\lambda_1 d\lambda_2 \, \Omega$$

Berry Curvature

$$\Omega = i \frac{\partial}{\partial \lambda_1} \langle \psi | \frac{\partial}{\partial \lambda_2} | \psi \rangle - i \frac{\partial}{\partial \lambda_2} \langle \psi | \frac{\partial}{\partial \lambda_1} | \psi \rangle$$

Analogies

Berry curvature

$$\Omega(\vec{\lambda})$$

Magnetic field

$$B(\vec{r})$$

Berry connection

$$\langle \psi | i \frac{\partial}{\partial \lambda} | \psi \rangle$$

Vector potential

$$A(\vec{r})$$

Geometric phase

$$\oint d\lambda \langle \psi | i \frac{\partial}{\partial \lambda} | \psi \rangle = \iint d^2\lambda \; \Omega(\vec{\lambda})$$

Aharonov-Bohm phase

$$\oint dr\, A(\vec{r}) = \iint d^2 r\, B(\vec{r})$$

Chern number (Simon, 1983)

$$\oiint d^2\lambda \; \Omega(\vec{\lambda}) = \text{integer}$$

Monopole (Dirac, 1931)

$$\oiint d^2 r \; B(\vec{r}) = \text{integer}\, h / e$$

Applications on Bloch electrons

- Zak's geometric phase

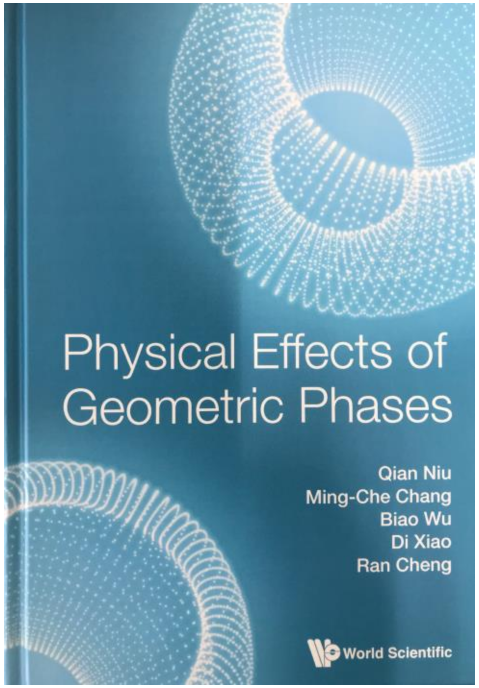
  polarization in crystals
- Berry curvature

  electron dynamics in Bloch bands
- Chern number

  quantum Hall effect,

  quantum charge pump

``Berry Phase Effects on Electronic Properties”,
by D. Xiao, M.C. Chang, Q. Niu,
Review of Modern Physics

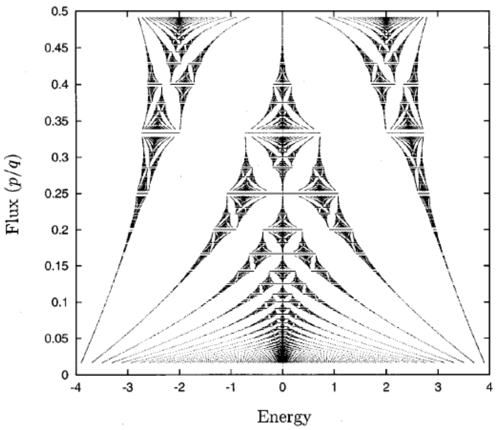


To Professor David J. Thouless

Hofstadter butterfly

- Spectral splitting

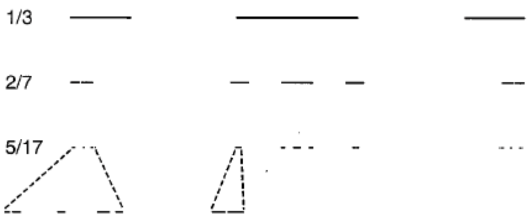
$$\phi = \cfrac{1}{f_1 + \cfrac{1}{f_2 + \cfrac{1}{f_3 + \cdots}}}$$



Azbel (1964): a band is split into f_1 sub-bands, each of which is split into f_2 sub-bands, …

- Example

$$\frac{1}{2+\sqrt{2}} = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

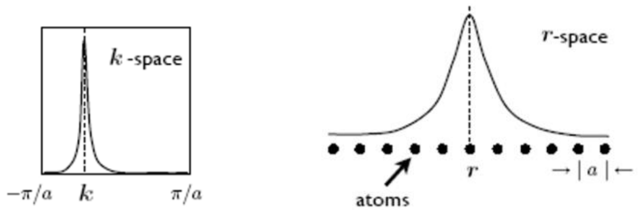


- Splitting anomaly: correction to the Azbel rule by Chern numbers

Semiclassical theory

Chang and Niu (1995)

- Wave packet



- Equations of Motion

$$\dot{\mathbf{r}} = \frac{\partial E_n(\mathbf{k})}{\hbar\,\partial \mathbf{k}} - \dot{\mathbf{k}} \times \mathbf{\Omega}_n(\mathbf{k})$$

$$\hbar \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \delta \mathbf{B}$$

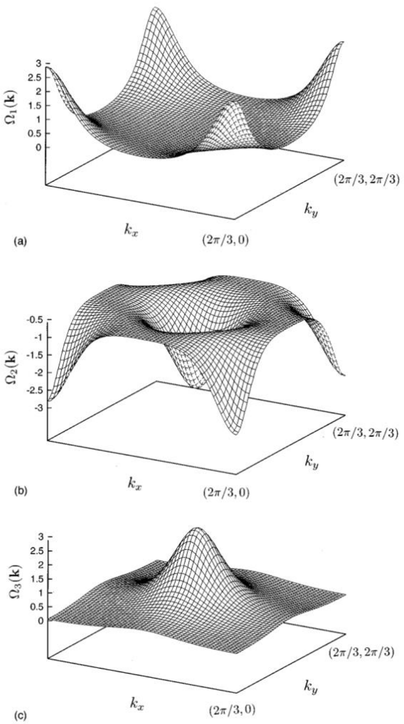


FIG. 5. Distributions of Berry curvature $\Omega_n(\mathbf{k})$ (in units of e^2/h). $\Omega_1(\mathbf{k})$ is equal to $\Omega_3(\mathbf{k})$ shifted by $(\pi/3, \pi/3)$.

+1

-2

+1

E_{Hofst}	0.0618	0.1067	0.1566	0.2124	0.2747	0.3443	0.4221	0.5098	0.6098	0.7266
	0.0678	0.1085	0.1570	0.2125						
E_{cyclo}	0.0632	0.1063	0.1558	0.2115	0.2738	0.3435	0.4212	0.5086	0.6082	0.7240