

课程基本信息

- 网址: http://home.ustc.edu.cn/~lishu/geometric_phase_2024.html
- 考核: 作业 (50%) + 期末 (50%)
- 作业完成时间: 2周, 大约3-4次作业交上之后由助教给习题课。
- 期末: 根据人数分组, 每组选定一个题目进行研究, 4周时间, 最后每组选出若干代表进行口头汇报, 汇报时间半小时。
- 不能补考。

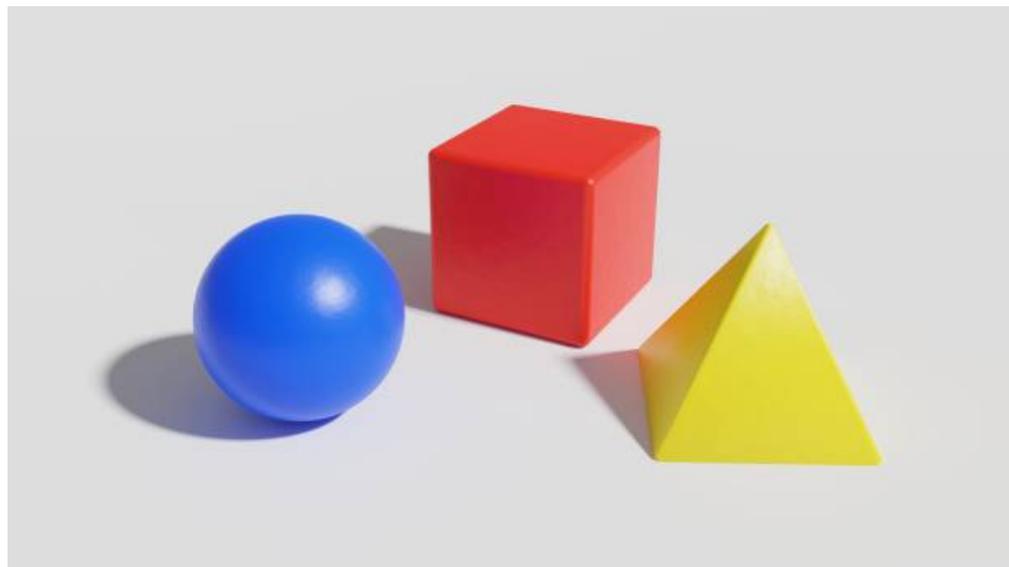
拓扑与几何

物理中的几何相位（专题1）

高阳

拓扑-橡胶膜的几何

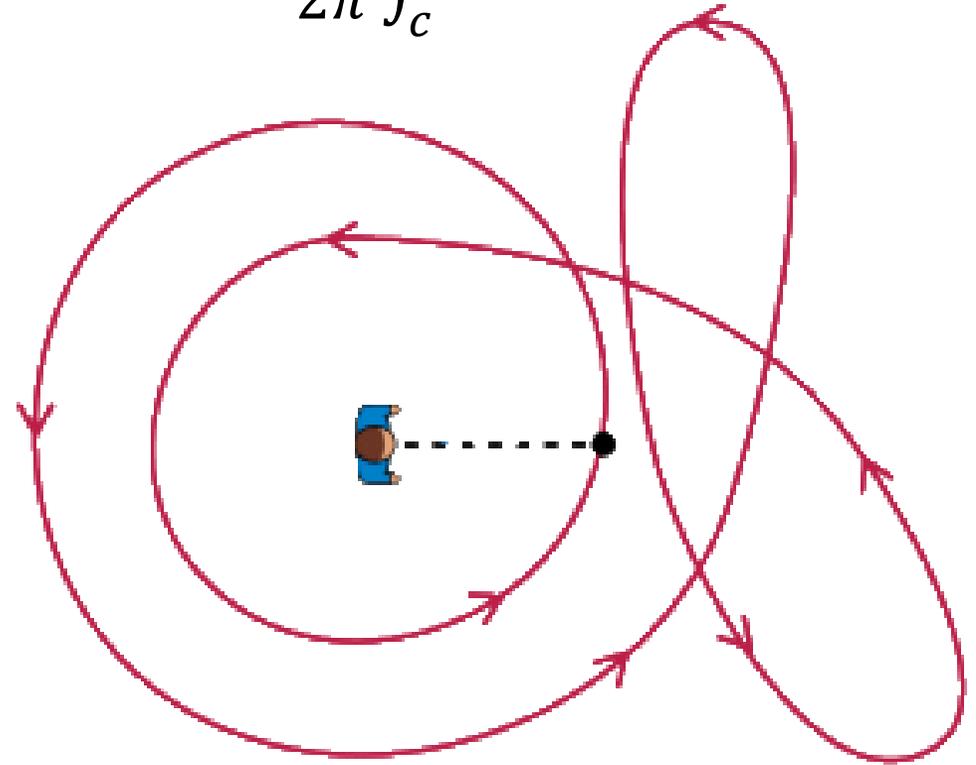
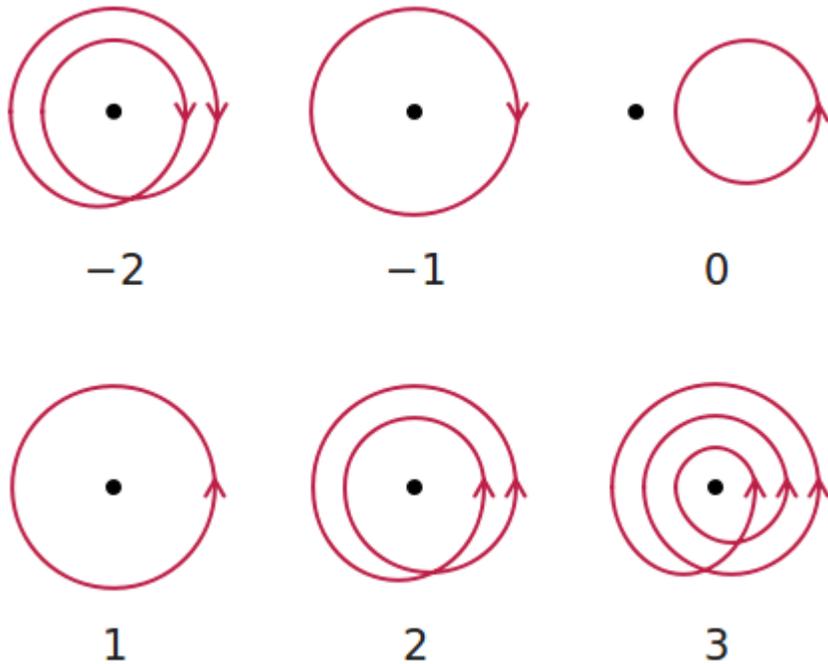
研究物体（特定空间）在连续形变（连续变换）下保持不变的性质



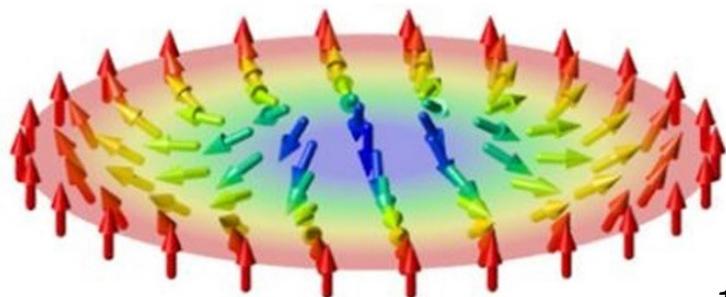
拓扑示例1：一维绕数 (winding number)

$$f: S^1 \rightarrow S^1$$

$$n = \frac{1}{2\pi} \int_c d\theta$$

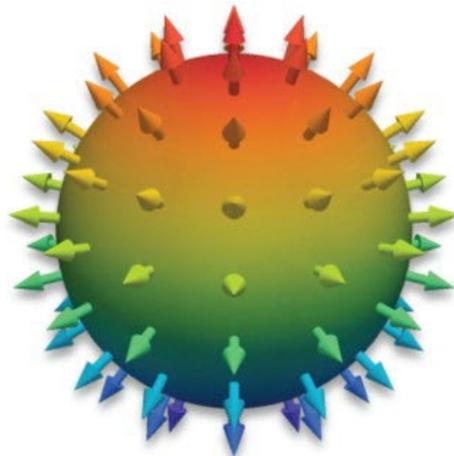


拓扑示例2：二维绕数 (winding number)

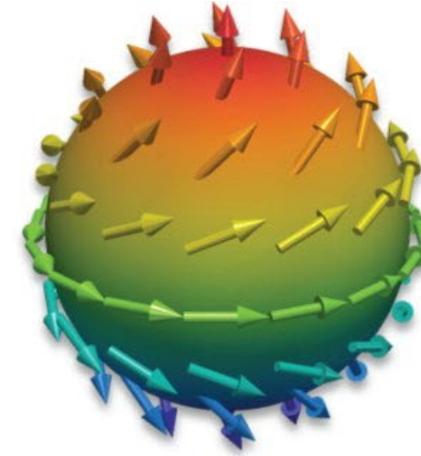


$$f: S^2 \rightarrow S^2$$

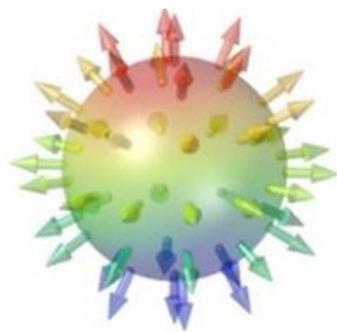
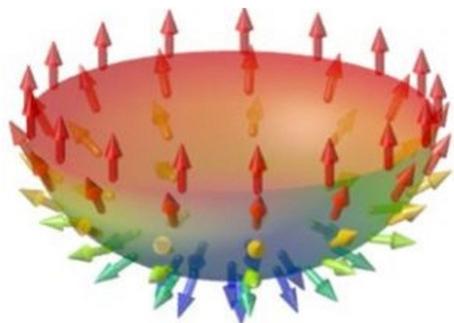
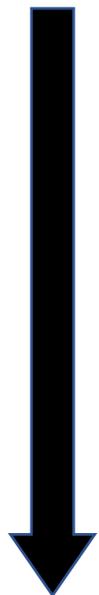
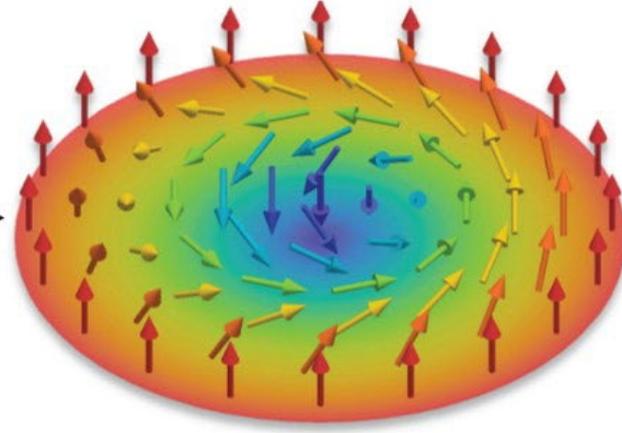
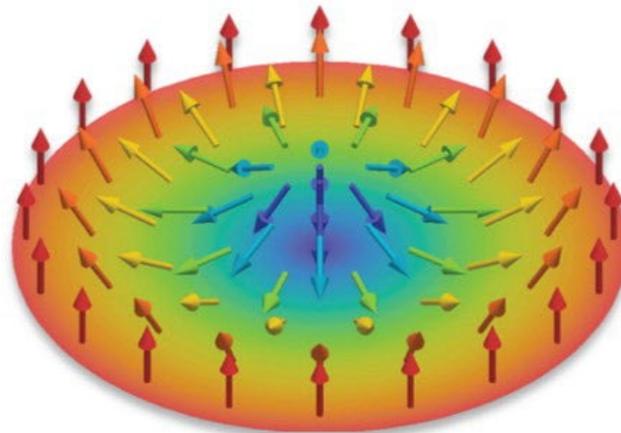
Neel-type skyrmion



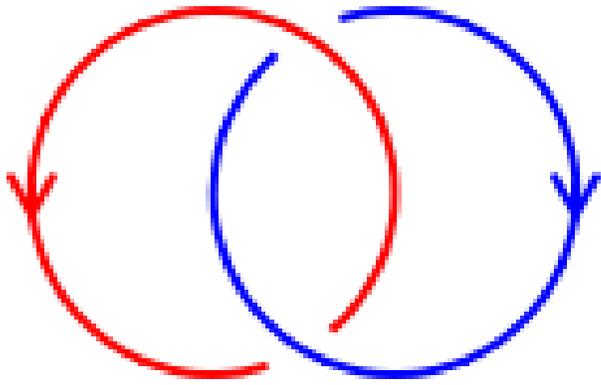
Bloch-type skyrmion



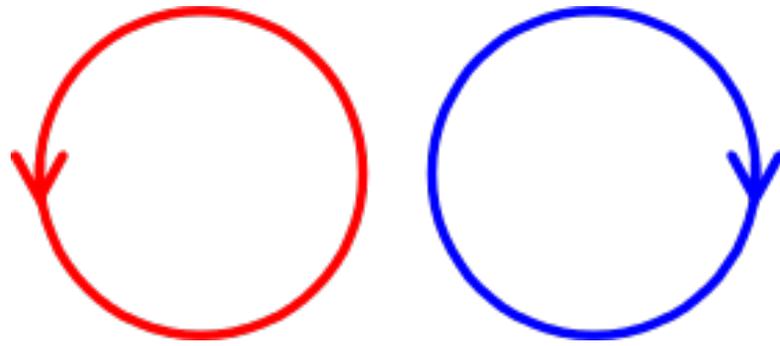
$$n = \frac{1}{4\pi} \int_c \hat{\mathbf{S}} \cdot \frac{\partial \hat{\mathbf{S}}}{\partial x} \times \frac{\partial \hat{\mathbf{S}}}{\partial y} dx dy$$



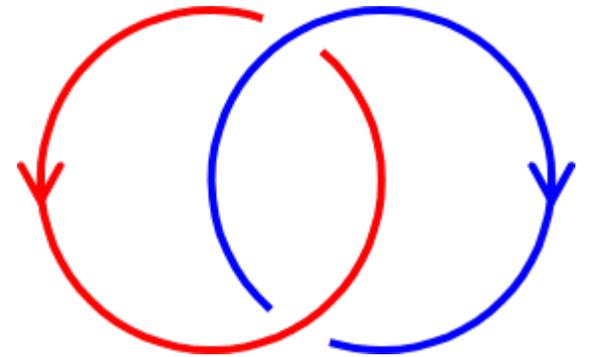
拓扑示例3：连环数 (linking number)



$$L = -1$$



$$L = 0$$



$$L = 1$$

Gauss Map: torus to sphere

$$\Gamma[\mathbf{r}_1(s), \mathbf{r}_2(t)] = \frac{\mathbf{r}_1(s) - \mathbf{r}_2(t)}{|\mathbf{r}_1(s) - \mathbf{r}_2(t)|}$$

Gauss integral: area of the Gauss map

$$L(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot d\mathbf{r}_1 \times d\mathbf{r}_2$$

作业1: 证明此公式给出了映射对应的面积

连环数的应用：磁场手性(helicity)

$$L(\gamma_1, \gamma_2) = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot d\mathbf{r}_1 \times d\mathbf{r}_2$$

$$H = \sum_{ij} L_{ij} \Phi_i \Phi_j = \int d^3r \mathbf{B} \cdot \mathbf{A}$$

在无耗散的等离子体中守恒

$$\begin{aligned} \frac{\partial H}{\partial t} &= -\int d^3r [(\nabla \times \mathbf{E}) \cdot \mathbf{A} + \mathbf{B} \cdot (\mathbf{E} + \nabla\Phi)] \\ &= -\int d^3r \nabla \cdot (\mathbf{E} \times \mathbf{A} + \Phi \mathbf{B}) - 2\int d^3r \mathbf{E} \cdot \mathbf{B} \end{aligned}$$

等离子体中的欧姆定律

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} = \rho \mathbf{J} = \mathbf{0}$$

手性守恒

$$\frac{\partial H}{\partial t} = 0$$

理想情况



$$H=0$$



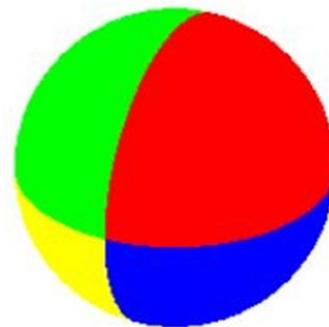
$$H=T\Phi^2$$



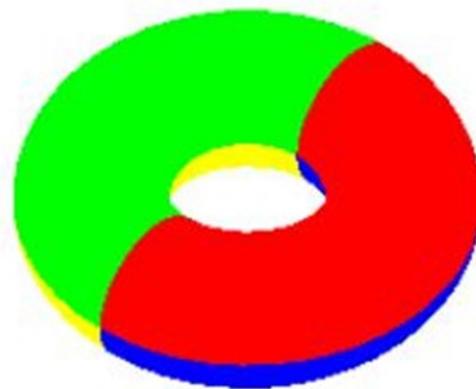
$$H=\pm 2\Phi_1\Phi_2$$

拓扑示例4：欧拉示性数

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2



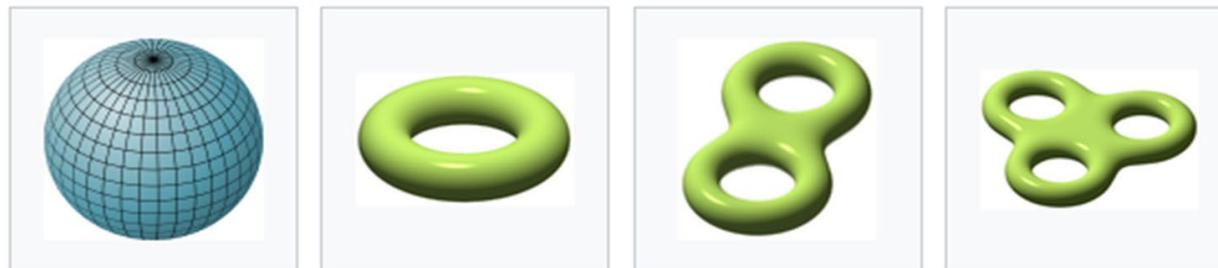
$$\begin{aligned}\chi &= V - E + F \\ &= 2 - 4 + 4 = 2\end{aligned}$$



$$\begin{aligned}\chi &= V - E + F \\ &= 4 - 8 + 4 = 0\end{aligned}$$

与剖分方式无关

欧拉示性数与亏格



$\chi = 2$

$\chi = 0$

$\chi = -2$

$\chi = -4$

$\chi(M) = 2 - 2g$

推广至N维曲面:

$$\chi(M) = \sum_0^N (-1)^k \beta_k$$

β_k 为k-单纯形的个数

$$k = 0, 1, 2, 3, \dots = \bullet, \text{---}, \triangle, \text{四面体}, \dots$$

高斯-博内定理：拓扑与几何的联系

闭合曲面

$$\chi(S) = \frac{1}{2\pi} \iint K dS$$

高斯曲率

有边界的曲面

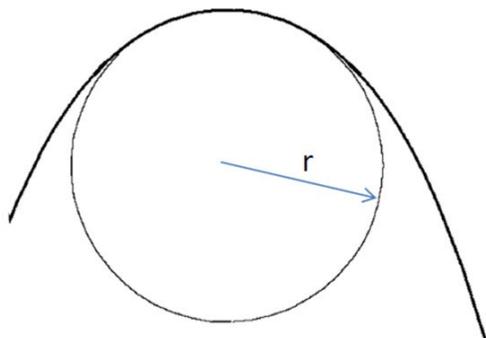
$$\chi(S) = \frac{1}{2\pi} \iint K dS + \frac{1}{2\pi} \int_{\partial S} k_g ds$$

测地线曲率

高斯曲率的一个分量
曲线在切平面的投影的曲率

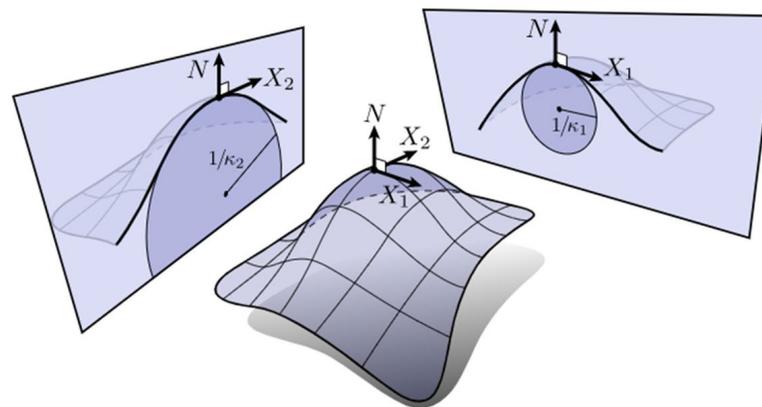
曲线与曲面的曲率

对直线的偏离



$$k = \frac{1}{r} = \frac{d\phi}{ds}$$

考察过P点的曲线



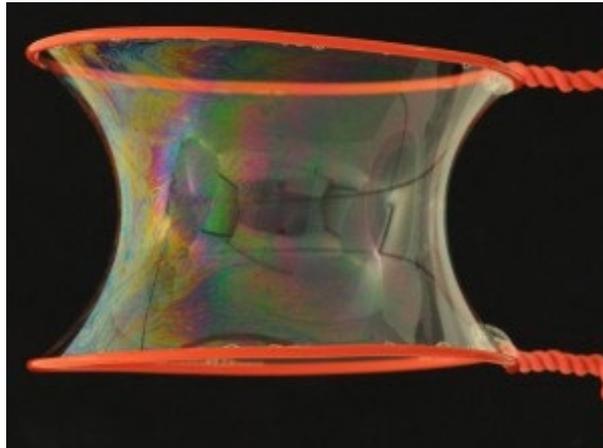
它们将具有不同的曲率，
其中的最大值和最小值为主曲率 k_1, k_2

平均曲率与最小曲面

平均曲率 (外禀) $k = k_1 + k_2$

最小曲面问题: 每个点附近的面积均取极小

决定肥皂膜的形状: 膜的表面张力对应的自由能正比于面积



高斯曲率

高斯曲率 (内禀) $k = k_1 k_2$

曲面的基本形式

曲面上两点的距离

$$\begin{aligned}x &= X(u, v) \\y &= Y(u, v) \\z &= Z(u, v)\end{aligned}$$

$$dr^2 = Edu + Fdudv + Fdvdu + Gdv^2$$

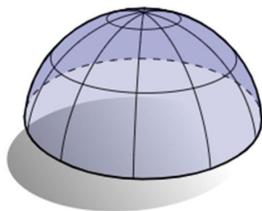
完全由度规决定

高斯极妙定理
除非产生拉伸或压缩,
否则高斯曲率不变

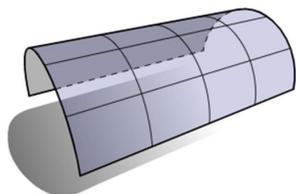


$$k = \frac{1}{2H} \left[\frac{\partial}{\partial u} \left(\frac{F}{EH} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{EH} \frac{\partial E}{\partial u} \right) \right] \quad H = \sqrt{EG - F^2}$$

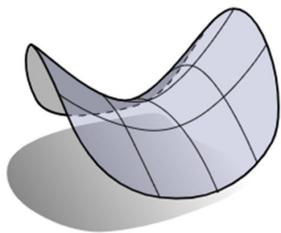
高斯曲率示例



$$k > 0$$

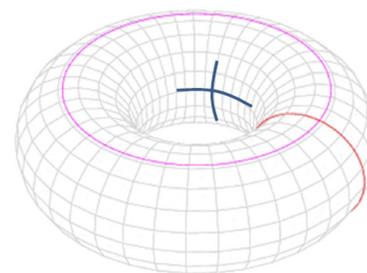


$$k = 0$$



$$k < 0$$

轮胎面（环面）



高斯-博内定理：拓扑与几何的联系(1)

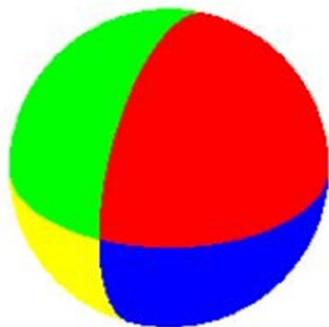
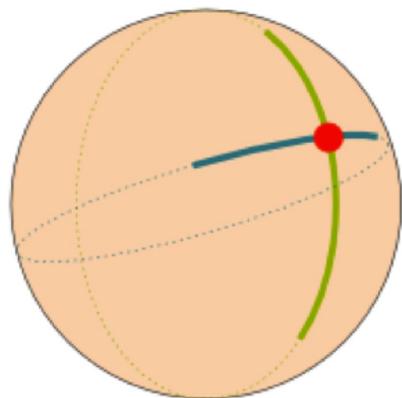
闭合曲面

$$\chi(S) = \frac{1}{2\pi} \iint K dS$$

高斯曲率

球面的曲率： $K = \frac{1}{R^2}$

$$\chi(S) = \frac{1}{2\pi} \iint K dS = \frac{1}{2\pi R^2} \times 4\pi R^2 = 2$$



$$\chi = V - E + F = 2 - 4 + 4 = 2$$

高斯-博内定理：拓扑与几何的联系(2)

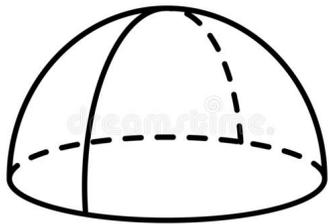
有边界的曲面

$$\chi(S) = \frac{1}{2\pi} \iint K dS + \frac{1}{2\pi} \oint_{\partial S} k_g ds$$

测地线曲率
衡量对测地线的偏离

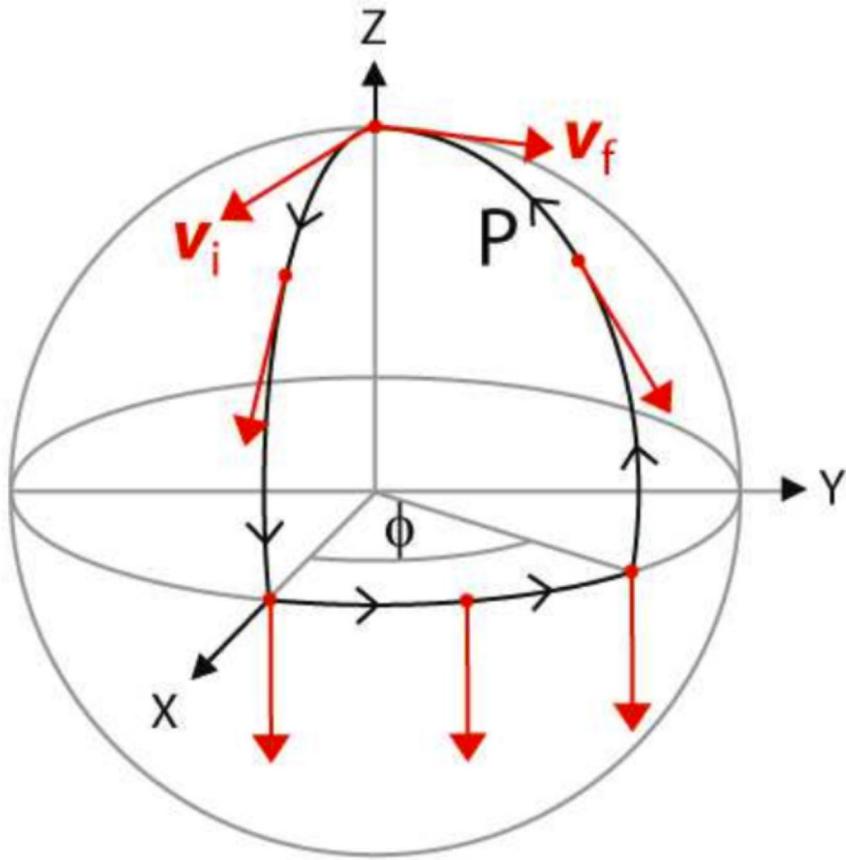
半球，边界为测地线

$$\chi(S) = \frac{1}{2\pi} \iint K dS = \frac{1}{2\pi R^2} \times 2\pi R^2 = 1$$



$$\chi = V - E + F = 5 - 8 + 4 = 1$$

球面三角形的内角和

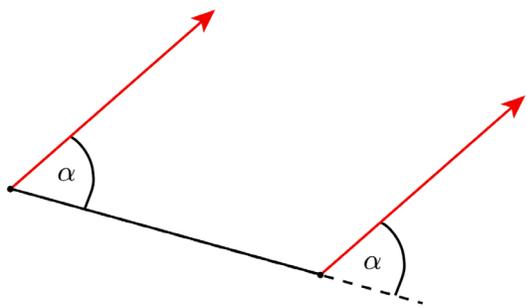


$$\phi = \frac{1}{2\pi} \iint K dS$$

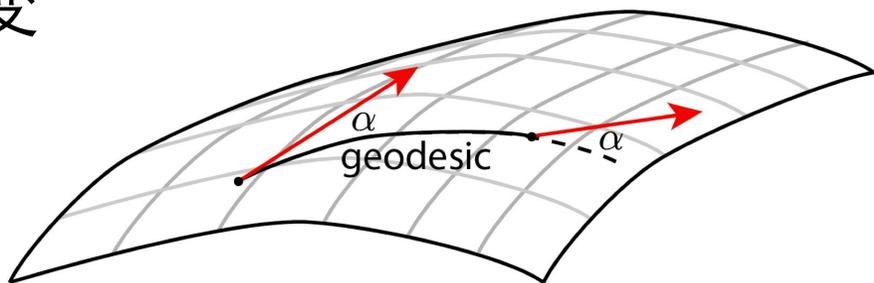
对平直空间的偏离

平行转移

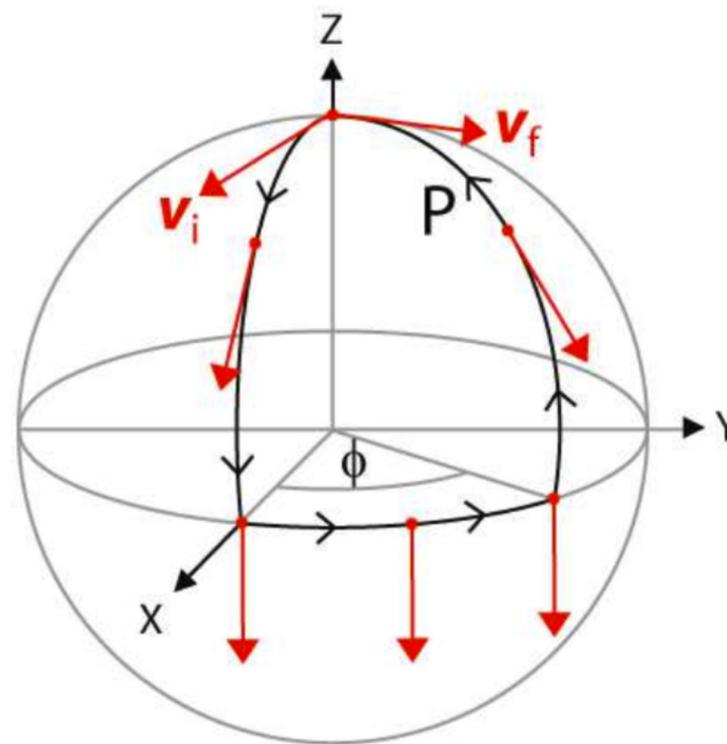
平直空间中的转移：
向量与两点连线的夹角不变



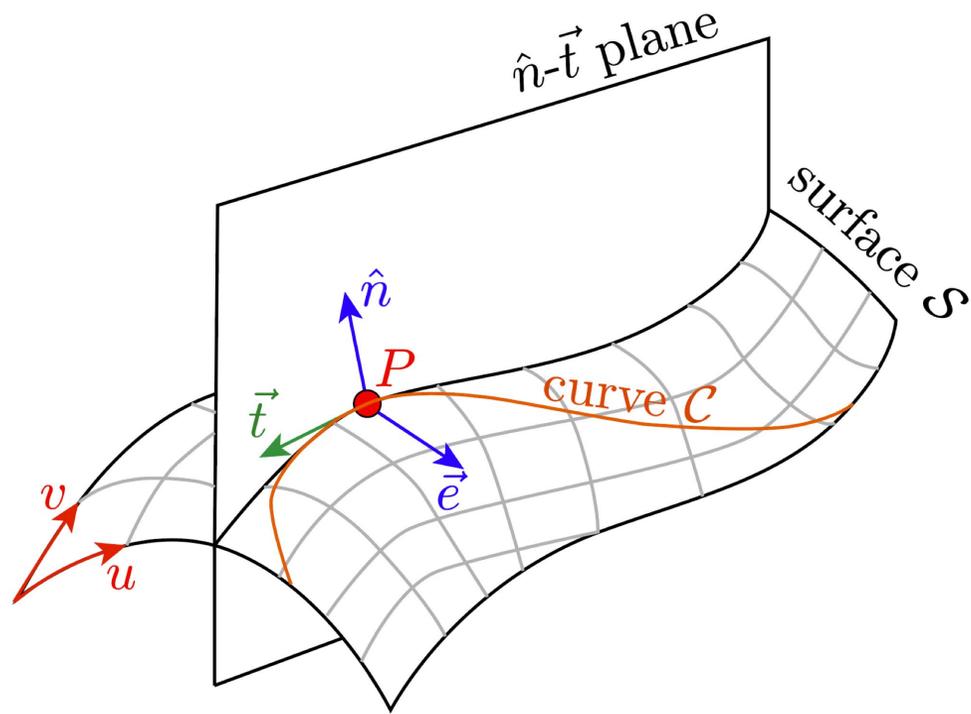
一般曲面空间中的转移：
向量与两点的测地线的夹角不变



球面示例



平行转移约束



平行转移切平面的矢量 \mathbf{a}

切平面的约束 $\mathbf{a} \cdot \mathbf{n} = 0$

矢量 \mathbf{a} 的变化 (转动) : $\dot{\mathbf{a}} = \boldsymbol{\Omega} \times \mathbf{a}$

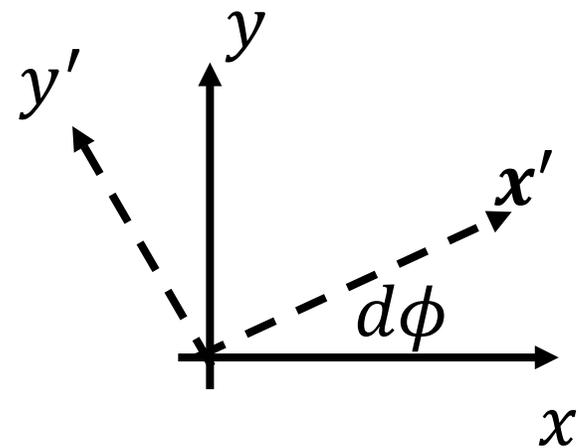
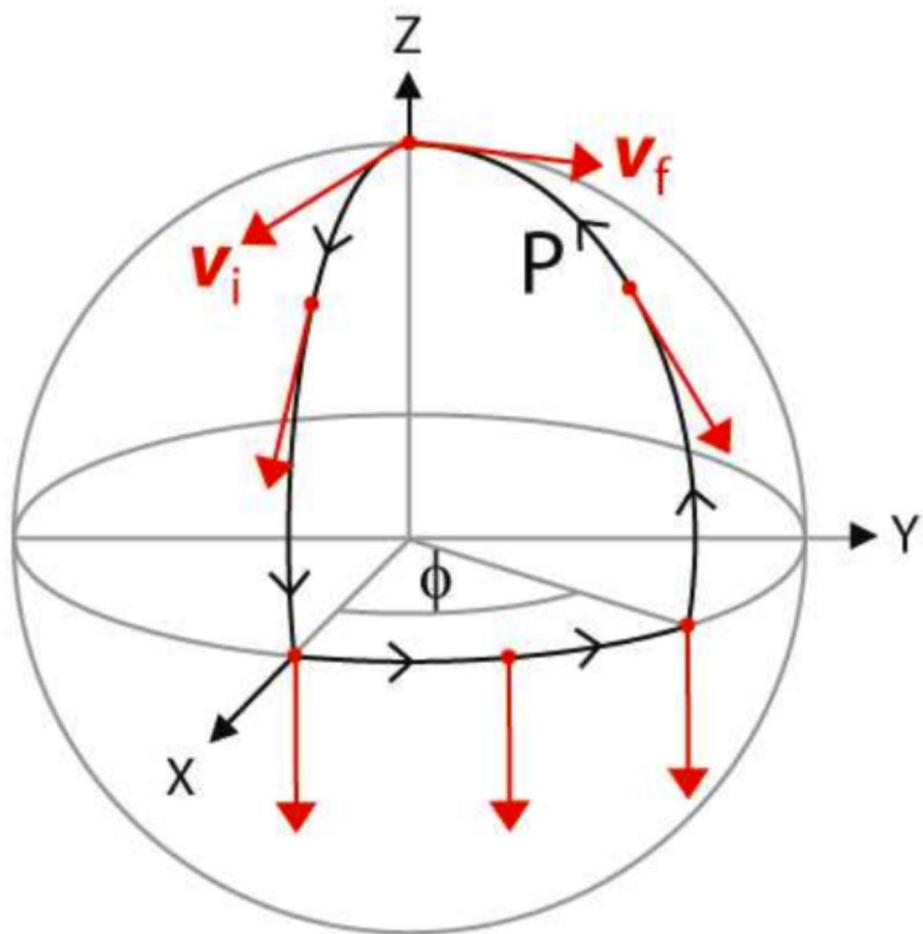
平行转移的约束: $\boldsymbol{\Omega} \cdot \mathbf{n} = 0$

补充局域坐标架: $\mathbf{b} = \mathbf{n} \times \mathbf{a}$

定义切空间的复矢量: $|\eta\rangle = \mathbf{a} + i \mathbf{b}$

平行转移 $\Leftrightarrow \langle \eta | i \partial | \eta \rangle = 0$

非和乐角



$$|\eta\rangle = x + i y$$

$$\begin{aligned} \langle \eta | i \partial | \eta \rangle d\lambda &= i \langle \eta | \eta' \rangle - i \\ &= i e^{-i d\phi} - i \\ &= d\phi \end{aligned}$$

$$\phi = \oint \langle \eta | i \partial | \eta \rangle d\lambda$$

假设在某个位置 局域固定坐标系=平行运输坐标系
则在下一个位置二者不等，也即固定坐标系不需被平行
运输。二者的夹角给出非和乐角。

量子力学中的对应

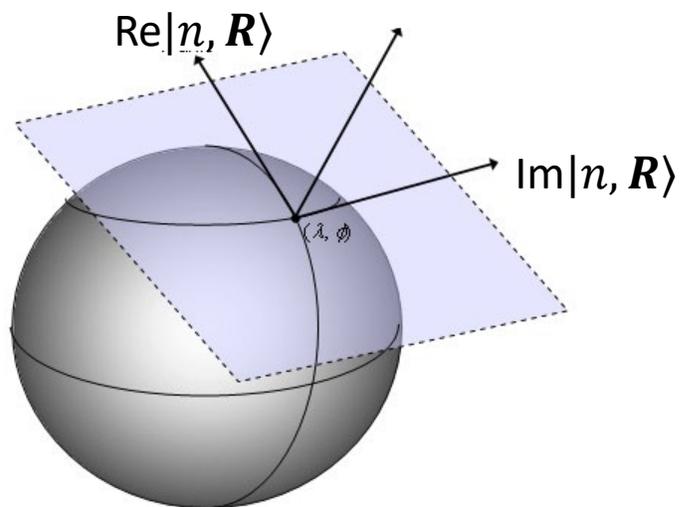
	曲面几何	量子力学中的对应
坐标架	局域坐标	参数空间
基矢	切平面的基矢	哈密顿量本征矢
平行转移	$\langle \eta i\partial \eta \rangle = 0$	$\langle \psi i\partial \psi \rangle = 0$
非和乐性	非和乐角	贝利相位
曲率	高斯曲率	贝利曲率
拓扑数	欧拉数	陈数

参数空间

哈密顿量依赖于参数，本征态也会有此性质

$$H(\mathbf{R})|n, \mathbf{R}\rangle = E_n(\mathbf{R})|n, \mathbf{R}\rangle$$

本征矢为复数，以其实部和虚部类比于参量空间的“切空间”



$$|\eta\rangle = \mathbf{a} + i \mathbf{b} \rightarrow |n, \mathbf{R}\rangle$$

量子态的演化

薛定谔方程

$$i\partial_t|\psi\rangle = H(\mathbf{R}(t))|\psi\rangle$$

瞬时本征矢分解

动力学相位

$$|\psi\rangle = \sum_n c_n(t) e^{-i\theta_n(t)} |n, \mathbf{R}(t)\rangle$$

$$\theta_n(t) = \int^t \frac{dt'}{\hbar} E_n[\mathbf{R}(t')]$$

求解

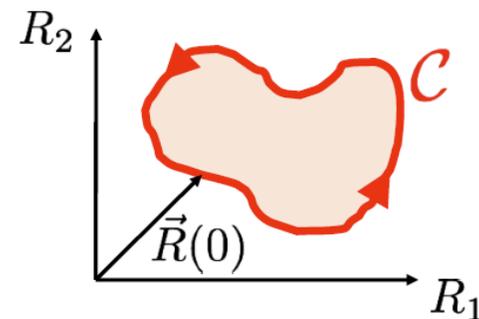
$$\begin{aligned} \partial_t c_n(t) &= - \sum_m c_m(t) \langle n, \mathbf{R}(t) | \partial_t |m, \mathbf{R}(t)\rangle \\ &= - c_n(t) \langle n, \mathbf{R}(t) | \partial_t |n, \mathbf{R}(t)\rangle \\ &\quad - \sum_{m \neq n} c_m(t) \langle n, \mathbf{R}(t) | \partial_t |m, \mathbf{R}(t)\rangle \end{aligned}$$

贝利相位：绝热循环演化

绝热条件： $\mathbf{R}(t)$ 变化足够缓慢， n 指标没有变化

$$\partial_t c_n(t) = -c_n(t) \langle n, \mathbf{R}(t) | \partial_t | n, \mathbf{R}(t) \rangle$$

循环演化： $\mathbf{R}(T) = \mathbf{R}(0)$



取如下解形式

$$c_n(t) = e^{i\gamma_n(t)}$$

相位满足如下方程

$$\partial_t \gamma_n(t) = i \langle n, \mathbf{R}(t) | \partial_t | n, \mathbf{R}(t) \rangle = i \langle n, \mathbf{R}(t) | \partial_{\mathbf{R}} | n, \mathbf{R}(t) \rangle \cdot \dot{\mathbf{R}}$$

贝利相位

$$\gamma_n(t) = i \oint \langle n, \mathbf{R}(t) | \partial_{\mathbf{R}} | n, \mathbf{R}(t) \rangle d\mathbf{R}$$

平行转移：量子版本

构造如下态矢量

$$|\eta_n\rangle = e^{i\gamma_n(t)} |n, \mathbf{R}\rangle$$

其将满足如下方程

$$\langle \eta_n | \partial_t | \eta_n \rangle = 0$$

此态矢量在参量空间中被平行转移

贝利相位即为非和乐角

