Jacobs' room-temperature shift[4] when corrected for the change in compressibility with thermal expansion.) If the two curves are superimposed, the low-energy side is almost identical, while the high-energy side differs slightly. The effect of this change is to increase the apparent total width at half-maximum under pressure by 3.5% to 0.017 eV. Jacobs[4] also measured an apparent increase in half-width in CsCl at room temperature of about 5% in 4000 atm.

Maisch and Drickamer[5] studied the effect of pressure to 50 000 atm on CsCl and found a new band, the K' band, with peak energy about 0.1 eV higher than the F band which grows at the expense of the F band with increasing pressure and is reversible upon decrease in pressure. It is thus possible that the high-energy component of the triplet is related to this K' band and is increasing in relative importance with pressure. However, no such K' band was seen in this range of hydrostatic pressures either for RbCl above or below its transition pressure, or for KBr, in which they observed a prominent K' band at considerably higher pressures. Either the K' band is not observable in the present range of pressures, or its appearance in Drickamer's experiments is due to shear and pressure inhomogeneities in his apparatus which are not present in our gas system.

SUMMARY

Using a novel technique a new absorption band has been formed by x irradiation of RbCl above its polymorphic transition pressure. The position, width, and temperature and pressure dependence of this band support its designation as the new F band in this CsCl-type phase. The band could be converted to the normal F band by reversing the transition at low temperatures. The shape of this new band has been examined under pressure at liquid-helium temperature, and shows no evidence of the multiplet structure observed in the cesium halides, thus indicating that the lattice structure is not responsible for this effect.

The pressure measurements on CsCl prove that for still another way of forming and observing the principal band, the triplet components all appear to be due to the F center.

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Magnetic Translation Group*

J. ZAK

National Magnet Laboratory,† Massachusetts Institute of Technology, Cambridge, Massachusetts

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In this paper a group-theoretical approach to the problem of a Bloch electron in a magnetic field is given. A magnetic translation group is defined and its properties, in particular its connection with the usual translation group, are established.

I. INTRODUCTION

THE translation symmetry of the Hamiltonian for an electron in a periodic potential leads to the classification of the solutions of the corresponding problem by means of a wave vector $\mathbf{k}$ and to the possibility of introducing Bloch functions. When a constant magnetic field is also present, the Hamiltonian is no longer invariant under the translation group. However, operators may be defined which commute with the Hamiltonian of a Bloch electron in a magnetic field.[1,2]

In this paper, a magnetic translation group which commutes with the Hamiltonian is defined and its general properties are established. In a following paper, we derive the irreducible representations of the magnetic translation group and give the classification of the solutions of the Schrödinger equation for an electron in both a periodic electric potential and a constant magnetic field.

II. DEFINITION OF THE MAGNETIC TRANSLATION GROUP

Let a Bravais lattice be represented by the vectors

$$\mathbf{R}_n = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3 \tag{1}$$

(where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are the unit cell vectors and n_1, n_2, n_3 are integers), each of which defines a point in the Bravais lattice. Let us define a path joining the origin O with the

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* Supported by the U. S. Air Force Office of Scientific Research.

† Permanent address: Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel.

[1] P. G. Harper, Proc. Phys. Soc. (London) **A68**, 879 (1955); H. J. Fischbeck, Phys. Stat. Solidi **3**, 1082 (1963).

[2] Recently, Brown [E. Brown, Bull. Am. Phys. Soc. **8**, 257 (1963); Phys. Rev. **133**, A1038 (1964)] has considered magnetic translation operators for constructing a ray representation of the usual translation group.

point defined by $\mathbf{R}_n$. We start from point O and move in a direction of another lattice point, say, given by $\mathbf{R}_1$; from $\mathbf{R}_1$ we move in a direction, say, $\mathbf{R}_2$ to the point given by $\mathbf{R}_1+\mathbf{R}_2$ and so on until we achieve the point given by $\mathbf{R}_n$. Clearly, the point given by $\mathbf{R}_n$ can be achieved by different choices of vectors $\mathbf{R}_1$, $\mathbf{R}_2$ and so on. Let *one* of the possibilities be given by the vectors $\mathbf{R}_1, \mathbf{R}_2, \cdots, \mathbf{R}_i$. We denote the described path by the symbol

$$|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i). \tag{2}$$

According to (2) we achieved the point of $\mathbf{R}_n$ by means of i steps. Also,

$$\mathbf{R}_n=\mathbf{R}_1+\mathbf{R}_2+\cdots+\mathbf{R}_i. \tag{2a}$$

In a similar way, we could choose another set of vectors satisfying (2a) and define a different path joining O with the point defined by $\mathbf{R}_n$.

Let us now define an operator which depends on both the vector $\mathbf{R}_n$ and the path joining the point O to the point defined by $\mathbf{R}_n$:

$$\begin{aligned}\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)&\\ =\exp\{(i/\hbar)\mathbf{R}_n\cdot[\mathbf{p}+(e/c)\mathbf{A}]\}&\\ \times\exp\{(i/2)[\mathbf{R}_1\times\mathbf{R}_2+\mathbf{R}_1\times\mathbf{R}_3+\cdots&\\ +\mathbf{R}_{i-1}\times\mathbf{R}_i]\cdot\mathbf{h}\},&\end{aligned} \tag{3}$$

where $\mathbf{h}=e\mathbf{H}/\hbar c$, $\mathbf{H}$ is the magnetic field, $\mathbf{A}$ is the vector potential of the magnetic field, $\mathbf{p}$ is the momentum operator of the electron, and e is its charge. The order of the vectors in the vector products in (3) is given by their order in path (2). The expression

$$\tfrac{1}{2}[\mathbf{R}_1\times\mathbf{R}_2+\mathbf{R}_1\times\mathbf{R}_3+\cdots+\mathbf{R}_{i-1}\times\mathbf{R}_i]\cdot\mathbf{h} \tag{4}$$

has a very simple meaning. To show this, let us take the projection of the path (2) onto the plane perpendicular to $\mathbf{h}$. This gives a path

$$|\mathbf{R}_1{}^p,\mathbf{R}_2{}^p,\cdots,\mathbf{R}_i{}^p), \tag{5}$$

where $\mathbf{R}_k{}^p$ is the projection of $\mathbf{R}_k$ onto the plane perpendicular to $\mathbf{h}$. It is easily seen that

$$\begin{aligned}&\tfrac{1}{2}[\mathbf{R}_1\times\mathbf{R}_2+\mathbf{R}_1\times\mathbf{R}_3+\cdots+\mathbf{R}_{i-1}\times\mathbf{R}_i]\cdot\mathbf{h}\\ &\quad=\tfrac{1}{2}[\mathbf{R}_1{}^p\times\mathbf{R}_2{}^p+\mathbf{R}_1{}^p\times\mathbf{R}_3{}^p+\cdots+\mathbf{R}_{i-1}{}^p\times\mathbf{R}_i{}^p]\cdot\mathbf{h}.\end{aligned} \tag{6}$$

The brackets with the factor $\frac{1}{2}$ on the right side of (6) give the area of the polygon enclosed by the vectors

$$\mathbf{R}_1{}^p,\ \mathbf{R}_2{}^p,\ \cdots,\ \mathbf{R}_i{}^p,\ -\mathbf{R}_n{}^p \tag{7}$$

(in which the vector $-\mathbf{R}_n{}^p$ was added in order to close the path). We thus see that the expression (4) gives the flux of the magnetic field (with the factor $e/\hbar c$) through the polygon which is obtained by projecting the vectors

$$\mathbf{R}_1,\ \mathbf{R}_2,\ \cdots,\ \mathbf{R}_i,\ -\mathbf{R}_n \tag{8}$$

onto the plane perpendicular to $\mathbf{H}$. Therefore, the elements (3) may be written in a different form

$$\begin{aligned}\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)=\exp\{(i/\hbar)\mathbf{R}_n\cdot[\mathbf{p}+(e/c)\mathbf{A}]\}&\\ \times\exp\{i\varphi(\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\},&\end{aligned} \tag{9}$$

where $\varphi(\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)$ is the flux of the magnetic field through the polygon (7) multiplied by $e/\hbar c$. It should be noted that the path (2), and therefore its projection onto the plane perpendicular to $\mathbf{H}$, can be very complicated. For example, from the point given by $\mathbf{R}_n$, which may be reached along the path (2), one may continue along the vector $-\mathbf{R}_n$ and then again turn back to point $\mathbf{R}_n$ by path (2). In such a case, the area in (6) will be doubled. The second exponent in the definition of the elements (3) or (9) thus expresses the curl nature of the magnetic field.

One may now show that the operators defined in (3) or (9) commute with the Hamiltonian of a Bloch electron in a magnetic field for a proper choice of the gauge. The Hamiltonian is

$$H=1/2m[\mathbf{p}-(e/c)\mathbf{A}]^2+V(\mathbf{r}), \tag{10}$$

m is the mass of the electron and $V(\mathbf{r})$ is the potential of the periodic field. It is easily shown that

$$\left[p_i+\frac{e}{c}A_i,\ p_k-\frac{e}{c}A_k\right]=i\hbar\frac{e}{c}\left[\frac{\partial A_k}{\partial x_i}+\frac{\partial A_i}{\partial x_k}\right]. \tag{11}$$

Therefore, if the gauge in (10) is chosen in a way that

$$\partial A_k/\partial x_i+\partial A_i/\partial x_k=0;\quad i,k=1,2,3, \tag{12}$$

the commutator (11) will vanish and the operators (3) will commute with the Hamiltonian (10). Relation (12) holds, for example, for a gauge

$$\mathbf{A}=\tfrac{1}{2}[\mathbf{H}\times\mathbf{r}]. \tag{13}$$

In order to show that the operators defined in (3) form a group, we shall check that a product of two such operators is again an operator of the same form. Let us take two operators

$$\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\quad\text{and}\quad\tau(\mathbf{R}_n{}'|\mathbf{R}_1{}',\mathbf{R}_2{}',\cdots,\mathbf{R}_k{}').$$

Their product is

$$\begin{aligned}&\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\tau(\mathbf{R}_n{}'|\mathbf{R}_1{}',\mathbf{R}_2{}',\cdots,\mathbf{R}_k{}')\\ &\quad=\exp\{(i/\hbar)(\mathbf{R}_n+\mathbf{R}_n{}')\cdot[\mathbf{p}+(e/c)\mathbf{A}]\}\\ &\qquad\times\exp\{(i/2)[\mathbf{R}_1\times\mathbf{R}_2+\cdots+\mathbf{R}_1\times\mathbf{R}_1{}'+\cdots\\ &\qquad\qquad+\mathbf{R}_{k-1}{}'\times\mathbf{R}_k{}']\cdot\mathbf{h}\},\end{aligned} \tag{14}$$

where the second exponent is defined by the path

$$|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i,\mathbf{R}_1{}',\mathbf{R}_2{}',\cdots,\mathbf{R}_k{}'). \tag{15}$$

In obtaining relation (14) we used the following expressions:

$$\left[\frac{i}{\hbar}\mathbf{R}_n\cdot\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right),\ \frac{i}{\hbar}\mathbf{R}_n{}'\cdot\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\right]=i(\mathbf{R}_n\times\mathbf{R}_n{}')\cdot\mathbf{h} \tag{16}$$

(without any gauge limitations) and

$$e^Ae^B=e^{A+B+\frac{1}{2}[A,B]}$$

for A and B such that $[A,B]$ commutes with both

A and B. The product (14) can be written according to the definition (3) as follows:

$$\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\tau(\mathbf{R}_n'|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k') \\ =\tau(\mathbf{R}_n+\mathbf{R}_n'|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i,\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k'). \quad (17)$$

We now have a multiplication rule for the elements defined in (3): In order to multiply two elements [left side of (17)], we have to add the vectors $\mathbf{R}_n$ and $\mathbf{R}_n'$, and the paths $|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)$ and $|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k')$; the vector on which the product [right side of (17)] depends is $\mathbf{R}_n+\mathbf{R}_n'$ and the path is $|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i,\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k')$ [the latter is obtained by displacing the path $|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k')$ by the vector $\mathbf{R}_n$, thus obtaining a path joining point O with the point defined by the vector $\mathbf{R}_n+\mathbf{R}_n'$]. Thus, the product of two elements of the form (3) is an element of the same form. One sees that the elements defined in (3) form a group because one may also easily show that $\tau(-\mathbf{R}_n|-\mathbf{R}_i,-\mathbf{R}_{i-1},\cdots,-\mathbf{R}_1)$ is reciprocal to the element $\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)$. We shall call this group the magnetic translation group (M.T.G.) and denote it by G.[2a] It should be noted that the operators

$$\exp\{(i/\hbar)\mathbf{R}_n\cdot(\mathbf{p}+(e/c)\mathbf{A})\} \quad (18)$$

do not form a group (as pointed out by Brown[2]), because a product of two such operators is not an operator of the form (18).

The group G defined in (3) is an infinite group for two reasons. First, there are an infinite number of vectors $\mathbf{R}_n$. Secondly, for any vector $\mathbf{R}_n$ an infinite number of different paths joining point O with $\mathbf{R}_n$ can be defined. We can, for instance, obtain from path P a different one P' by starting at and returning to any point belonging to the Bravais lattice on path P and adding a closed path to the path P at the mentioned point.

The structure of the group G becomes more apparent after its connection with the usual translation group, denoted by R, is established. To do this, let us define, for every vector $\mathbf{R}_n$, a set of elements $H(\mathbf{R}_n)$ to which all the elements $\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)$ of G belong. Each such set $H(\mathbf{R}_n)$ consists of an infinite number of elements. It is easy to see that we can look at the sets $H(\mathbf{R}_n)$ as at elements of a group. If we multiply an element of the set $H(\mathbf{R}_n)$ by an element of set $H(\mathbf{R}_n')$, we shall get an element of the set $H(\mathbf{R}_n+\mathbf{R}_n')$. $H(\mathbf{O})$ is the identity and $H(-\mathbf{R}_n)$ is reciprocal to $H(\mathbf{R}_n)$. This group, denoted by H, is an infinite commutative group, and is isomorphic to the translation group R. The isomorphism follows from the one-to-one correspondence between $\mathbf{R}_n$ and $H(\mathbf{R}_n)$. The group G is homomorphic to H,

$$\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\rightarrow H(\mathbf{R}_n), \quad (19)$$

where the arrow shows the one-sided correspondence of the elements of G with those of H. From the homomorphism of G to H and the isomorphism of H and R, it follows that the magnetic translation group G is homomorphic to the usual translation group R. The last fact will be used to introduce the Born-von Karman boundary conditions for the representations of the magnetic translation group.

The group G has an invariant subgroup which is of great importance for constructing its irreducible representations. Consider the vectors r_m in the plane $\mathbf{a}_1$, $\mathbf{a}_3$ and construct elements of the group G:

$$\tau(\mathbf{r}_m|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_f') \\ =\exp\{(i/\hbar)\mathbf{r}_m\cdot[\mathbf{p}+(e/c)\mathbf{A}]\} \\ \times\exp\{(i/2)[\mathbf{R}_1'\times\mathbf{R}_2'+\mathbf{R}_1'\times\mathbf{R}_3'+\cdots \\ +\mathbf{R}_{f-1}'\times\mathbf{R}_f']\cdot\mathbf{h}\}, \quad (20)$$

where

$$\mathbf{r}_m=m_1\mathbf{a}_1+m_3\mathbf{a}_3 \quad (21)$$

and $\mathbf{R}_1'$, $\mathbf{R}_2'$, $\cdots$, $\mathbf{R}_f'$ are arbitrary vectors of (1) except for the requirement that they form a path joining point O with the point defined by $\mathbf{r}_m$. The elements in (20) form a subgroup of G because a product of any two elements from (20) is also an element belonging to (20). Let us denote this subgroup by F. A similarity transformation by means of any element $\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)$ of G when applied to an element $\tau(\mathbf{r}_m|\mathbf{R}_1',\mathbf{R}_2',\cdots\mathbf{R}_f')$ of F gives the following result:

$$\tau^{-1}(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\tau(\mathbf{r}_m|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_f') \\ \times\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i) \\ =\tau(\mathbf{r}_m|-\mathbf{R}_i,-\mathbf{R}_{i-1},\cdots,-\mathbf{R}_1,\mathbf{R}_1',\mathbf{R}_2',\cdots, \\ \mathbf{R}_f',\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i). \quad (22)$$

The relation (22) shows us that subgroup F is an invariant subgroup of group G. It is clear that another invariant subgroup of G can be obtained by taking in (20) vectors $\mathbf{r}_m'$ in the plane $\mathbf{a}_2$, $\mathbf{a}_3$. This latter invariant subgroup is isomorphic to F. With respect to the invariant subgroup F, group G can be written as follows:

$$G=\tau(\mathbf{O}|\mathbf{O})F+\tau(\mathbf{a}_2|\mathbf{a}_2)F+\cdots \\ +\tau(n_2\mathbf{a}_2|n_2\mathbf{a}_2)F+\cdots. \quad (23)$$

It is of interest to note that when the magnetic field $\mathbf{H}$ is in the direction $\mathbf{a}_3$ the subgroup F is a commutative one, while the elements of F do not commute with those elements of G that do not belong to F.

So far, the groups have all been infinite. For constructing their irreducible representation, it is convenient to deal with finite groups. To do this, the Born-von Karman boundary conditions can be applied.

III. FINITE MAGNETIC TRANSLATION GROUP

The finiteness of the usual translation group is achieved by looking for its special representations, namely, by seeking representations which satisfy the

[2a] After the abstract of this paper appeared in Phys. Rev. Letters, Gerald A. Peterson pointed out (private communication) that in his Ph.D. thesis he has defined a closed set of magnetic translation operators and has called it magnetic translation group [G. A. Peterson, Ph.D. thesis, Cornell University, 1960 (unpublished)].

Born-von Karman boundary conditions

$$D\{(\epsilon|\mathbf{a}_1)^N\}=D\{(\epsilon|\mathbf{a}_2)^N\}=D\{(\epsilon|\mathbf{a}_3)^N\}=D\{(\epsilon|0)\}\,,\quad(24)$$

where $(\epsilon|\mathbf{a}_1)$, $(\epsilon|\mathbf{a}_2)$, $(\epsilon|\mathbf{a}_3)$ are translations in the directions of the unit cell vectors (ϵ is the identity of the rotation group in the usual notation[3]), N is a large integer, and D denotes a representation of the translation group R. When seeking representations of R which satisfy the conditions (24), we can consider the group R as being a finite group $\bar{R}$ of order N^3, the elements of which are given by the direct product of the following three groups:

$$(\epsilon|\mathbf{a}_1)^i;\ (\epsilon|\mathbf{a}_2)^j;\ (\epsilon|\mathbf{a}_3)^k;\quad i,\ j,\ k=1,\,2,\,\cdots,\,N.\quad(25)$$

Similarly, in order to make G finite, we require similar boundary conditions to those imposed for the representations of the translation group (24). To do this, we use the fact that the translation group R is isomorphic to H and require the following conditions on the representations of H:

$$D\{H^N(\mathbf{a}_1)\}=D\{H^N(\mathbf{a}_2)\}=D\{H^N(\mathbf{a}_3)\}=D\{H(\mathbf{0})\}\,.\quad(26)$$

With the conditions (26), the group H can be considered as being finite and given by the direct product of the following three groups:

$$\{H^i(\mathbf{a}_1)\}\,;\ \{H^j(\mathbf{a}_2)\}\,;\ \{H^k(\mathbf{a}_3)\}\,;\quad i,\ j,\ k=1,\,2,\,\cdots,\,N.\quad(27)$$

Let us denote the last group by $\bar{H}$ and the corresponding M.T.G. by $\bar{G}$. It is clear that again $\bar{R}$ and $\bar{H}$ are isomorphic.

The requirements (26) mean that elements

$$\tau(Na_k|\mathbf{R}_1,\mathbf{R}_2,\cdots\mathbf{R}_f)\,;\quad k=1,\,2,\,3\,,\quad(28)$$

of G can be considered as constant factors. From this it follows that the elements (28) commute with all the elements of G. It is easy to show that

$$\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\tau(\mathbf{R}_n'|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k)=\tau(\mathbf{R}_n'|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_k')\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)\times\exp\{i(\mathbf{R}_n\times\mathbf{R}_n')\cdot\mathbf{h}\}\,.\quad(29)$$

Hence, the elements (28) commute with all the elements of G if the exponential term in (29) equals 1 for $\mathbf{R}_n=N\mathbf{a}_k$ ($k=1$, 2, 3). We have therefore the following condition:

$$(N\mathbf{a}_k\times\mathbf{R}_n')\cdot\mathbf{h}=2\pi m\,;\quad k=1,\,2,\,3\,,\quad(30)$$

where m is an integer and $\mathbf{R}_n'$ is any vector from (1). Before discussing condition (30), let us derive another condition from the requirements (26). Consider two elements of the form (28) and take their product

$$\tau(N\mathbf{a}_i|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_f)\tau(N\mathbf{a}_k|\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_g')=\tau(N\mathbf{a}_i+N\mathbf{a}_k|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_f,\mathbf{R}_1',\mathbf{R}_2',\cdots,\mathbf{R}_g')\,.\quad(31)$$

[3] G. F. Koster, *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc.. New York, 1957), Vol. 5.

But since the elements on the left side of (31) should behave as constant factors, the additional factor on the right side of (31),

$$\exp\{\tfrac{1}{2}(N\mathbf{a}_i\times N\mathbf{a}_k\cdot\mathbf{h}\}\,,\quad(32)$$

should be equal to 1. We have thus

$$(N\mathbf{a}_i\times N\mathbf{a}_k)\cdot\mathbf{h}=4\pi m\,;\quad i,\,k=1,\,2,\,3\,,\quad(33)$$

for odd N [for even N there is no additional condition to the one expressed by (30)]. Combining (30) and (33) we get for odd N the condition

$$(N\mathbf{a}_k\times\mathbf{R}_n')\cdot\mathbf{h}=4\pi m\,;\quad k=1,\,2,\,3\,,\quad(34)$$

which is stronger than the requirement (30). We now write the conditions (30) and (34) in a more convenient form, using the fact that (30) and (34) will be satisfied for any vector $\mathbf{R}_n'$ if they hold for the unit cell vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$:

$$N(\mathbf{a}_k\times\mathbf{a}_l)\cdot\mathbf{h}=2\pi n_{kl}\quad\text{(for even }N)\,,\quad(35\text{a})$$

$$N(\mathbf{a}_k\times\mathbf{a}_l)\cdot\mathbf{h}=4\pi n_{kl}\quad\text{(for odd }N)\,,\quad(35\text{b})$$

where n_{kl} is an integer and k, $l=1$, 2, 3. By using the definition of the reciprocal lattice vectors

$$K_i=(2\pi/V)\mathbf{a}_j\times\mathbf{a}_k\,,\quad(36)$$

where i, j, k form the cycle (123) and V is the volume of this unit cell, we can write the conditions (35a), (35b) as follows:

$$N(V/2\pi)\mathbf{K}_m\cdot\mathbf{h}=2\pi m\quad\text{(for even }N)\,,\quad(37\text{a})$$

$$N(V/2\pi)\mathbf{K}_m\cdot\mathbf{h}=4\pi m\quad\text{(for odd }N)\,.\quad(37\text{b})$$

Here m is an integer. Equations (37a) and (37b) show that

$$\mathbf{h}=(2\pi/V)(\mathbf{R}_n/N)\quad\text{(for even }N)\,,\quad(38\text{a})$$

$$\mathbf{h}=(4\pi/V)(\mathbf{R}_n/N)\quad\text{(for odd }N)\,.\quad(38\text{b})$$

The requirements (38a), (38b) which follow from (26) are limitations on both the possible directions and values of the magnetic field. Let us choose the magnetic field in the direction $\mathbf{a}_3$ [this can always be done when conditions (38a), (38b) hold] and let $\mathbf{R}_n=n\mathbf{a}_3$; then

$$\mathbf{h}=(2\pi/V)(n/N)\mathbf{a}_3\quad\text{(for even }N)\,,\quad(39\text{a})$$

$$\mathbf{h}=(4\pi/V)(n/N)\mathbf{a}_3\quad\text{(for odd }N)\,.\quad(39\text{b})$$

For N sufficiently large the limitations (39a), (39b) on the magnetic field $\mathbf{H}$ are not essential. It should, however, be noted that when the crystal has dimensions of the order of 1 cm ($N\approx10^8$), the fields satisfying conditions (39a), (39b) differ by quanta of the order of 10 G (i.e., the lowest nonzero magnetic field satisfying the above conditions is of the order of 10 G). Unlike the boundary conditions for the usual translation group (which lead to no physical consequences), in the case of the M.T.G. we have restrictions on the magnetic field even for crystals of the dimensions of 1 cm on which bulk experiments have been carried out.

We now show that the Born-von Karman conditions (26) on the representations of H, which led to requirements (39a), (39b), turn the group G into a finite one $\bar{G}$. By using (39a), (39b), we find that the elements of $\bar{G}$ are of the form

$$\tau(\mathbf{R}_n|\mathbf{R}_1,\mathbf{R}_2,\cdots,\mathbf{R}_i)=\exp\{(i/\hbar)\mathbf{R}_n\cdot(\mathbf{p}+(e/c)\mathbf{A})\}\times\begin{cases}\exp\{2\pi i m(n/N)\} & (\text{for odd } N), \quad (40a)\\ \exp\{\pi i m(n/N)\} & (\text{for even } N), \quad (40b)\end{cases}$$

where m is the coefficient by the product $\mathbf{a}_1\times\mathbf{a}_2$ in the expression

$$\mathbf{R}_1\times\mathbf{R}_2+\mathbf{R}_1\times\mathbf{R}_3+\cdots+\mathbf{R}_{i-1}\times\mathbf{R}_i. \qquad (41)$$

From Eqs. (40a), (40b), we see that the number of elements of $\bar{G}$ is finite. In order to count the number of elements of $\bar{G}$, we treat the cases for even and odd N separately. For odd N, condition (39b) is imposed and the elements of $\bar{G}$ are given by (40a). The number of different values of the vector $\mathbf{R}_n$ is N^3. For each $\mathbf{R}_n$ there are different elements in (40a) arising from the second exponential factor. We first show that the number m in (40a) [and also in (40b)] can take all integer values. This follows from the fact that we can add to the path joining the point O with point $\mathbf{R}_n$ a closed path which will change (41) by an elementary area $\mathbf{a}_1\times\mathbf{a}_2$ and will change m in the expressions (40a), (40b) by unity. Let us now assume that p is the largest common factor of n and N (when there is no common factor, $p=1$), and let $N/p=N'$, $n/p=n'$. In order to count the different elements of (40a) for a given $\mathbf{R}_n$, let us check that

$$\exp\{2\pi i m(n/N)\}\neq\exp\{2\pi i m'(n/N)\} \qquad (42)$$

for $m'\neq m$ (we exclude here those cases for which m' differs from m by the number N'). By assuming the equality sign in (42) to hold, we get

$$\exp\{2\pi i(m-m')(n/N)\}=1. \qquad (43)$$

Relation (43) holds only when

$$(m-m')(n/N)=(m-m')(n'/N')=\text{integer}. \qquad (44)$$

But since n' and N' have no common factor, the relation (44) is possible only for $m-m'\geq N'$, which means that m' should differ from m at least by N' in order to get the equality sign in (42). The second exponential in (40a) thus takes on N' different values for a given $\mathbf{R}_n$. Since $\mathbf{R}_n$ itself has N^3 different values, there are N^3N' different elements in $\bar{G}$ in the case of odd N. In a similar way, it is easy to show that there are $N^3(2N')$ different elements in $\bar{G}$ for even N.

It is of interest to note that when the magnetic field $\mathbf{H}$ is very strong (of the order of 10^{10} G), we can have the case that

$$\mathbf{h}=(4\pi/V)\mathbf{a}_3, \qquad (45)$$

in which case the group $\bar{G}$ consists of N^3 commutative elements of the form

$$\exp\{(i/\hbar)\mathbf{R}_n\cdot(\mathbf{p}+(e/c)\mathbf{A})\} \qquad (46)$$

and is isomorphic to the usual translation group.

IV. CONCLUSION

We have defined here a magnetic translation group G which commutes with the Hamiltonian for a Bloch electron in a magnetic field. In a following paper, we construct the irreducible representations of this group and give the classification of the solutions of Schrödinger's equation for an electron in both a periodic electric potential and a constant magnetic field.

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