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LETTER TO THE EDITOR

Linear momentum in ferromagnets

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Abstract. The paradox of linear momentum in ferromagnets is explained in terms of the exchange of the momentum between coherent and incoherent degrees of freedom in condensed matter.

In some continual field theories, both in condensed matter and in particle physics, there is the energy–momentum-tensor problem: the linear momentum is either not well defined, or is not conserved. A typical example is general relativity where it is impossible to construct the energy–momentum tensor which is co-variant under general coordinate transformations (Einstein 1916, Faddeev 1982). The analogous phenomenon takes place in ferromagnets where the canonical momentum of the magnetisation motion is not invariant under spin rotations (Haldane 1986) and in superfluid ^{3}He–A. The linear momentum of the coherent superfluid motion at $T = 0$ in ^{3}He–A does not coincide with canonical momentum and is not conserved, while the angular momentum is not well defined.

The reason for the anomalous behaviour of the linear momentum in ^{3}He–A is now understood: both linear and angular momenta of coherent motion transfer to the incoherent degrees of freedom in the subsystem of fermionic excitations (Volovik 1986a, b) producing a strong analogy with chiral anomaly in quantum electrodynamics. The source of the fermionic excitation momentum corresponds to the source of the chiral current in QED and is described by the same Schwinger term (on the chiral anomaly in QED see e.g. Huang 1982).

Here with the simple ferromagnetism model of de-localised electrons (see e.g. Ziman 1972) we consider how the canonical momentum in ferromagnets becomes well defined due to a fermionic background. As in ^{3}He–A the anomaly in the linear momentum of coherent motion is related with momentum transfer to the fermionic excitations and may be described in terms of the Wess–Zumino type action. This scenario seems to be the general case for those condensed media where the energy–momentum tensor for soft (hydrodynamical) modes is not well defined. Possibly the explanation in terms of the energy–momentum transfer to the incoherent background may be applied to the energy–momentum problem in general relativity, especially if one takes into account that the analogue of gravitation appears in condensed matter. In particular among the variables describing the coherent motion of superfluid vacuum in ^{3}He–A there are components of the metric tensor of the medium which define the dynamics of fermions in curved space (Volovik 1986c).

The simple model of ferromagnetism in the system of de-localised fermions with spin $\frac{1}{2}$ is described by:

$$S = \int d^3x\, dt\left(- i\Psi^+_\alpha\, \partial_t \Psi_\alpha + \frac{1}{2m}\partial_i \Psi^+_\alpha\, \partial_i \Psi_\alpha - \tfrac{1}{4}g(\Psi^+_\alpha \sigma_{\alpha\beta}\Psi_\beta)^2\right) \qquad (1)$$

where $\Psi_\alpha(\boldsymbol{r}, t)$ is the fermionic field and $\boldsymbol{\sigma}$ are the Pauli matrices. If the coupling constant g is large enough ($gN_F > 1$, where N_F is the density of states on the Fermi surface, see Ziman 1972) the ferromagnetic transition occurs at which the 'quasi-average' appears, the density of magnetic moment

$$\boldsymbol{M}(\boldsymbol{r}, t) = \tfrac{1}{2}\hbar\gamma\langle\Psi^+ \boldsymbol{\sigma}\Psi\rangle \qquad (2)$$

where γ is the gyromagnetic ratio.

We are interested in the dynamics of the soft variable, the unit vector $\boldsymbol{m}(\boldsymbol{r}, t) = \boldsymbol{M}/|\boldsymbol{M}|$. If $\boldsymbol{m}(\boldsymbol{r}, t)$ changes slowly in space and time the fermionic ground state may be considered as two local unequally populated Fermi spheres of fermions with spins oriented along the local direction of $\boldsymbol{m}(\boldsymbol{r}, t)$. The difference in the population is $n_+ - n_- = 2|\boldsymbol{M}|/\gamma\hbar$, where the modulus of the magnetisation may be considered as constant in this approximation. The spin structure of the fermionic wavefunction in such a state is defined by a spin rotation matrix $\boldsymbol{U}(\boldsymbol{r}, t)$ which couples the local direction $\boldsymbol{m}(\boldsymbol{r}, t)$ with some fixed direction:

$$\Psi_\alpha(\boldsymbol{r}, t) = \Psi_\pm(\boldsymbol{r}, t)U_{\alpha\beta}(\boldsymbol{r}, t)\eta_{\beta\pm} \qquad \eta_+ = \begin{pmatrix}1\\0\end{pmatrix} \qquad \eta_- = \begin{pmatrix}0\\1\end{pmatrix} \qquad (3)$$

where Ψ_+ and Ψ_- are the local wavefunctions for the fermions in two local Fermi spheres. Inserting equation (3) in equation (1) and discarding the last term in equation (1) which defines the magnitude of $|\boldsymbol{M}|$ one obtains the action which describes the dynamics of two species of particles with opposite 'charges' $e = \pm$ in the 'electromagnetic' field $\boldsymbol{A}$, A_0:

$$S = \sum_e \int d^3x\, dt\, \Psi^+_e\left((-i\,\partial_t + eA_0) + \frac{1}{2m}(-i\,\partial_i - eA_i)^2 + \frac{1}{8m}(\partial_i \boldsymbol{m})^2\right)\Psi_e. \qquad (4)$$

There is the surplus of the positive 'charge' due to the different population of the Fermi spheres:

$$\sum_e en_e = \sum_e e\langle\Psi^+_e \Psi_e\rangle = 2|\boldsymbol{M}|/\gamma\hbar. \qquad (5)$$

The electromagnetic potentials $\boldsymbol{A}$ and A_0 are expressed in terms of the spin-rotation solid angles θ_a ($a = 1, 2, 3$)

$$A_0 = -\tfrac{1}{2}m_a\partial_t\theta_a \qquad A_i = \tfrac{1}{2}m_a\partial_i\theta_a. \qquad (6)$$

Among these three angles only two are physical, those which define the orientation of the $\boldsymbol{m}$ field. The third angle of the rotation about vector $\boldsymbol{m}$ may be chosen arbitrarily. The arbitrariness of this angle is equivalent to the choice of the gauge: the rotation of angle f about $\boldsymbol{m}$ corresponds to the gauge transformation in this 'electrodynamics':

$$A_i \to A_i + \tfrac{1}{2}\partial_i f \qquad A_0 \to A_0 - \tfrac{1}{2}\partial_t f \qquad \Psi_e \to \Psi_e \exp(\tfrac{1}{2}ief). \qquad (7)$$

The 'magnetic' and 'electric' fields do not depend on the gauge and are expressed in terms of the gradients of the $\boldsymbol{m}$ field:

$$\begin{aligned} F_{ik} &= \partial_i A_k - \partial_k A_i = -\tfrac{1}{2}\boldsymbol{m}\cdot(\partial_i \boldsymbol{m} \times \partial_k \boldsymbol{m}) \\ E_i &= -\partial_t A_i - \partial_i A_0 = \tfrac{1}{2}\boldsymbol{m}\cdot(\partial_t \boldsymbol{m}\times\partial_i \boldsymbol{m}). \end{aligned} \tag{8}$$

Thus we obtained the action (equation (4)) for the bosonic field $\boldsymbol{m}(\boldsymbol{r}, t)$ which defines the coherent motion in ferromagnets and for the rest incoherent fermionic fields. The dynamics of the coherent motion are obtained by the variation of S by $\boldsymbol{m}$. Taking into account that

$$\frac{\delta A_0}{\delta \boldsymbol{m}} = -\tfrac{1}{2}\boldsymbol{m}\times\partial_t \boldsymbol{m} \qquad \frac{\delta A_i}{\delta \boldsymbol{m}} = \tfrac{1}{2}\boldsymbol{m}\times\partial_i \boldsymbol{m} \tag{9}$$

one obtains the Landau–Lifshitz equation for $\boldsymbol{m}$

$$\frac{M}{\gamma}(\partial_t \boldsymbol{m} + v_i \partial_i \boldsymbol{m}) + \boldsymbol{m}\times\left(\frac{\partial F}{\partial \boldsymbol{m}} - \partial_i \frac{\partial F}{\partial(\partial_i \boldsymbol{m})}\right) = 0 \qquad F = n(\partial_i \boldsymbol{m})^2/8m \tag{10}$$

Here $n(\boldsymbol{r}, t) = n_+ + n_-$ and $\boldsymbol{v}(\boldsymbol{r}, t)$ are the local density and the local mean velocity of the fermions respectively:

$$n = \sum_e \langle \Psi_e^+ \Psi_e \rangle \qquad mnv_i = \sum_e \langle \Psi_e^+ (-\mathrm{i}\partial_i - eA_i)\Psi_e\rangle. \tag{11}$$

Usually equation (10) is written with $v = 0$ since the electronic liquid is at rest with respect to the crystal lattice due to the viscosity. At $v = 0$ and $n =$ constant the Landau–Lifshitz equation for $\boldsymbol{m}$ becomes closed. However the linear momentum density of this coherent subsystem is not well defined. For example the canonical definition of the momentum

$$P_i^{\text{coherent}} = \frac{\partial L}{\partial(\partial_t \theta_a)}\partial_i \theta_a = \frac{M}{\gamma} m_a \partial_i \theta_a = 2\frac{M}{\gamma}A_i \tag{12}$$

though it satisfies the momentum conservation law is not invariant under a gauge transformation. This known paradox in the Landau–Lifshitz theory (see e.g. Haldane 1986) is a result of the separation of the coherent motion from the other degrees of freedom. The correct definition of the linear momentum is obtained if one takes the incoherent motion into account.

The canonical momentum of the fermionic subsystem

$$P_i^{\text{incoherent}} = \sum_e \langle \Psi_e^+ (-\mathrm{i}\partial_i)\Psi_e\rangle \tag{13}$$

is also badly defined, as is well known, for the charged particles in an electromagnetic field; it is not gauge invariant. However the mass current of fermions, $mn\boldsymbol{v}$, is a well defined linear momentum of the total system being the sum of the canonical momenta of the subsystems:

$$mn\boldsymbol{v} = \boldsymbol{P}^{\text{coherent}} + \boldsymbol{P}^{\text{incoherent}} \tag{14}$$

This momentum obeys the following equation (neglecting dissipation, quadratic terms in $\boldsymbol{v}$ and compressibility of the fermionic liquid)

$$\partial_t(mn\boldsymbol{v}) = (n_+ - n_-)(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H}) \tag{15}$$

describing the motion of the 'charged' particles in 'electric' and 'magnetic' fields. According to Landau–Lifshitz equation (10) the right-hand side of (15) is a pure derivative:

$$(n_+ - n_-)(E_i + (\boldsymbol{v} \times \boldsymbol{H})_i) = - \partial_k \pi_{ik} \qquad \pi_{ik} = F\delta_{ik} - \frac{\partial F}{\partial(\partial_i \boldsymbol{m})} \partial_k \boldsymbol{m} \tag{16}$$

therefore the momentum (14) is conserved.

Thus for restoration of the correctly defined linear momentum density of the ferromagnets, additional hydrodynamical variables, describing the background (fermionic vacuum), were introduced: the density n and the normal velocity $\boldsymbol{v}$ of the fermionic liquid. Analogous variables, related to the normal motion of fermions, are introduced in ^{3}He–A at $T = 0$. Such hydrodynamical variables may be used only at small enough frequency, ω, as compared with the inverse relaxation time τ^{-1}. In the opposite limit case ($\omega\tau \gg 1$) the Landau–Lifshitz equation should be supplemented by a kinetic equation for the fermionic excitations.

Such incompleteness of the dynamical equations for coherent subsystem in condensed matter (the system of equations may be closed but according to the symmetry limitations is not self-consistent in the sense that the energy–momentum tensor is not well defined) leads to the unusual form of the action S, if it is expressed in terms of coherent variables only, i.e. after integration over the incoherent background. In the case of the ferromagnets the integration over the fermionic variables Ψ_e, Ψ_e^+ gives the following action:

$$S_{\text{eff}}[\boldsymbol{m}] = \int \mathrm{d}^3x \, \mathrm{d}t \, F[\boldsymbol{m}] + S_{\text{WZ}}. \tag{17}$$

Here S_{WZ} is the Wess–Zumino type action (for the Wess–Zumino term, see Wess and Zumino 1971, Witten 1983, Balachandran *et al* 1983 and in ^{3}He–A Volovik 1986b) defined in the five-dimensional space with the boundary being in the four-dimensional space–time continuum. The variation of the Wess–Zumino action is however defined in physical space–time:

$$\delta S_{\text{WZ}} = \int \mathrm{d}^3x \, \mathrm{d}t \frac{M}{\gamma} (\partial_t \boldsymbol{m} \times \boldsymbol{m}) \cdot \delta \boldsymbol{m} \tag{18}$$

and this leads to the Landau–Lifshitz equation. The precise form of the Wess–Zumino term, which provides the variation (18) is:

$$S_{\text{WZ}} = \int \mathrm{d}^3x \, \mathrm{d}t \, \mathrm{d}x^5 \frac{M}{\gamma} \boldsymbol{m} \cdot (\partial_t \boldsymbol{m} \times \partial_5 \boldsymbol{m}). \tag{19}$$

This action depends on the choice of the five-dimensional manifold and the requirement of the independence of $\exp(\mathrm{i}S_{\text{WZ}}/\hbar)$ on the choice gives the quantisation of the factor in S_{WZ}. In the case of (19) this results in the ordinary quantisation of spin: $\int \mathrm{d}^3x M/\gamma = \frac{1}{2}k\hbar$ with an integer k (see Volovik 1986b).

Note also an interesting behaviour of the fermionic subsystem in the presence of the hedgehog in the ferromagnet with the integer topological charge N of the homotopy group π_2:

$$N = \int_S \mathrm{d}S_i e_{ikl} \boldsymbol{m} \cdot (\partial_k \boldsymbol{m} \times \partial_l \boldsymbol{m})/8\pi. \tag{20}$$

Here the integral is over the closed surface S around the hedgehog. According to (8) the hedgehog plays the part of the Dirac magnetic monopole (Dirac 1931) with the magnetic charge $g = \frac{1}{2}N$ producing the vector potential A with the singularity on string.

In conclusion, the correct linear momentum in ferromagnets is restored after the introduction of the 'hidden' variables of the fermionic background. The fermions of the background interact with the coherent magnetisation motion via the Abelian gauge field. The separation of the coherent motion from the incoherent background violates the gauge invariance and gives rise to the anomalous behaviour of linear momentum and to the Wess–Zumino term in action.

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