Resonant-line Radiative Transfer Process of Lyman-alpha Radiation

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Introduction of Lyman-alpha Radiation

• What’s Lyman-alpha radiation?
• Why Lyman-alpha radiation is important? (What can it provide for us?)

• Reference: https://ui.adsabs.harvard.edu/abs/2010arXiv1012.3175L/abstract
  https://ui.adsabs.harvard.edu/abs/2017arXiv170403416D/abstract
General Discussion of Lyman-alpha Radiation

• Actually the physics of Lyman-alpha radiation has been built.

• Optical depth: \( \tau = \int_0^z k_\nu \, dz \quad \tau_0 > 10^4 \)

• Specific intensity: \( dE = I(\hat{r}, \hat{n}, \nu, t) \, dA \, dt \, d\Omega \, d\nu \)

• Fundamental Equation:

\[
\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \hat{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu + \int k_{\nu'} I_{\nu'} \, R_{\nu', \nu} \, d\Omega' \, d\nu',
\]

emission absorption scattering redistribution

• Change Notation:

\[
x = \frac{\nu - \nu_0}{\Delta \nu_D} \quad \nu_0 = 2.466 \times 10^{15} \text{Hz} \quad \Delta \nu_D = (v_{\text{th}}/c) \nu_0 \quad v_{\text{th}} = (2k_B T/m_H)^{1/2}
\]

\[
\Delta \nu_L = 9.936 \times 10^7 \text{Hz} \quad a = \Delta \nu_L / 2 \Delta \nu_D
\]

• Important function: Describe the frequency dependence of the absorption coefficient

\[
H(a, x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{a^2 + (y - x)^2} \approx \begin{cases} 
  a \frac{e^{-x^2}}{\sqrt{\pi} x^2} & \text{`wing'} \\
  \frac{a}{\sqrt{\pi} x^2} & \text{`core'}
\end{cases}
\]
General Discussion of Lyman-alpha Radiation

\[ \frac{1}{c} \frac{\partial I_\nu}{\partial t} + \hat{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu + \int k_\nu' I_{\nu'} R_{\nu',\nu', \rightarrow \nu, \hat{n}, \hat{n'}} \, d\Omega' \, d\nu', \quad H(a, x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{a^2 + (y - x)^2} \approx \begin{cases} \frac{a}{\sqrt{\pi} x^2} & \text{`core'} \\ \frac{a}{\sqrt{\pi} x^2} & \text{`wing'} \end{cases} \]

\[ J_x = \frac{1}{4\pi} \int d\Omega I_x \quad H_x = \frac{1}{4\pi} \int d\Omega I_x \, n \]

• After integral:

\[ \frac{1}{c} \frac{\partial J_x}{\partial t} + \nabla \cdot H_x = j'_x - k_x J_x + \int k_x' J_{x'} R_{x' \rightarrow x} \, dx' \]

\[ R_{x' \rightarrow x} \equiv (4\pi)^{-2} \int d\Omega' d\Omega R_{x', n' \rightarrow x, n} \quad j'_x = \frac{1}{4\pi} \int d\Omega \]

• Fick's Law:

\[ H_x \approx -\frac{\nabla J_x}{3k_x} \]

• Fokker-Planck approximation:

\[ -k_x J_x + \int k_x' J_{x'} R_{x' \rightarrow x} \, dx' \approx \frac{\partial}{\partial x} \left( \frac{k_x}{2} \frac{\partial J_x}{\partial x} \right) \]

\[ \frac{1}{c} \frac{\partial J_x}{\partial t} = j'_x + \nabla \cdot \left( \frac{\nabla J_x}{3k_x} \right) + \frac{\partial}{\partial x} \left( \frac{k_x}{2} \frac{\partial J_x}{\partial x} \right) \]
General Discussion of Lyman-alpha Radiation

• Assumption: static (time-independent), isothermal, optically thick \( \tau_0 > 10^4 \)

\[ k_x(\mathbf{r}, x) = k(\mathbf{r}) H(x) \]

\[ \frac{1}{k(\mathbf{r})} \nabla \cdot \left( \frac{\nabla J}{k(\mathbf{r})} \right) + \frac{3}{2} H(x) \frac{\partial}{\partial x} \left( H(x) \frac{\partial J}{\partial x} \right) = - \frac{3\mathcal{L}}{4\pi} \frac{\eta(\mathbf{r})}{k(\mathbf{r})} \frac{H^2(x)}{\sqrt{\pi}} \int \int \int j'_x dV dxd\Omega = \mathcal{L} \]

\[ \frac{d\tilde{x}}{dx} = \sqrt{\frac{2}{3}} \frac{1}{H(x)} \quad J = \tilde{J} \mathcal{L} \sqrt{6}/(4\pi) \quad 3H^2(x) \approx \sqrt{6\pi} \delta(\tilde{x}) \]

• Key equation:

\[ \frac{1}{k(\mathbf{r})} \nabla \cdot \left( \frac{\nabla \tilde{J}}{k(\mathbf{r})} \right) + \frac{\partial^2 \tilde{J}}{\partial \tilde{x}^2} = - \frac{\eta(\mathbf{r})}{k(\mathbf{r})} \delta(\tilde{x}) \]
Gridless Monte-Carlo Radiative Transfer (GMCRT) Method

- New method to simulate the radiative process, Not necessary to discretize!
- Initial equation: \[
\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \bar{n} \cdot \nabla I_\nu = j_\nu - k_\nu I_\nu + \int \int k_{\nu'} I_{\nu'} R_{\nu',\bar{n}' \rightarrow \nu,\bar{n}} \ d\Omega' d\nu',
\]

The region we solve the equation. If the photon cross the boundary, we stop tracking this photon and record the frequency.

Two key questions:
1. How does the photon change its frequency?
2. How long does the photon transfer between two scatter events?

Use some mathematical techniques to transform this question into finding the root of a transcendent equation. And we use Halley‘ method to find the root.

[Link to paper](https://ui.adsabs.harvard.edu/abs/2015MNRAS.449.4336S/abstract)
Slab Geometry

\[ \frac{1}{k(r)} \nabla \cdot \left( \frac{\nabla \tilde{J}}{k(r)} \right) + \frac{\partial^2 \tilde{J}}{\partial x^2} = -\frac{\eta(r)}{k(r)} \delta(\tilde{x}) \]

\[ \eta(z) = \delta(z) \]

\[ z = 0 \]

Central Source

\[ \frac{\tilde{J}(\tau_0, x)}{\int_{-\infty}^{+\infty} \tilde{J}(\tau_0, x) dx} = \sqrt{\frac{\pi}{6}} \frac{x^2}{a\tau_0} \text{sech} \left( \sqrt{\frac{\pi^2}{54}} \frac{|x|}{a\tau_0} \right) \]

\[ (\text{Harrington 1973}) \]

Can we generalize it? Give an analytic expression of between these two solution?

We got two important results here.

Consider it more physically.

\[ \eta(z) = (\alpha + 1) |z/Z|^\alpha / (2Z) = \eta_0 |z|^\alpha \]

\[ k(z) = k_0 |z|^\beta \]
Slab Geometry

- Result \( \delta = (\alpha + 1)/(\beta + 1) > 0 \)

\[
\eta(z) = (\alpha + 1)|z/Z|^\alpha/(2Z) = \eta_0|z|^\alpha
\]

\[
k(z) = k_0|z|^\beta
\]

\[
\tilde{J}(\tau_0, x) = \frac{e^{-\sqrt{\frac{\pi^2 x^4}{54 a \tau_0}}}}{-2 f \pi^{\delta-1/2} a \tau_0} C(\delta) \frac{x^2}{a \tau_0} \mathcal{L}\left(-e^{-\sqrt{\frac{\pi^2 x^4}{54 a \tau_0}}}, \delta, \frac{1}{2}\right)
\]

\[
+ \frac{\delta}{-\sqrt{\pi} f a \tau_0} \tanh^{-1}\left(e^{-\sqrt{\frac{\pi^2 x^4}{54 a \tau_0}}}ight)
\]

\[
\mathcal{L}(z, s, a) = \sum_{n=0}^{+\infty} \frac{z^n}{(n+a)^s}
\]

\[
C(\delta) = \frac{\sqrt{\pi} 2^{\delta-1} \delta \Gamma(\delta/2)}{\Gamma(1/2 - \delta/2)}
\]
Spherical Geometry: Two-Cell Model

• Maybe Spherical coordinate is closer to our universe, and we assume that our system has spherical symmetry.

• The average intensity within the homogeneous density profile in spherical coordinate (One cell) has been found by Dijkstra in 2006

\[ J(x) = \frac{1}{8} \sqrt{\frac{\pi}{6}} \frac{x^2}{a\tau_0} \text{sech}^2 \left( \sqrt{\frac{\pi^3}{54}} \frac{x^3}{a\tau_0} \right) \]

What about the two cell model, which means that the absorption coefficient is piece-wise?

The calculation is technically difficult because of the discontinuity at the \( r = R_1 \)

\[ g_r = \frac{\tau_1}{\tau_0} \quad g_R = \frac{R_1}{R_2} \]
Spherical Geometry: Two-Cell Model

- Numerical Result

Only the thin shell model has a clear deviation!
Thin-Shell Model

• Analytic solution

\[ j(R_2, x) = \frac{R_1^2}{4\pi R_2 \left( \frac{1}{3} k_1 R_1^2 + k_2 R_2^4 (R_2 - R_1) \right)} \sqrt{\frac{\pi}{x^2}} \left( -\frac{\tau_2^2 e^{-\sqrt{\frac{\pi}{4x^2}}}}{k_2^2 R_2 \pi^2} \right) \frac{e^{\sqrt{\frac{\pi}{2x^2}}}}{\cosh\left(\sqrt{\frac{\pi}{54 \alpha_{\tau_2}}}\right)} \]

• Numerical Verification

\( g_R = 0.5 \) \( g_\tau = 0 \) \( g_\tau / g_R = 0 \)

\( g_R = 0.5 \) \( g_\tau = 0.01 \) \( g_\tau / g_R = 0.02 \)

\( g_R = 0.5 \) \( g_\tau = 0.1 \) \( g_\tau / g_R = 0.2 \)

\( g_R = 0.9 \) \( g_\tau = 0 \) \( g_\tau / g_R = 0 \)

\( g_R = 0.9 \) \( g_\tau = 0.01 \) \( g_\tau / g_R = 0.0111111 \)

\( g_R = 0.9 \) \( g_\tau = 0.1 \) \( g_\tau / g_R = 0.111111 \)
Spherical Geometry: Power-law Density Profile

- The physical profile of number density is similar to power-law. So we want to see what happen if we set the absorption coefficient $k(r) = k_0 r^\beta$ ($-1 < \beta < 0$)
- I got a analytic solution:
  \[
  J(\tau_0, x) = \frac{\pi^{\frac{\theta+1}{2}}}{-f R^2 \Gamma\left(\frac{\theta+1}{2}\right)} x^2 \exp\left(-\frac{\theta^2 + 2\theta}{2} \frac{|x|^2}{x_0^2 \alpha^2}\right) L\left(-e^{-\frac{\theta^2 + 2\theta}{2}}, -\frac{\theta}{2}, \frac{2 + \theta}{4}\right)
  \]
- Numerical Verification

- But there are still some drawbacks of the solution. When $\beta < -0.5$, the solution is not accurate anymore.
Thank you for your listening!