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1 History

First a brief history, extracted from [18]. Hodge theory, named after W.V.D.Hodge, has its origin in works by Abel, Jacobi, Gauss, Legendre and Weierstrass among many others on the periods of integrals of rational one-forms. In 1931, Hodge assimilated de Rham’s theorem and defined the Hodge star operator. It would allow him to define harmonic forms and so fine the de Rham theory. Hodge’s major contribution, as Atiyah put in [1], was in the conception of harmonic integrals and their relevance to algebraic geometry.

The relative theory appeared in the late 1960’s with the work of Griffiths [11, 12]. He found that higher weights generalization of the ordinary
Jacobian, the intermediate Jacobian, need not be polarized. He generalized Abelian-Jacobi maps in this set-up and used these to explain the difference of cycles and divisors. The important insight that any algebraic variety has a generalized notion of Hodge structure was worked out in Hodge II,III [6, 7]. In the relative setting, if the family acquires singularities, the Hodge structure on the cohomology of a fiber may degenerate when the base point goes to the singular locus, leading to the so-called limit mixed Hodge structure. Morihiko Saito introduced the theory of Mixed Hodge Modules around 1985, which unifies many theories: algebraic D-modules and perverse sheaves.

2 Hodge structure

A quick review of Hodge theory for real manifolds. Let \((M, g)\) be a compact orientable Riemann manifold. For \(k \in \mathbb{N}\), let \(\Omega^k(M)\) be the real vector space of smooth differential \(k\)-forms on \(M\) and \(d\) denotes the exterior derivative on \(\Omega^*(M)\). The metric \(g\) induces an inner product on \(\Omega^k(M)\) for each \(k\).

Consider the adjoint operator \(d^* : \Omega^{k+1}(M) \rightarrow \Omega^k(M)\) of \(d\) with respect to these inner products. Then the Laplacian is defined by

\[
\Delta_d = dd^* + d^*d \quad (1)
\]

Hodge theory interprets \(H^*_{dR}(X, \mathbb{R})\) in terms of the kernel of the Laplacian \(\Delta_d\). Explicitly, the Hodge theorem states that the space of harmonic \(k\)-forms, \(\mathcal{H}^k_{\Delta_d}(M) = \ker(\Delta|_{\Omega^k(M)})\), is canonically isomorphic to \(H^k_{dR}(M, \mathbb{R})\).

Is there a complex match of the above real theory? On a complex manifold \(X\), a Hermitian metric is a complex analogue of Riemannian metric. To be precise, a Hermitian metric \(h\) is a smoothly varying Hermitian inner product on each holomorphic tangent space. Note that its real part \(\text{Re} h\) is a usual Riemannian metric. Let \(A^k(X)\) be the space of global smooth differential \(k\)-forms with complex coefficients. Recall the bigrading \(A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X)\) and the corresponding splitting \(d = \partial + \overline{\partial}\). With regard to \(h\), we may define \(\overline{\partial}\)-Laplacian by \(\Delta_{\overline{\partial}} = \partial^\dagger \overline{\partial} + \overline{\partial} \partial^\dagger\) in a similar way to (1). One may expect a refinement of the classical Hodge theory in this case. A priori, however, \(\Delta_{\overline{\partial}}\) is not related to \(\Delta_d\). The remedy is the following definition.

**Definition 1** (Kähler manifold). A Kähler manifold is a complex manifold with a Hermitian metric \(h\) such that the 2-form \(\omega\) defined by \(\omega(u,v) = \frac{1}{2} (du(u) dv(v) - dv(u) du(v))\)
Im\(h(u,v)\) is closed. Such a special metric \(h\) is called a Kähler metric and \(\omega\) a Kähler form.

For a Kähler manifold \((X, \omega)\), \(\omega\) is a symplectic form. Hence we find three mutually compatible structures: a complex structure, a Riemannian structure, and a symplectic structure. Every complex submanifold of a Kähler manifold is again Kähler with the induced metric. The Fubini-Study metric [14, Example 3.1.9] on a projective space \(\mathbb{C}P^n\) is Kähler. As a result, a complex projective manifold is Kähler. Any Hermitian metric on a Riemann surface is Kähler, since \(d\omega \in A^3(X) = 0\). Thus we find various examples of Kähler manifolds.

From now until Section 5, let \((X, w)\) be a compact Kähler manifold with \(\dim_{\mathbb{C}} X = n\) unless otherwise stated.

A landmark in Hodge theory is the following theorem.

**Theorem 1** (Hodge’s decomposition theorem). [14, Corollary 3.2.12] Let \(X\) be a compact Kähler manifold. Define \(H^{p,q}(X) = \{[\alpha] \in H^{p,q}_{dR}(X, \mathbb{C}) : \alpha \in A^{p,q}(X), d\alpha = 0\}\) to be the subspace represented by \((p,q)\)-forms. Then

\[H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X).\]

Furthermore, we have Hodge symmetry: \(\overline{H^{p,q}(X)} = H^{q,p}(X)\).

A key argument in the proof is \(\Delta_d = 2\Delta_{dR}\) on \(X\). The Hodge decomposition is actually independent of the choice of the Kähler metric, which can be proved using Bott-Chern cohomology. Hodge decomposition induces a filtration on each \(H^k(X, \mathbb{C})\), called the Hodge filtration, by \(F^p H^k(X, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X)\). Hodge decomposition motives the definition of (pure) Hodge structure.

**Definition 2** (pure \(\mathbb{Q}\)-Hodge structure). Let \(n \in \mathbb{N}\). A pure \(\mathbb{Q}\)-Hodge structure of weight \(n\) consists of a finite dimensional \(\mathbb{Q}\)-vector space \(V\), a finite decreasing filtration \(F^*\) on \(V = \mathbb{C} \otimes_{\mathbb{Q}} V\) such that for any \(p \in \mathbb{Z}\), \(F^p V \cap F^{n+1-p} V = 0\) and \(V^c = F^p V \oplus F^{n+1-p} V\).

The existence of such a filtration is equivalent to that of a decomposition: to pass between these two definitions, given the Hodge filtration \(F^*V\), for \(p \in \mathbb{Z}\), define \(V^{p,n-p} = F^p V \cap F^{n-p} V\), then \(V^c = \bigoplus_{p \in \mathbb{Z}} V^{p,n-p}\). The filtration is preferred as in the relative situation, the filtration varies holomorphically.
while the bigradings do not (by Hodge symmetry). Moreover, the filtration satisfies a remarkable result, Proposition 3.

In the definition of Hodge structure, we may replace \( \mathbb{Q} \) by other coefficient rings like \( \mathbb{Z}, \mathbb{R} \).

Now that Hodge structure on \( H^*(X, \mathbb{C}) \) is determined by the complex structure of \( X \), a natural question is the converse: Does the Hodge structure on cohomology determines the complex structure? The Global Torelli theorem is said to hold for a particular class of compact complex algebraic or Kähler manifolds if any two manifolds of the given type can be distinguished by their integral Hodge structures.

**Theorem 2.** Two complex tori \( T \) and \( T' \) are biholomorphic if and only if there exists an isomorphism of weight one integral Hodge structures \( H^1(T, \mathbb{Z}) \to H^1(T', \mathbb{Z}) \).

Hodge decomposition may fail for a general compact complex manifold, as shown by Example 1. This potential failure is encoded in the following spectral sequence. (The readers who are not familiar with spectral sequences can skip when they make appearances.)

**Definition 3** (Frölicher spectral sequence). Let \( X \) be a complex manifold. The decreasing filtration \( F^pA^k = \bigoplus_{i \geq p} A^{i,k-i} \) induces a spectral sequence \( E_{p,q}^1 = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}(X, \mathbb{C}) \).

We call \( h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p_X) \) the Hodge numbers of the complex manifold \( X \). The Hodge numbers of a compact complex manifold are finite.

**Definition 4.** For a complex manifold \( X \), the putative Hodge filtration on \( H^k(X, \mathbb{C}) \) is given by

\[
F^p H^k(X, \mathbb{C}) = \text{Im}(\mathbb{H}^k(X, \Omega^{\geq p}_X) \to \mathbb{H}^k(X, \Omega^*_X)).
\]

**Corollary 1.** For a compact Kähler manifold \( X \), its Frölicher spectral sequence degenerates at \( E_1 \). For any \( p, q \in \mathbb{Z} \), \( H^q(X, \Omega^p_X) \) is canonically isomorphic to \( H^{p+q}(X) \). The putative Hodge filtration coincides with the actual Hodge filtration.

**Theorem 3.** [15] For a compact complex surface, the Frölicher spectral sequence degenerates at \( E_1 \).

A consequence of the degeneration of the Frölicher spectral sequence is that any holomorphic global form is closed. Iwasawa manifold is a compact complex manifold of (complex) dimension 3 admitting a non-closed holomorphic 1-form. (See [13], Example VI.8.10.)
3 Topological constraints

Hodge theory imposes strong restrictions on the topology of the manifold, as the following results show. Recall that Betti numbers $b_j(X) = \dim_\mathbb{C} H^j(X, \mathbb{C})$.

**Corollary 2.** The Betti numbers $b_{2k-1}(X)$ are even.

**Corollary 3.** If the fundamental group $\pi_1(X)$ is a free group, then it is trivial.

*Proof.* Suppose that $\pi_1(X)$ is a free group on $m(\geq 1)$ generators. We can find a subgroup $H \leq \pi_1(X)$ of index 2. By [17, Theorem 2.10], $H$ is a free group on $2m - 1$ generators. This corresponds to a two-sheeted cover $\pi: Y \to X$ where $\pi_1(Y)$ is isomorphic to $H$. The space $Y$ with pullback structures is also compact Kähler. However, the Betti number $b_1(Y) = 2m - 1$ is odd, which contradicts Corollary 2. \qed

The first example of compact complex surface with no Kähler metric is the following.

**Example 1 (Hopf surface).** Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$ by $(k, z) \mapsto 2^k z$. The quotient space $X$ is a compact complex manifold, which is diffeomorphic to $S^1 \times S^3$. In particular, the Betti number $b_1(X) = \dim_\mathbb{C} H^1(X, \mathbb{C}) = 1$ is odd. Therefore, $H^1(X, \mathbb{C})$ admits no (strong) Hodge decomposition. By Corollary 2, $X$ is not homeomorphic to any Kähler manifold. By Corollary 5, $X$ cannot be algebraic. The Dolbeault cohomology group $H^1(X, O_X) = H^0_{\overline{\partial}}(X)$ is nonzero since the global form

$$\alpha = \frac{z_1 \overline{d}z_1 + z_2 \overline{d}z_2}{|z_1|^2 + |z_2|^2}$$

is $\overline{\partial}$-closed but not $\overline{\partial}$-exact. By Theorem 9, $b_1 = h^{1,0} + h^{0,1}$, so we find the Hodge numbers $h^{0,1} = 1$ and $h^{1,0} = 0$.

As Example 1 illustrates, the existence of Hodge decomposition is strictly stronger than the degeneration of Frölicher spectral sequence.

We turn to another aspect of classical Hodge theory. Define the Lefschetz operator $L: H^*_{dR}(X, \mathbb{R}) \to H^{*+2}_{dR}(X, \mathbb{R})$ by $[\eta] \mapsto [\eta \wedge \omega]$.
Theorem 4 (Hard Lefschetz). For $0 \leq k \leq n$, 

$$L^{n-k} : H^k_{dR}(X, \mathbb{R}) \to H^{2n-k}_{dR}(X, \mathbb{R})$$

is an isomorphism. For $k \leq j \leq n$, 

$$L^{n-j} : H^k_{dR}(X, \mathbb{R}) \to H^{2n-2j+k}_{dR}(X, \mathbb{R})$$

is injective.

Corollary 4. The even Betti numbers $b_{2i}(X)$ are increasing in the range $2i \leq n$ and similarly the odd Betti numbers $b_{2i+1}(X)$ are increasing in the range $2i + 1 \leq n$.

For $k \leq n$, define the primitive part of its cohomology as

$$H^k(X, \mathbb{R})_{\text{prim}} = \ker[L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R})].$$

When $p, q \geq 0$ and $p + q \leq n$, we can similarly define 

$$H^{p,q}(X, \mathbb{C})_{\text{prim}} = \{[\alpha] \in H^{p,q}(X, \mathbb{C}) : L^{n-p-q+1}\alpha = 0\}.$$ 

Then Theorem 4 gives an isomorphism $L^{n-p-q} : H^{p,q}(X, \mathbb{C}) \to H^{n-q,n-p}(X, \mathbb{C})$ and another decomposition theorem of the cohomology groups.

Theorem 5 (Lefschetz decomposition theorem).

$$H^{p,q}(X, \mathbb{C}) = H^{p,q}(X, \mathbb{C})_{\text{prim}} \oplus L(H^{p-1,q-1}(X, \mathbb{C})) \quad (2)$$

$$H^k(X, \mathbb{C}) = \bigoplus_{k - 2r \leq n} L^r H^{k-2r}(X, \mathbb{C})_{\text{prim}} \quad (3)$$

Since the Kähler form $\omega \in A^{1,1}$ is real, the above theorem remains true with real coefficients.

We mention a striking theorem in passing. For a compact simply connected Kähler manifold, its real homotopy type is determined by its cohomology ring (cf.[8]). This is later improved to rational homotopy type in [22].
4 Polarized Hodge structure

Let $X$ be a compact Kähler manifold as before. For $k \leq n$, define a bilinear form on $H^k_{dR}(X, \mathbb{R})$ by

$$Q([\alpha], [\beta]) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$ 

Then $Q$ is $(-1)^k$-symmetric. This form is called the intersection form. If the Kähler class $[\omega] \in H^2(X, \mathbb{Z})$, i.e., $X$ is projective (by Kodaira embedding theorem) this pairing is defined integrally.

**Definition 5.** A polarized $\mathbb{R}$-Hodge structure of weight $k$ is a $\mathbb{R}$-Hodge structure $(V, V^{p,q})$ of weight $k$, with a $(-1)^k$-symmetric bilinear form $Q : V \times V \to \mathbb{R}$, such that its extension to $V_C$ satisfies Hodge-Riemann bilinear relations:

1. the Hodge decomposition is orthogonal with respect to $Q$
2. for any $\alpha \in V^{p,q}$ nonzero,
   $$i^{p-q}(-1)^{(k-1)/2}Q(\alpha, \bar{\alpha}) > 0.$$

**Proposition 1.** $[23, \text{Theorem 6.32}]$ $(H^k(X, \mathbb{R})_{prim}, Q)$ is a polarized real Hodge structure of weight $k$.

**Theorem 6** (Riemann). Let $L \subseteq \mathbb{C}^n$ be a lattice. Then the complex torus $X = \mathbb{C}^n/L$ is algebraic if and only if the Hodge structure of $H^1(X, \mathbb{Z})$ admits a polarization.

With polarized Hodge structure we can state the following two Torelli type results.

**Theorem 7.** Two compact Riemann surfaces $M$ and $N$ are isomorphic if and only if there exists an isomorphism of weight one Hodge structures $H^1(M, \mathbb{Z}) \to H^1(N, \mathbb{Z})$ that respects the intersection pairing.

**Definition 6** ((analytic) K3 surface). A compact complex surface $X$ with trivial canonical bundle and $h^{0,1}(X) = 0$ is called a K3 surface.

Every K3 surface is Kähler (cf. [20]).

**Theorem 8.** Two complex K3 surfaces $S$ and $S'$ are isomorphic if and only if there exists an isomorphism of Hodge structures $H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ respecting the intersection pairing.
5 Mixed Hodge structure

By variety we mean a quasi-projective variety (cf. [13, p. 10]) over \( \mathbb{C} \). Recall Chow’s theorem: a projective manifold is algebraic. We are thus led to the following question: can Hodge theory be extended to complex algebraic varieties?

For a general complex variety \( X \), its analytification \( X^{an} \) may not be compact nor smooth. For example, consider the punctured line \( X = \text{Spec} \mathbb{C}[t, t^{-1}] \), then \( X^{an} = \mathbb{C}^* \) and \( H^1(X^{an}, \mathbb{C}) = \mathbb{C} \) is of odd dimension. Here \( X \) is smooth but not proper. On the other hand, take \( C \) to be the plane projective curve \( Y^2Z = X^2(X - Z) \). Then still \( \dim_{\mathbb{C}} H^1(C, \mathbb{C}) = 1 \). In this case, \( C \) has a nodal singularity. The cohomology groups cannot be expected to have a Hodge decomposition in both cases. We are forced to seek a weaker notion than Hodge structure.

The heuristic evidence for the existence of such a weaker notion comes from the properties of étale cohomology of varieties over fields with positive characteristic. A dictionary between \( l \)-adic cohomology and Hodge theory is in [5]. Deligne was also the first to give an affirmative answer to the question. More precisely, he established the existence of mixed Hodge structure on the cohomology of complex algebraic varieties.

**Definition 7** (mixed Hodge structure). Let \( H \) be a finite \( \mathbb{Z} \)-module. A mixed Hodge structure on \( H \) consists of

1. an increasing filtration \( W_* \) on \( H_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H \), called the weight filtration;
2. a decreasing filtration \( F_* \) on \( H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H \), called the Hodge filtration

with the property that \( F_* \) induces a pure \( \mathbb{Q} \)-Hodge structure of weight \( k \) on the graded piece

\[
\text{Gr}^W_k(H_{\mathbb{Q}}) = W_k/W_{k-1}.
\]

**Theorem 9.** [5, Theorem 3.2.5] Let \( X/\mathbb{C} \) be a smooth algebraic variety. Then \( H^k(X, \mathbb{C}) \) has a canonical mixed Hodge structure. This structure is functorial. If \( X \) is furthermore complete (proper), then this structure is pure of weight \( k \).

Sketch of the proof in Hodge II: a smooth variety \( X/\mathbb{C} \) has a smooth compactification \( j : X \to \bar{X} \) such that \( D = \bar{X} - X \) is a divisor of normal crossing by Nagata’s theorem [21, Tag 0F41] and Hironaka’s resolution of singularity.
Then Deligne used differential forms with logarithmic singularities along the boundary and the residue maps between them to show Theorem 9.

In Hodge III, the result is generalized to proper varieties and finally to any complex algebraic variety. In the singular case, varieties are replaced by simplicial schemes, leading to more complicated homological algebra. Using the theory of motives, it is possible to refine the weight filtration on the cohomology with rational coefficients to one with integral coefficients.

Remark 1. The weight filtration on $H^k(X)$ satisfies the following properties:

1. $W_{-1} = 0$ and $W_{2k} = H^k$.

2. If $X/\mathbb{C}$ is proper, then $W_k = H^k(X)$ and for any resolution of singularity $\tilde{X} \to X$, we have $W_{k-1}H^k(X) = \ker(H^k(X) \to H^k(\tilde{X}))$.

3. If $X/\mathbb{C}$ is smooth, then $W_{k-1}H^k(X) = 0$ and for any smooth compactification $i : X \to \tilde{X}$, then $i^*H^k(\tilde{X}) = W_kH^k(X)$.

In particular, when $X/\mathbb{C}$ is a smooth projective variety, the weight filtration is trivial $0 = W_{k-1} \subset W_k = H^k(X)$, that is, we recover the pure Hodge structure given by Theorem 1.

Corollary 5. Let $X/\mathbb{C}$ be a proper smooth variety, then $X$ admits a strong Hodge decomposition.

Example 2. [10, Section 4] Let $X_1, X_2 \subset \mathbb{C}P^N$ be two complex submanifolds intersecting transversally. Let $X = X_1 \cup X_2$. The Mayer-Vietoris sequence is

$$H^{m-1}(X_1 \cap X_2) \xrightarrow{\delta} H^m(X) \to H^m(X_1) \oplus H^m(X_2) \to H^m(X_1 \cap X_2) \to$$

The weight filtration of $H^m(X)$ is $W_{m-2} = 0$, $W_{m-1} = \text{Im} \delta$ and $W_m = H^m(X)$. (Note that the category of mixed Hodge structures is abelian and any morphism is strict. See [18, Corollary 3.6])

Example 3. For the projective curve $C : Y^2Z = X^2(X - Z)$, then $0 = W_{-1} \subset W_0H^1(C) = H^1(C)$.

Example 4. $X = \mathbb{A}^1_{\mathbb{C}} - 0$, then $0 = W_1H^1 \subset W_2H^1 = H^1$

We present one application.
Theorem 10 (Weight principle). Let $Z \subset U \subset X$ be inclusions, where $X/\mathbb{C}$ is a proper smooth variety, $U \subset X$ is a Zariski dense open and $Z \subset X$ is a closed subvariety, for each $l$,\[H^l(X, \mathbb{Q}) \xrightarrow{a} H^l(U, \mathbb{Q}) \xrightarrow{b} H^l(Z, \mathbb{Q})\]have $\text{Im}(ba) = \text{Im}(b)$

Example 5 shows that the real counterpart is false.

Example 5. Let $X = \mathbb{C}P^1$, $U = \mathbb{C}^*$, $Z = S^1$, then $H^1(X, \mathbb{Q}) \to H^1(Z, \mathbb{Q})$ has trivial image while $H^1(U, \mathbb{Q}) \to H^1(Z, \mathbb{Q})$ is an isomorphism.

6 Variation of Hodge structures

We turn to a relative version of Hodge theory.

Let $\pi : X \to S$ be a smooth proper analytic morphism of relative dimension $n$, such that the Frölicher spectral sequence degenerates at each fiber $X_s$. (This happens if all the fibers admit a Kähler metric. Another example is that $\pi : X \to S$ is algebraic, then the degeneration is guaranteed by [4, Theorem 5.5].) We can ask how the Hodge structures (if exist) on $H^k(X_s, \mathbb{C})$ vary. The variation of Hodge structures is closely related to monodromy action and has important application in arithmetic.

Proposition 2. The Hodge numbers of $X_s$ are locally constant.

For $k \geq 0$, let $\mathcal{H}^k = R^k\pi_*\mathbb{C}O_S$ be the holomorphic bundle on $S$. It is the sheaf associated to the presheaf on $S$ defined by

\[U \mapsto H^k(\pi^{-1}(U), \mathbb{Q}).\]

Riemann-Hilbert correspondence (cf.[18, Corollary 10.4]) shows that $\mathcal{H}^k$ is equipped with a flat connection

\[\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega^1_S,\]

which is known as Gauss-Manin connection. In terms of parallel transport, we may identify nearby fibers of $\mathcal{H}^k$. Explicitly, the bundle $\mathcal{H}^k$ admits natural local $\nabla$-flat trivialisations

\[\mathcal{H}^k|_U \to H^k(X_{s_0}, \mathbb{C}) \otimes_{\mathbb{C}} O_S\]
and hence a local period map

\[ \mathcal{P}^{p,k} : U \to \text{Grass}(\mathcal{V}^{p,k}, H^k(X_{s_0}, \mathbb{C})). \]

This map is holomorphic. For details, see [23, p. 239].

**Corollary 6.** [23, p. 250] There exists a holomorphic subbundle \( F^p \mathcal{H}^k \), called the Hodge subbundle, such that for every \( s \in S \), \( F^p \mathcal{H}^k_s \) can be identified with \( F^p H^k(X_s, \mathbb{C}) \).

**Proposition 3** (Griffiths’ transversality). [18, Corollary 10.31]

\[ \nabla(F^p \mathcal{H}^i) \subset F^{p-1} \mathcal{H}^i \otimes \Omega^1_S. \]

The properties above lead to an abstract definition.

**Definition 8** (variation of Hodge structure). Let \( S \) be a complex manifold. A variation of Hodge structure of weight \( k \) on \( S \) consists of the following data:

1. a local system \( \mathcal{V} \) of finite dimensional \( \mathbb{Q} \)-vector spaces on \( S \);

2. the Hodge filtration: a finite decreasing filtration \( F^* \) of the holomorphic vector bundle \( \mathcal{V} = \mathcal{V} \otimes \mathbb{Q} \mathcal{O}_S \) by holomorphic subbundles.

These data must satisfy the following conditions:

- for any \( s \in S \), \( F^*(s) \) of \( \mathcal{V}_s \otimes \mathbb{Q} \mathbb{C} \) defines a pure Hodge structure of weight \( k \) on \( \mathcal{V}_s \).

- The induced connection satisfies the Griffith transversality: \( \nabla(F^p) \subset F^{p-1} \otimes \Omega^1_S \)

**Proposition 4.** [18, Proposition 1.38] For a smooth projective morphism \( \pi : X \to S \), the restriction maps

\[ H^m(X, \mathbb{Q}) \to H^0(S, R^m \pi_* \mathbb{Q}) \]

are surjective.
The right hand side is exactly the invariants under the monodromy action. Therefore, the proposition has the interpretation: for any \(s \in S\), an invariant of \(H^m(X_S, \mathbb{Q})\) come from a global class.

In Hodge II, Deligne considered variation of polarized Hodge structure, and showed (\([6] \text{, Theorem 4.2.6}\)) the semi-simplicity of monodromy representation under suitable conditions. The underlying variation of Hodge structures of a local system can be used to show that the monodromy group is big in suitable sense. An example is in \([18] \text{, Theorem 10.22}\).

**Theorem 11.** \([19] \text{, Theorem 7.22}\) Let \(H\) be a complex polarized variation of Hodge structure over a quasi-projective base \(S\). Let \(e\) be a global flat section of \(H\), and write \(e = \sum e^{p,q}\) for \(e^{p,q} \in H^{p,q}\), then each \(e^{p,q}\) is again flat.

The interaction of period mappings and monodromy finds application in number theory, for example, the proof of Mordell conjecture given in \([16]\).

7 Open problems

Let \(X\) be a projective manifold with \(\dim_{\mathbb{C}} X = n\). Define the Hodge class of \(H^{2k}(X)\) to be \(Hdg^{2k}(X, \mathbb{Q}) = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)\). If \(j : C \to X\) is a complex submanifold of codimension \(k\), consider \(j_* : H_{2n-2k}(C, \mathbb{Z}) \to H_{2n-2k}(X, \mathbb{Z})\) and Poincaré duality \(H_{2n-2k}(X, \mathbb{Z}) \to H^{2k}(X, \mathbb{Z})\). The fundamental class of \(C\) is mapped to \([C] \in H^{2k}(X, \mathbb{Z})\). By passing to desingularization, for any algebraic subvariety \(C\) (not necessarily smooth), we may also define \([C] \in H^{2k}(X, \mathbb{Z})\). In fact, \([C] \in Hdg^{k,k}(X, \mathbb{Z})\). This construction extends to a cycle class map starting from the Chow group

\[
cl : CH^k(X) \to H^{2k}(X, \mathbb{Z}).
\] (4)

For a complex variety \(Y\), let \(Z^k(Y)\) be the group of cycles of codimension \(k\) on \(Y\).

**Conjecture 1** (Hodge conjecture). Let \(X\) be a projective complex manifold, then \(Hdg^k(X, \mathbb{Q}) = \{[Z]_B : Z \in Z^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}\}\). Equivalently, the cycle class map \(cl : CH^k(X)_\mathbb{Q} \to Hdg^{2k}(X, \mathbb{Q})\) is surjective.

Informally, the conjecture is that every Hodge class is algebraic. The theory of Mixed Hodge structures was used by Cattani, Deligne and Kaplan to prove an algebraicity theorem that provides strong evidence for the Hodge conjecture (cf.\([3]\)). Hodge conjecture is related to generalized Bloch conjecture, cf.\([24]\).
Conjecture 2 (generalized Bloch conjecture). If the Hodge numbers $h^{p,q} = 0$ for $p \neq q$ and $p < c$ or $q < c$, then for any $i < c - 1$, the cycle class map

$$cl : CH_i(X) \otimes \mathbb{Q} \rightarrow H^{2n-2i}(X, \mathbb{Q})$$

is injective.

Complex subvarieties are rather rigid, making it difficult to construct them.

References


