Chapter 9

Introduction to Singular Perturbation Theory

Section 9.1

Roots of Polynomial Equations

9.1.0 Introduction

Boundary layer in Solid mechanics

Boundary layer in bent thin shells:

- H. Reissner 1912, Spannungen in Kugelschalen-fKuppeln, *Festschrift MueUer-Breslau* .
- Kelvin and Tait in 1867, Boussinesq in 1871. Love AEH: *A Treatise on the Mathematical Theory of Elasticity* (Cambridge, 1892)

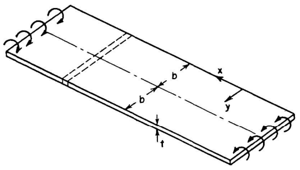
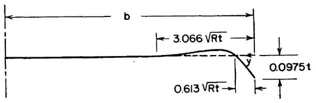


FIG. 1



- Fung, Y. C., & Wittrick, W. H. (1955). *A boundary layer phenomenon in the large deflexion of thin plates.* The Quarterly Journal of Mechanics and Applied Mathematics, 8(2), 191-210.

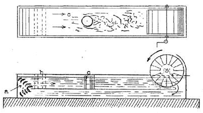
$$\zeta = -\frac{\mu t e^{-ay}\left(\cos ay - \sin ay\right)}{\sqrt{12\left(1-\mu^2\right)}}$$

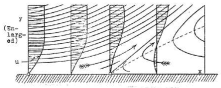
boundary layer length $a = 1/\sqrt{Rt}$

9.1.0 Introduction

Boundary layer in Fluid mechanics

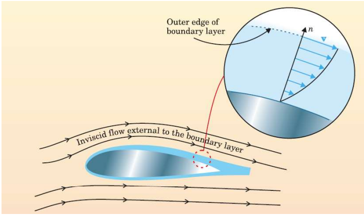
Prandtl Ludwig 1928. *Motion of fluids with very little viscosity*. Vier Abhandlungen zur Hydrodynamik und Aerodynamik.





Prandtl's paper gave the first description of the boundary-layer concept in fluid dynamics.

- The viscous effects only exist in a thin region (boundary layer) near the surface.
- Outside the boundary layer, the flow was essentially the inviscid flow.



9.1.1 Roots of Polynomial Equations

Example: regular quadratic

$$x^2 + \varepsilon x - 1 = 0$$

ε : a small constant, say ε =0.0000001.

Exact solutions $x = \dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2}$

As ε=0, we have the unperturbed solution

$$x^2 + \cancel{\varepsilon x} - 1 = 0 \Rightarrow x = \pm 1$$

9.1.1 Roots of Polynomial Equations

- Taylor expansion of the exact solution:

$$x = \begin{cases} 1 - \dfrac{\varepsilon}{2} + \dfrac{\varepsilon^2}{8} - \dfrac{\varepsilon^4}{128} + O\left(\varepsilon^6\right) \\ -1 - \dfrac{\varepsilon}{2} - \dfrac{\varepsilon^2}{8} + \dfrac{\varepsilon^4}{128} + O\left(\varepsilon^6\right) \end{cases}$$

converge if $|\varepsilon| < 2$

- Series method

power series expansion around $x = \pm 1$

$$x = \pm 1 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots$$

The same Taylor expansions can be reproduced.

9.1.1 Roots of Polynomial Equations

Example: singular quadratic

$$\epsilon x^2 + x - 1 = 0$$

ε is a small constant, say ε =0.0000001.

Exact solutions $x = \dfrac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}$

As ε=0, we have the unperturbed solution

$$\cancel{\epsilon x}^2 + x - 1 = 0 \quad ⇨ \quad x = 1$$

Only one root !

9.1.1 Roots of Polynomial Equations

- Taylor expansion of the exact solution:

$$x = \begin{cases} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + O\left(\varepsilon^4\right) \\ -\dfrac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + 5\varepsilon^3 + O\left(\varepsilon^4\right) \end{cases}$$

converge if $|\varepsilon| < 1/4$

- Expansion method

Assuming power series

Why? be patient

$$x = \boxed{\frac{a_{-1}}{\varepsilon}} + a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

Substituting into $\epsilon x^2 + x - 1 = 0$

9.1.1 Roots of Polynomial Equations

$$\varepsilon^{-1}\left(a_{-1}^2 + a_{-1}\right) + \varepsilon^0\left(2a_{-1}a_0 + a_0 - 1\right) + \varepsilon\left(2a_{-1}a_1 + a_0^2 + a_1\right)$$
$$+ \cdots = 0$$

Comparing coefficients of ε of same order

ε^{-1} : $a_{-1}^2 + a_{-1} = 0$ $\quad a_{-1} = -1 \quad$ or $\quad a_{-1} = 0$

ε^{0} : $2a_{-1}a_0 + a_0 - 1 = 0$ $\quad a_0 = -1 \quad\quad a_0 = 1$

ε^{1} : $2a_{-1}a_1 + a_0^2 + a_1 = 0$ $\quad a_1 = 1 \quad\quad a_1 = -1$

$$\boxed{x = -\frac{1}{\varepsilon} - 1 + \varepsilon + \cdots} \qquad \boxed{\begin{array}{l} x = 1 - \varepsilon \\ +2\varepsilon^2 \cdots \end{array}}$$

Singular root

Regular root

9.1.1 Roots of Polynomial Equations

- Balancing terms

balance the three terms $\varepsilon x^2 + x - 1 = 0$

(1) x and −1 is comparable, assuming ε x^2 is smaller than other two terms.

$$\varepsilon x^2 + x - 1 = 0 \;\Rightarrow\; x \approx 1$$

$$\varepsilon x^2 \sim O(\varepsilon) \quad << \quad x \sim O(1)$$

Good guess

9.1.1 Roots of Polynomial Equations

(2) ε x^2 and −1 is comparable, assuming x is smaller than other two terms.

$$\varepsilon x^2 + x - 1 = 0 \;\Rightarrow\; |x| \approx \frac{1}{\sqrt{\varepsilon}} >> 1$$

$$\varepsilon x^2 \sim O(1) \quad << \quad x \sim \mathrm{O}(\varepsilon^{-1/2})$$

Bad guess

9.1.1 Roots of Polynomial Equations

(3) ε x^2 and x is comparable, assuming both terms >> 1.

$$\varepsilon x^2 + x - 1 = 0 \;\Rightarrow\; |x| \approx \frac{1}{\varepsilon} >> 1$$

$$\varepsilon x^2 \sim \mathrm{O}(\varepsilon^{-1}) \quad x \sim O\left(\varepsilon^{-1}\right) >> O(1)$$

Self consistent

When ε x^2 and x balance, x is very large $x \sim O(\varepsilon^{-1})$

rescale x, $\quad x = \dfrac{X}{\varepsilon} \quad$ with $X \sim O(1)$

9.1.1 Roots of Polynomial Equations

We get a regular looking problem

$$X^2 + X - \varepsilon = 0$$

Using regular expansion

$$X = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

Comparing coefficients of ε of same order, we get

$$\boxed{x = -\frac{1}{\varepsilon} - 1 + \varepsilon + \cdots}$$

9.1.1 Roots of Polynomial Equations

Consider a quadratic equation

$$m^2 + \varepsilon m - 4 = 0, \qquad 0 < \varepsilon \ll 1$$

Naïve simplification leads to

$$m^2 - 4 = 0 \;\rightarrow\; m_0 \approx \pm 2$$

assume a series solution around $m_0=\pm 2$

$$m = \pm 2 + m_1\varepsilon^1 + m_2\varepsilon^2 + \cdots$$

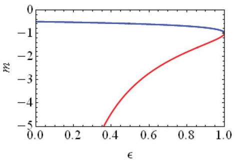
We can find

$$m_1 = 2 - \frac{1}{4}\varepsilon + \cdots, \quad m_2 = -2 - \frac{3}{4}\varepsilon + \cdots$$

9.1.1 Roots of Polynomial Equations

Consider another quadratic equation

$$\varepsilon m^2 + 2m + 1 = 0, \qquad 0 < \varepsilon \ll 1$$



Naïve simplification leads to

$$2m + 1 = 0 \;\rightarrow\; m_0 \approx -\frac{1}{2}$$

assume a series solution around $m_0=-1/2$

$$m = -\frac{1}{2} + m_1\varepsilon^1 + m_2\varepsilon^2 + \cdots$$

We can find

$$m = -\frac{1}{2} - \frac{1}{8}\varepsilon^1 - \frac{1}{16}\varepsilon^2 + \cdots$$

One root lost

The naively approximated problem has a different *qualitative nature from the original problem.*

9.1.1 Roots of Polynomial Equations

The Naïve simplification implies the root we are look for is

$$m \sim O(1), \quad as \;\; \varepsilon \to 0 \qquad \varepsilon m^2 \sim O(\varepsilon)$$

so we can find the root $m_0 \approx -1/2$.

But, we may lost the second root if

$$m \to \infty, \quad as \;\; \varepsilon \to 0 \qquad \varepsilon m^2 >> 1$$

Let's compare the magnitude of three terms systemically

$$\varepsilon m^2 + 2m + 1 = 0$$

Case 0. All three terms have the same magnitude

- all three terms are important
- we have to solve the original full equation.

9.1.1 Roots of Polynomial Equations

Case 1. terms 1 ~ term 3

$$\varepsilon m^2 + 1 = 0 \quad \rightarrow \quad m = i\varepsilon^{-1/2}$$

Check: $\varepsilon m^2 + 2m + 1 = 0$

$$\downarrow$$

$$\left| i2\varepsilon^{-1/2} \right| \gg 1$$

- bigger than the rest two.
- Inconsistent.

Case 2. terms 1 ~ term 2

$$\varepsilon m^2 + 2m = 0 \quad \rightarrow \quad m = -2\varepsilon^{-1}, \;\; m = 0$$

$$1 \ll \left|\varepsilon m^2\right| \approx 4\varepsilon^{-1} \qquad 1 \ll \left|2m\right| \approx 4\varepsilon^{-1}$$

consistent

root m=0 -> inconsistent

9.1.1 Roots of Polynomial Equations

To improve the accuracy of root m， we use iterative method.

$F(x) = G(x) \qquad m \to \infty$

$$\varepsilon m^2 + 2m = 0 \;\Longrightarrow\; \varepsilon m + 2 = -m^{-1}$$

$$\Downarrow$$

$$m_n = -2\varepsilon^{-1} - \varepsilon^{-1}m_{n-1}^{-1} \;\Longleftarrow\; m = -2\varepsilon^{-1} - \varepsilon^{-1}m^{-1}$$

$$\Downarrow$$

$$m_1 = -\frac{2}{\varepsilon} - \frac{1}{\varepsilon m_0} = -\frac{2}{\varepsilon} - \frac{1}{\varepsilon\left(-2\varepsilon^{-1}\right)} = -\frac{2}{\varepsilon} + \frac{1}{2}$$

$$\Downarrow$$

$$m_2 = -\frac{2}{\varepsilon} - \frac{1}{\varepsilon\left(-2\varepsilon^{-1} + 2^{-1}\right)} = -\frac{2}{\varepsilon} + \frac{1}{2 - \varepsilon/2}$$

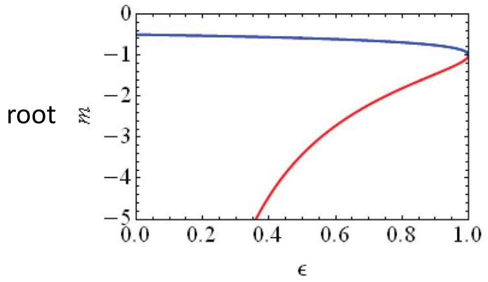
$$= -\frac{2}{\varepsilon} + \frac{1}{2} + \frac{1}{8}\varepsilon + \cdots$$

9.1.1 Roots of Polynomial Equations

This indicates we may use series expansion to look for the two roots

$$m_1 = -\frac{1}{2} + m_1\varepsilon + \cdots$$

$$m_2 = -\frac{2}{\varepsilon} + m_0 + m_1\varepsilon + \cdots$$

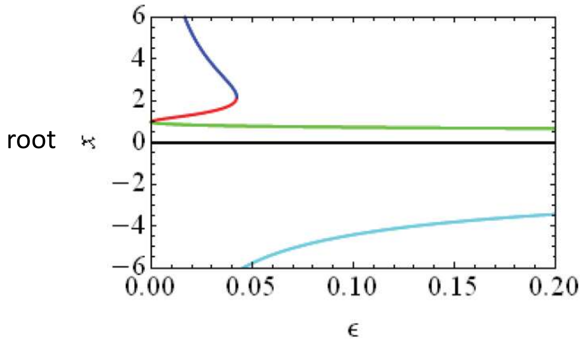


9.1.2 A more complicated problem

A more complicated problem

Consider a quartic equation

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0, \qquad 0 < \varepsilon << 1$$



9.1.2 A more complicated problem

Example 1. Find approximations to those roots ~ **O(1)**

Assume naively $x = 1 + x_1\varepsilon + O(\varepsilon^2)$

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$$

$$\Downarrow$$

$$\varepsilon\left(1 + x_1\varepsilon + O(\varepsilon^2)\right)^4 + \varepsilon\left(1 + x_1\varepsilon + O(\varepsilon^2)\right)^3$$
$$-\left(1 + x_1\varepsilon + O(\varepsilon^2)\right)^2 + 2\left(1 + x_1\varepsilon + O(\varepsilon^2)\right) - 1 = 0$$

$$\Downarrow$$

$$\varepsilon\left(2 + O(\varepsilon)\right) - x_1\varepsilon^2 + O(\varepsilon^3) = 0$$

$$\Downarrow$$

$$\varepsilon^1 :\ \cancel{2 = 0} \quad \Rightarrow \quad \cancel{x = 1 + x_1\varepsilon} + O(\varepsilon^2)$$

Assumption is naive

9.1.2 A more complicated problem

Let's try with iterative method: $F(x) = G(x)$

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0 \quad \to \quad (x-1)^2 = \varepsilon x^4 + \varepsilon x^3$$

We can reach for the two roots iteratively

$$x_0 = 1, \quad x_n = 1 \pm \sqrt{\varepsilon x_{n-1}^4 + \varepsilon x_{n-1}^3}$$

For example

$$x_0 = 1, \quad x_1 = 1 + \sqrt{\varepsilon x_0^4 + \varepsilon x_0^3} = 1 + \sqrt{2}\varepsilon^{1/2}$$

$$x_2 = 1 + \sqrt{2}\varepsilon^{1/2} + \frac{7}{2}\varepsilon + 2\sqrt{2}\varepsilon^{3/2} + \varepsilon^2$$

Therefore we may use regular series expansion

$$x = 1 + a_1\varepsilon^{1/2} + a_2\varepsilon + a_3\varepsilon^{3/2} + \cdots$$

Instead of $x = 1 + x_1\varepsilon + O(\varepsilon^2)$

9.1.2 A more complicated problem

balancing procedure: look for the rest two roots

- The case that the first two terms are negligible has been considered.
- We consider the following balance:

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$$

$$\text{(a)} \quad \varepsilon x^4 + \varepsilon x^3 = 0 \qquad \text{(b)} \quad \varepsilon x^4 - x^2 = 0$$

$$\text{(c)} \quad \varepsilon x^4 + 2x = 0 \qquad (d) \quad \varepsilon x^4 - 1 = 0$$

$$(e) \quad \varepsilon x^3 - x^2 = 0 \qquad (f) \quad \varepsilon x^3 + 2x = 0$$

$$(g) \quad \varepsilon x^3 - 1 = 0$$

9.1.2 A more complicated problem

For example, check case (a)

$$\text{(a)} \quad \varepsilon x^4 + \varepsilon x^3 = 0 \to \cancel{x = 0} \qquad x = -1$$

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$$

$$\varepsilon x^4 = O(\varepsilon) \qquad \varepsilon x^3 = O(\varepsilon)$$

$$x^2 = O(1) \qquad 2x = O(1) \qquad 1 = O(1)$$

inconsistency

9.1.2 A more complicated problem

Check case (b)

$$\text{(b)}\quad \varepsilon x^4 - x^2 = 0$$

$$\Downarrow$$

$$p^{(0)} = \frac{1}{\sqrt{\varepsilon}}, \qquad n^{(0)} = -\frac{1}{\sqrt{\varepsilon}}$$

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0$$

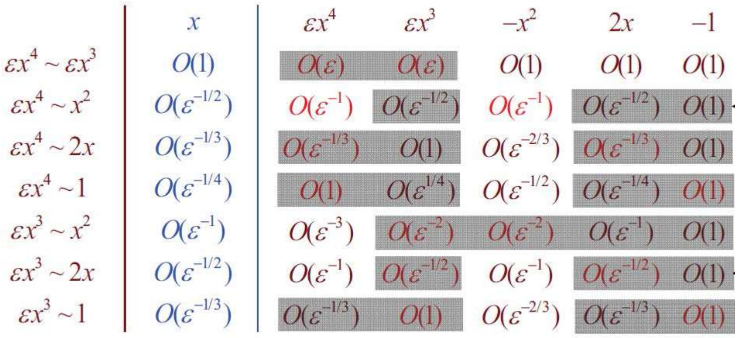
$$\varepsilon x^4 = O\left(\varepsilon^{-1}\right) \qquad x^2 = O\left(\varepsilon^{-1}\right)$$

$$\varepsilon x^3 = O\left(\varepsilon^{-1/2}\right) \qquad 2x = O\left(\varepsilon^{-1/2}\right) \qquad 1 = O(1)$$

consistency

9.1.2 A more complicated problem

Check all cases for consistency



	x	εx^4	εx^3	$-x^2$	$2x$	-1
$\varepsilon x^4 \sim \varepsilon x^3$	$O(1)$	$O(\varepsilon)$	$O(\varepsilon)$	$O(1)$	$O(1)$	$O(1)$
$\varepsilon x^4 \sim x^2$	$O(\varepsilon^{-1/2})$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-1/2})$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-1/2})$	$O(1)$
$\varepsilon x^4 \sim 2x$	$O(\varepsilon^{-1/3})$	$O(\varepsilon^{-1/3})$	$O(1)$	$O(\varepsilon^{-2/3})$	$O(\varepsilon^{-1/3})$	$O(1)$
$\varepsilon x^4 \sim 1$	$O(\varepsilon^{-1/4})$	$O(1)$	$O(\varepsilon^{1/4})$	$O(\varepsilon^{-1/2})$	$O(\varepsilon^{-1/4})$	$O(1)$
$\varepsilon x^3 \sim x^2$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-3})$	$O(\varepsilon^{-2})$	$O(\varepsilon^{-2})$	$O(\varepsilon^{-1})$	$O(1)$
$\varepsilon x^3 \sim 2x$	$O(\varepsilon^{-1/2})$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-1/2})$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-1/2})$	$O(1)$
$\varepsilon x^3 \sim 1$	$O(\varepsilon^{-1/3})$	$O(\varepsilon^{-1/3})$	$O(1)$	$O(\varepsilon^{-2/3})$	$O(\varepsilon^{-1/3})$	$O(1)$

- (a), (c)-(g) are all inconsistent.
- Only (b) is consistent:

9.1.2 A more complicated problem

iterative method for more accurate roots:

$$\varepsilon x^4 + \varepsilon x^3 - x^2 + 2x - 1 = 0 \quad \rightarrow \quad \varepsilon x^2 - 1 = x^{-2} - 2x^{-1} - \varepsilon x$$

$$\varepsilon\left[x^{(0)}\right]^2 - 1 = 0 \qquad\qquad F(x) = G(x)$$

$$\varepsilon\left[x^{(n)}\right]^2 - 1 = \left[x^{(n-1)}\right]^{-2} - 2\left[x^{(n-1)}\right]^{-1} - \varepsilon\left[x^{(n-1)}\right]$$

$$\Downarrow$$

$$p^{(0)} = \frac{1}{\sqrt{\varepsilon}}, \quad p^{(1)} = \frac{1}{\sqrt{\varepsilon}} - \frac{3}{2} + \cdots$$

$$n^{(0)} = -\frac{1}{\sqrt{\varepsilon}}, \quad n^{(1)} = \frac{-1}{\sqrt{\varepsilon}} - \frac{3}{2} + \cdots$$

p and n can be expanded in power series of $\sqrt{\varepsilon}$

9.1.3 The use of scaling

For quadratic equation

$$\varepsilon m^2 + 2m + 1 = 0, \qquad 0 < \varepsilon \ll 1$$

root-1 $\quad m_1 = -\frac{1}{2} - \frac{1}{8}\varepsilon^1 + \cdots \sim O(1)$

$$\underset{\uparrow}{\varepsilon} m^2 + 2m + 1 = 0 \qquad \text{correct scaling}$$

Indicate the magnitude

root-2 $\quad m_2 = -\frac{2}{\varepsilon} + \frac{1}{2} + \frac{1}{8}\varepsilon + \cdots \sim O\left(\varepsilon^{-1}\right)$

To get correct scaling, introduce $\quad m = \varepsilon^{-1}\hat{m}$

$$\hat{m}^2 + 2\hat{m} + \varepsilon = 0 \qquad \hat{m} \sim O(1)$$

no single scale in singular perturbation problem.

Section 9.2

Boundary Value Problems for Ordinary Differential Equations

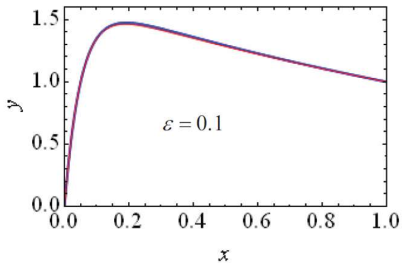
9.2.1 Examination of the exact solution to a model problem

Consider a second order ordinary differential equation

$$\varepsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0, \quad 0 < x < 1, \quad 0 < \varepsilon \ll 1$$

B.C. $\quad y(0) = 0, \quad y(1) = 1$

solution



9.2.1 Examination of the exact solution to a model problem

Consider a second order ordinary differential equation

$$\varepsilon \frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0, \quad 0 < x < 1, \quad 0 < \varepsilon \ll 1$$

B.C. $\quad y(0) = 0, \quad y(1) = 1$

Naïve simplification

$$2\frac{dy}{dx} + y = 0 \ \rightarrow \ y = K\exp\left(-\frac{1}{2}x\right)$$

$$y(0) = 0 \ \rightarrow \ y \equiv 0$$

$$y(1) = 1 \ \rightarrow \ y = \exp\left(\frac{1}{2} - \frac{x}{2}\right)$$

Which boundary conditions should be used?

9.2.1 Examination of the exact solution to a model problem

We see in this BVP problem:

- singular features similar to those of the algebraic equations discussed above.
- neither simplified solutions can satisfy both boundary conditions
- Perhaps, the simplified solutions is a good in certain subdomain ?

9.2.1 Examination of the exact solution to a model problem

Exact solution:

$$\varepsilon m^2 + 2m + 1 = 0$$

Characteristic equation

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

$$m_{1,2} = \left(-1 \pm \sqrt{1-\varepsilon}\right)/\varepsilon$$

$$\begin{cases} y(0) = A + B = 0 \\ y(1) = Ae^{m_1} + Be^{m_2} = 1 \end{cases} \Rightarrow y = \frac{\exp(m_1 x) - \exp(m_2 x)}{\exp(m_1) - \exp(m_2)}$$

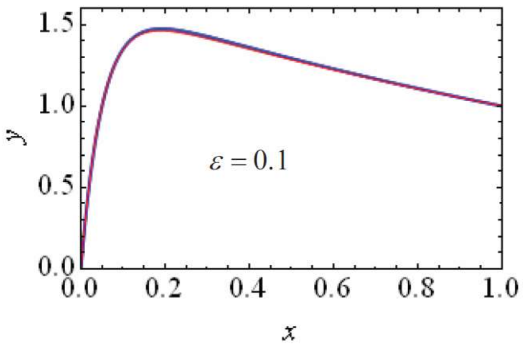
Approximate solution:

$$m_1 \approx -\frac{1}{2}, \quad m_2 \approx -\frac{2}{\varepsilon} \qquad \boxed{y(x,\varepsilon) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right)}$$

9.2.1 Examination of the exact solution to a model problem

$$\boxed{y = \frac{\exp(m_1 x) - \exp(m_2 x)}{\exp(m_1) - \exp(m_2)}} \qquad \boxed{y \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right)}$$

Very good approximate solution



9.2.1 Examination of the exact solution to a model problem

Exam the behaviors of the approximate solution:

$$\boxed{y(x,\varepsilon) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right)}$$

- B.C. at $x = 0$ is satisfied

$$y(0) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right)\Big|_{x=0} = 0$$

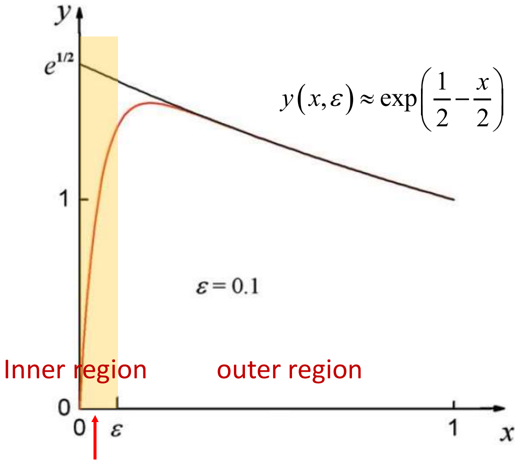
- when $x \gg \varepsilon$, the second term decays rapidly.

$$y(x,\varepsilon) \approx e^{1/2}\left(e^{-x/2} \cancel{- e^{-2x/\varepsilon}}\right) = \exp\left(\frac{1}{2} - \frac{x}{2}\right)$$

Coincident with one of the naïve solutions

$$y(1) = 1 \ \rightarrow \ y = \exp\left(\frac{1}{2} - \frac{x}{2}\right)$$

9.2.1 Examination of the exact solution to a model problem



Inner region outer region

Boundary layer: rapid change

9.2.1 Examination of the exact solution to a model problem

Consider the case as $\varepsilon<0$

$$\varepsilon\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0,\quad 0<x<1,\quad 0<|\varepsilon|\ll 1,\ \varepsilon<0$$

B.C. $y(0)=0,\quad y(1)=1$

$$\Downarrow$$

$$e\frac{d^2y}{dx^2}-2\frac{dy}{dx}-y=0,\quad 0<e\ll 1,\ e=-\varepsilon>0$$

B.C. $y(0)=0,\quad y(1)=1$

9.2.1 Examination of the exact solution to a model problem

$$em^2-2m-1=0$$

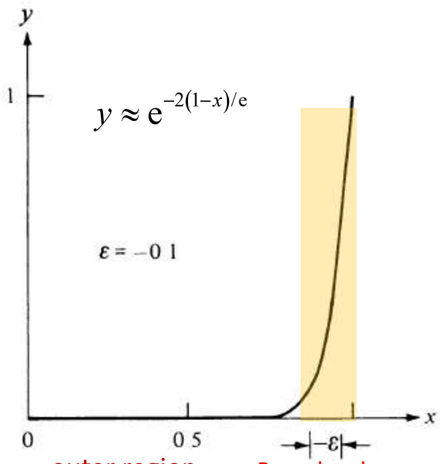
$$m_1\approx-\frac{1}{2},\quad m_2\approx\frac{2}{e}=-\frac{2}{\varepsilon}>0$$

$$y=\frac{e^{-x/2}-e^{2x/\mathrm{e}}}{\cancel{e^{-1/2}}-e^{2/\mathrm{e}}}\approx-e^{-2/\mathrm{e}}\left(\cancel{e^{-x/2}}-e^{2x/\mathrm{e}}\right)\approx \mathrm{e}^{-2(1-x)/\mathrm{e}}$$

Coincident with one of the naïve solutions as $\varepsilon>0$

$$y\approx \mathrm{e}^{-2(1-x)/\mathrm{e}}$$

9.2.1 Examination of the exact solution to a model problem



outer region

Boundary layer

9.2.1 Examination of the exact solution to a model problem

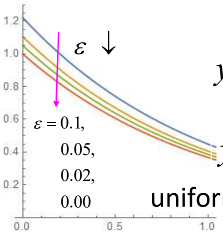
uniform limit 一致收敛

We say $\boxed{\lim_{\varepsilon\to0} y(x,\varepsilon)=y_O(x),\ x\in[a,1],\ a>0}$

is an uniform limit, if

$$\forall E>0,\ \exists D(E)>0,\ |y(x,\varepsilon)-y_O(x)|<E,$$

$$as\ 0<\varepsilon<D$$

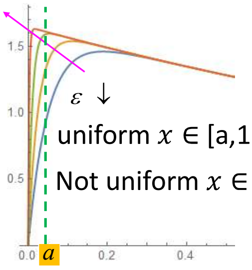


$$y(x,\varepsilon)=e^{-x+\varepsilon}$$

$$\downarrow$$

$$y_O(x)=e^{-x}$$

uniform $x\in[0,1]$



uniform $x\in[a,1]$

Not uniform $x\in(0,1]$

9.2.1 Examination of the exact solution to a model problem

Consider the following limit

$$y_O(x)=\exp\left[\frac{1}{2}(1-x)\right]$$

$$\lim_{x\to0}\left[\lim_{\varepsilon\to0} y(x,\varepsilon)\right]=\lim_{x\to0}\left[\lim_{\varepsilon\to0} e^{1/2}\left(e^{-x/2}-e^{-2x/\varepsilon}\right)\right]$$

$$=\lim_{x\to0} e^{1/2}e^{-x/2}=\lim_{x\to0} y_O(x)=y_O(0)=e^{1/2}$$

but

$$\lim_{\varepsilon\to0}\left[\lim_{x\to0} y(x,\varepsilon)\right]=\lim_{\varepsilon\to0}\left[\lim_{x\to0} e^{1/2}\left(e^{-x/2}-e^{-2x/\varepsilon}\right)\right]$$

$$=\lim_{\varepsilon\to0} 0=0$$

The limit operator can be interchanged only if

$$\lim_{\varepsilon\to0} y(x,\varepsilon)=y_O(x),\ x\in(0,b),\ b>0$$

is an uniform limit

9.2.1 Examination of the exact solution to a model problem

Summary

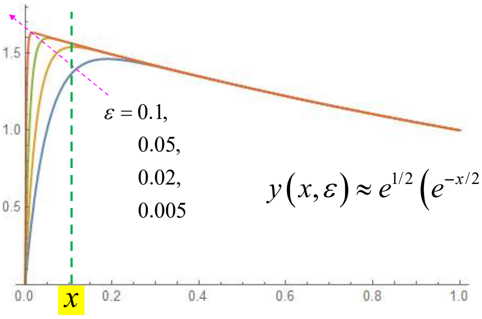
- the outer solution $y_0(x)$ does not satisfy the boundary condition at $x = 0$.
- $y_0(x)$ *is* the limit of $y(x,\varepsilon)$ in (0, 1] as ε->0
- $y_0(x)$ *is* the not the uniform limit of $y(x,\varepsilon)$ in (0, 1] as ε->0

what is an appropriate approximation for small ε *inside* the boundary layer?

- Inner solution

9.2.1 Examination of the exact solution to a model problem

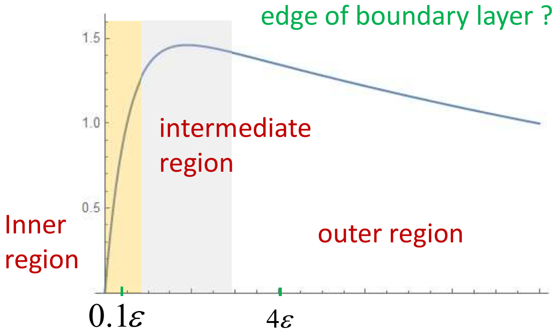
For a fixed x close to the origin, it will be out of the boundary layer as ε small enough



$\varepsilon = 0.1,$
$0.05,$
$0.02,$
0.005

$$y(x,\varepsilon) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right)$$

9.2.1 Examination of the exact solution to a model problem



edge of boundary layer ?

intermediate region

Inner region

outer region

$$\xi = \frac{x}{\varepsilon} \ ?$$

The so-called " edge" of boundary layer is obscure, but

- x=0.1ε is well inside the boundary layer
- x=4ε is well outside the boundary layer

9.2.1 Examination of the exact solution to a model problem

Inner solution $y_I(\xi)$

Define $\xi = \dfrac{x}{\varepsilon}$, for x in boundary layer

then $y(x,\varepsilon) = y(\xi\varepsilon,\varepsilon) = Y(\xi,\varepsilon)$

Inner solution $y_I(\xi) = \lim\limits_{\substack{\varepsilon\to 0\\ \xi \text{ fixed}}} Y(\xi,\varepsilon)$

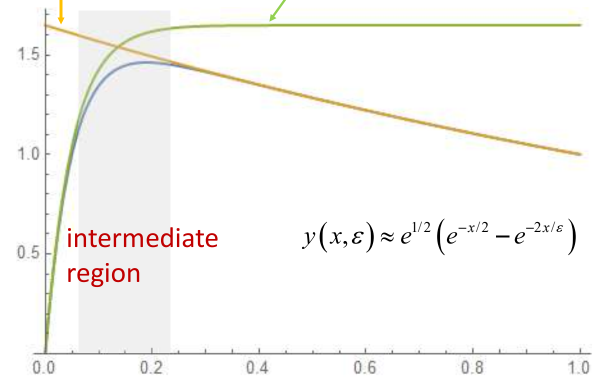
In our problem:

$$y(x,\varepsilon) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right) = e^{1/2}\left(e^{-\varepsilon\xi/2} - e^{-2\xi}\right) = Y(\xi,\varepsilon)$$

$$y_I(\xi) = \lim_{\substack{\varepsilon\to 0\\ \xi \text{ fixed}}} Y(\xi,\varepsilon) = e^{1/2}\left(1 - e^{-2\xi}\right)$$

9.2.1 Examination of the exact solution to a model problem

$y_O(x) = e^{(1-x)/2}$ $\qquad$ $y_I(x) = e^{1/2}\left(1 - e^{-2x/\varepsilon}\right)$



intermediate region

$$y(x,\varepsilon) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right)$$

9.2.2 Finding an approximate solution by singular perturbation methods

Let's solve ODE

$$\varepsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0, \quad 0 < x < 1, \quad 0 < \varepsilon \ll 1$$

B.C. $\quad y(0) = 0, \quad y(1) = 1$

- outer solution

  $2\dfrac{dy_O}{dx} + y_O = 0, \quad y_O(1) = 1 \ \Rightarrow \ y_O(x) = e^{(1-x)/2}$

- inner solution

  Let $\xi = \dfrac{x}{\delta}$ $\quad y(x,\varepsilon) = y(\xi\delta,\varepsilon) = Y(\xi,\varepsilon)$

  thus $\dfrac{\varepsilon}{\delta^2}\dfrac{d^2Y}{d\xi^2} + \dfrac{2}{\delta}\dfrac{dY}{d\xi} + Y = 0$

9.2.2 Finding an approximate solution by singular perturbation methods

Pairwise balance:

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y}{d\xi^2} + \frac{2}{\delta}\frac{dY}{d\xi} + Y = 0$$

$\boxed{\delta = \sqrt{\varepsilon}}$

$O(1) \quad O(\varepsilon^{-1/2}) \quad O(1)$

inconsistent

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y}{d\xi^2} + \frac{2}{\delta}\frac{dY}{d\xi} + Y = 0$$

$\boxed{\delta = \varepsilon}$

$O(\varepsilon^{-1}) \quad O(\varepsilon^{-1}) \quad O(1)$

consistent

$$\frac{d^2Y}{d\xi^2} + 2\frac{dY}{d\xi} + \varepsilon Y = 0$$

$$y_I(\xi) = \lim_{\substack{\varepsilon\to 0\\ \xi \text{ fixed}}} Y(\xi,\varepsilon)$$

$$\frac{d^2y_I}{d\xi^2} + 2\frac{dy_I}{d\xi} = 0 \qquad y_I(0) = 0$$

inner solution $y_I(x) = C\left(1 - e^{-2\xi}\right)$

9.2.3 Matching

The scales of three sub-domain :

- Inner region $x=O(\delta)$
- Intermediate region $x=O\left[\Theta(\varepsilon)\right]$
- Outer region $x=O(1)$

$$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \quad \lim_{\varepsilon\to0}\frac{\Theta}{\delta}=\infty, \qquad \lim_{\varepsilon\to0}\frac{\Theta}{1}=0$$

In intermediate region, *x* can be rescaled by

$$\eta=\frac{x}{\Theta(\varepsilon)} \longrightarrow \quad x=\eta\Theta, \qquad \xi=\frac{x}{\delta}=\frac{\eta\Theta}{\delta}$$

The inner and outer solutions matches if

$$\lim_{\varepsilon\to0}\left[y_O(x)\big|_{x=\eta\Theta}\right]=\lim_{\varepsilon\to0}\left[y_I(\xi)\big|_{\xi=\eta\Theta/\delta}\right],\qquad \eta\text{ fixed}$$

9.2.3 Matching

In the present problem

$$\left.\begin{array}{l}\displaystyle\lim_{\substack{\varepsilon\to0\\ \eta\text{ fixed}}}\left[y_O(x)\big|_{x=\eta\Theta}\right]=\lim_{\substack{\varepsilon\to0\\ \eta\text{ fixed}}}e^{(1-\eta\Theta)/2}=e^{1/2}\\[2ex] \displaystyle\lim_{\substack{\varepsilon\to0\\ \eta\text{ fixed}}}\left[y_I(\xi)\big|_{\xi=\eta\Theta/\delta}\right]=\lim_{\substack{\varepsilon\to0\\ \eta\text{ fixed}}}C\left(1-e^{-2\eta\Theta/\delta}\right)=C\end{array}\right\}\quad C=e^{1/2}$$

We find the uniform approximation for the whole domain

$$\begin{aligned}y_U(x)&=y_I\left(\frac{x}{\delta}\right)+y_O(x)-\lim_{\varepsilon\to0}y_O(\eta\Theta)\\&=e^{1/2}\left(1-e^{-2x/\varepsilon}\right)+e^{(1-x)/2}-e^{1/2}\\&=e^{1/2}\left(e^{-x/2}-e^{-2x/\varepsilon}\right)\end{aligned}$$

9.2.3 Matching

Another way for the uniform approximation

$$\begin{aligned}y_U(x)&=y_I\left(\frac{x}{\delta}\right)y_O(x)/\lim_{\varepsilon\to0}y_O(\eta\Theta)\\&=e^{1/2}\left(1-e^{-2x/\varepsilon}\right)e^{(1-x)/2}/e^{1/2}\\&=e^{1/2}\left(e^{-\frac{x}{2}}-e^{-\frac{2x}{\varepsilon}-\frac{x}{2}}\right)\\&\approx e^{1/2}\left(e^{-\frac{x}{2}}-e^{-\frac{2x}{\varepsilon}}\right)\end{aligned}$$

9.2.4 Further examples

Example 1. as $\varepsilon<0$, we know the BL is at x=1. But if we still pretend BL locates at x=0, matching will be impossible

$$e\frac{d^2y}{dx^2}-2\frac{dy}{dx}-y=0,\qquad 0<e\ll1,\ \ e=-\varepsilon>0$$

B.C. $y(0)=0,\quad y(1)=1$

9.2.4 Further examples

- outer solution

$$2\frac{dy_O}{dx}+y_O=0,\quad y_O(1)=1 \quad\Rightarrow\quad y_O(x)=e^{(1-x)/2}$$

- inner solution

Let $\xi=\dfrac{x}{\delta}$ $\qquad y(x,e)=y(\xi\delta,e)=Y(\xi,e)$

thus $\dfrac{e}{\delta^2}\dfrac{d^2Y}{d\xi^2}-\dfrac{2}{\delta}\dfrac{dY}{d\xi}-Y=0$

$\boxed{\delta=e}\qquad O(e^{-1})\qquad O(e^{-1})\qquad O(1)$

9.2.4 Further examples

$$\frac{d^2Y}{d\xi^2}-2\frac{dY}{d\xi}-eY=0$$

$$\Big\downarrow$$

$$\frac{d^2y_I}{d\xi^2}-2\frac{dy_I}{d\xi}=0$$

$$y_I(\xi)=\lim_{\substack{e\to0\\ \xi\text{ fixed}}}Y(\xi,e)$$

$$y_I(0)=0$$

$$y_I(x)=C\left(1-e^{2\xi}\right)$$

In intermediate region, *x* can be rescaled by

$$\eta=\frac{x}{\Theta(e)} \longrightarrow \quad x=\eta\Theta \qquad \xi=\frac{e}{\delta}=\frac{\eta\Theta}{\delta}$$

9.2.4 Further examples

$$\lim_{e\to 0}\frac{\Theta}{\delta}=\infty \qquad \lim_{e\to 0}\frac{\Theta}{1}=0$$

$$\left.\begin{aligned}\lim_{\substack{e\to 0\\ \eta\text{ fixed}}}\left[y_O(x)\big|_{x=\eta\Theta}\right] &= \lim_{\substack{e\to 0\\ \eta\text{ fixed}}} e^{(1-\eta\Theta)/2}=e^{1/2}\\ \lim_{\substack{e\to 0\\ \eta\text{ fixed}}}\left[y_I(\xi)\big|_{\xi=\eta\Theta/\delta}\right] &= \lim_{\substack{e\to 0\\ \eta\text{ fixed}}} C\left(1-e^{2\eta\Theta/\delta}\right)=\infty\end{aligned}\right\}\ \infty\neq e^{1/2}$$

matching is impossible

9.2.4 Further examples

Example 2. as ε<0, we know the BL is at x=1.

$$e\frac{d^2y}{dx^2}-2\frac{dy}{dx}-y=0,\quad 0<e\ll 1,\ e=-\varepsilon>0$$

B.C. $\quad y(0)=0,\quad y(1)=1$

9.2.4 Further examples

- outer solution

$$2\frac{dy_O}{dx}+y_O=0,\quad y_O(0)=0 \quad\Rightarrow\quad y_O(x)\equiv 0$$

- inner solution

Let $\xi=\dfrac{1-x}{\delta}$ $\qquad y(x,e)=y(\xi\delta,e)=Y(\xi,e)$

thus $\dfrac{e}{\delta^2}\dfrac{d^2Y}{d\xi^2}+\dfrac{2}{\delta}\dfrac{dY}{d\xi}-Y=0$

$\boxed{\delta=e}$ $\qquad O(e^{-1}) \qquad O(e^{-1}) \qquad O(1)$

9.2.4 Further examples

$$\frac{d^2Y}{d\xi^2}+2\frac{dY}{d\xi}-eY=0$$

$$\Downarrow$$

$$\frac{d^2y_I}{d\xi^2}+2\frac{dy_I}{d\xi}=0$$

$$y_I(\xi)=\lim_{\substack{e\to 0\\ \xi\text{ fixed}}}Y(\xi,e)$$

$$y_I(1)=1$$

$$y_I(x)=C\left(e^{-2\xi}-1\right)+1$$

In intermediate region, *x* can be rescaled by

$$\eta=\frac{1-x}{\Theta(e)} \longrightarrow \begin{aligned} x&=1-\eta\Theta\\ \xi&=\frac{1-x}{\delta}=\frac{\eta\Theta}{\delta}\end{aligned}$$

9.2.4 Further examples

$$\lim_{e\to 0}\frac{\Theta}{\delta}=\infty \qquad \lim_{e\to 0}\frac{\Theta}{1}=0$$

$$\left.\begin{aligned}\lim_{\substack{e\to 0\\ \eta\text{ fixed}}}\left[y_O(x)\big|_{x=1-\eta\Theta}\right] &= 0\\ \lim_{\substack{e\to 0\\ \eta\text{ fixed}}}\left[y_I(\xi)\big|_{\xi=\eta\Theta/\delta}\right] &= \lim_{\substack{e\to 0\\ \eta\text{ fixed}}} C\left(e^{-2\frac{\eta\Theta}{\delta}}-1\right)+1=C-1\end{aligned}\right\}\ C=1$$

the uniform approximation

$$y_U(x)=y_I\left(\frac{1-x}{\delta}\right)+y_O(x)-\lim_{\varepsilon\to 0}y_O(\eta\Theta)$$

$$=e^{-2(1-x)/e}=e^{2(1-x)/\varepsilon}$$

9.2.4 Further examples

General procedure:

- for singular perturbation BVP for 2nd ODE
- works often but may fail

Consider the boundary value problem

$$\varepsilon y''+f(x,y,y')=0,\ \ y(a)=A,\ y(b)=B,\ \ \varepsilon>0$$

Suppose boundary layer locates at $x=a$

(a) Determine the outer approximation y_0 by solving

$$f(x,y,y')=0,\ \ y(b)=B$$

9.2.4 Further examples

(b) Introduce the boundary layer variable

$$\xi = \pm\frac{x-a}{\delta(\varepsilon)}$$

where the sign is chosen to make ξ positive

With the notation $Y(\xi,\varepsilon)=y(a\pm\xi\delta,\varepsilon)$

$$\frac{\varepsilon}{\delta^2}\frac{d^2y}{d\xi^2}+f\left(a\pm\xi\delta,Y,\pm\frac{1}{\delta}\frac{dy}{d\xi}\right)=0$$

Suppose the second term has the form $\delta^s F\left(\xi,Y,\frac{dY}{d\xi}\right)$

Balance: $\frac{\varepsilon}{\delta^2}=\delta^s \to \delta=\varepsilon^{1/(2+s)} \qquad s>-2$

9.2.4 Further examples

(c) Determine $y_I(\xi)$ by solving

$$\frac{d^2y_I}{d\xi^2}+F\left(\xi,y_I,\frac{dy_I}{d\xi}\right)=0,\ y_I(0)=A$$

(c) To find the arbitrary constant in the above solution introduce the intermediate variable

$$\eta=\pm\frac{x-a}{\Theta(\varepsilon)}$$

$$\lim_{\varepsilon\to 0}\frac{\Theta}{\delta}=\infty \qquad \lim_{\varepsilon\to 0}\frac{\Theta}{1}=0$$

9.2.4 Further examples

(c) Determine $y_I(\xi)$ by solving

$$\frac{d^2y_I}{d\xi^2}+F\left(\xi,y_I,\frac{dy_I}{d\xi}\right)=0,\ y_I(0)=A$$

(c) To find the arbitrary constant in the above solution introduce the intermediate variable

$$\eta=\pm\frac{x-a}{\Theta(\varepsilon)} \qquad \lim_{e\to 0}\frac{\Theta}{\delta}=\infty \qquad \lim_{e\to 0}\frac{\Theta}{1}=0$$

Impose the matching requirement

$$\lim_{\varepsilon\to 0}\left[y_O(x)\Big|_{x=a\pm\eta\Theta}\right]=\lim_{\varepsilon\to 0}\left[y_I(\xi)\Big|_{\xi=\eta\Theta/\delta}\right], \qquad \eta \text{ fixed}$$

9.2.4 Further examples

Example 3. Find outer and inner approximations for small positive ε to the solution of

$$\varepsilon\frac{d^2y}{dx^2}+\alpha(x)\frac{dy}{dx}+\beta(x)y=0, \qquad \text{BL: } x=0$$

$$y(0)=0,\ y(1)=1,\ \alpha(0)\equiv a_0>0,\ \beta(0) \text{ finite}$$

- outer solution

$$\alpha(x)\frac{dy}{dx}+\beta(x)y=0,\ \ y(1)=1$$

$$y_O(x)=\exp\left[\int_x^1\frac{\beta}{\alpha}ds\right]$$

9.2.4 Further examples

- inner solution

Let $\xi=\frac{x}{\delta} \qquad \delta(\varepsilon)\to 0 \ \ as\ \varepsilon\to 0$

$$\frac{\varepsilon}{\delta^2}\frac{d^2Y}{d\xi^2}+\frac{\alpha(\delta\xi)}{\delta}\frac{dY}{d\xi}+\beta(\delta\xi)Y=0$$

$$\boxed{\delta=\varepsilon} \quad O(\varepsilon^{-1}) \qquad O(\varepsilon^{-1}) \qquad O(1)$$

$$\frac{d^2y_I}{d\xi^2}+\alpha_0\frac{dy_I}{d\xi}=0,\ \ y_I(0)=0$$

$$y_I(x)=C\left(1-e^{-\alpha_0\xi}\right)+1$$

9.2.4 Further examples

Matching

$$C=\exp\left[\int_0^1\frac{\beta}{\alpha}ds\right]$$