

Ch. 8 一个生理流动问题的求解及其所示明的技巧

Chapter 8

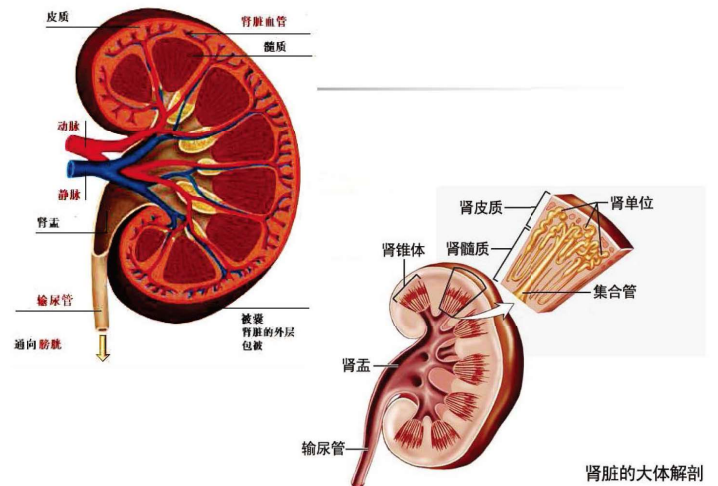
Illustration of Techniques on a Physiological Flow Problem

- 8.1 “固定梯度”渗流模型的物理表述和量纲分析
- 8.2 一个数学模型及其量纲分析
- 8.3 求得最终尺度化后的无量纲数学模型
- 8.4 解和解释

用量纲分析、尺度化和正则摄动方法处理一个生理流动的简化模型

8.1 “固定梯度”渗流模型的物理表述和量纲分析

1. 一些生理学事实
2. 渗透作用和渗透压摩尔
3. 影响固定梯度流动的因素
4. 函数关系的量纲分析
5. 建立固定梯度流尺度化模型的可能性

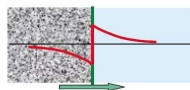


肾脏的大体解剖

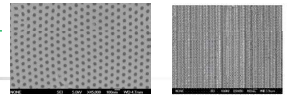
1. 一些生理学事实

- 各种组织将溶质分泌到它们附近的或“浸泡”它们的液体中去 盐分通过肾脏排出
- 在感兴趣的情况下，浸泡液体中溶质的浓度近似等于或略大于溶质在分泌组织中的平均浓度
- 扩散不可能是溶质分泌的主要机理
- 另一种可能的机理是由组织和浸泡液间薄膜的主动运输（化学“泵”）
- 对溶质浓度随与薄膜距离的关系的测量表明溶质不可能直接通过薄膜注入浸泡液

$$J = -k \frac{\partial c}{\partial x} \approx -k \frac{\Delta c}{t}$$



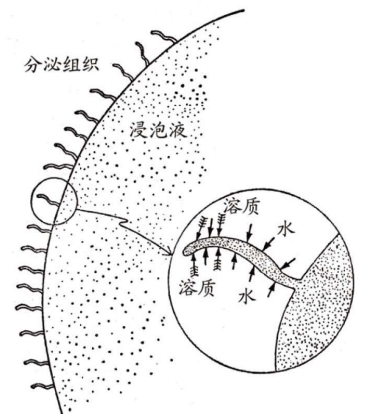
氧化铝模板SEM照片



J. Diamond 等的假设：
溶质被主动运输到细管的“远”端，当高浓度液体流向细管开口时，逐渐被渗入的水稀释

定常流动或固定梯度流动

也适用于吸收组织



2. 渗透作用和渗透压摩尔

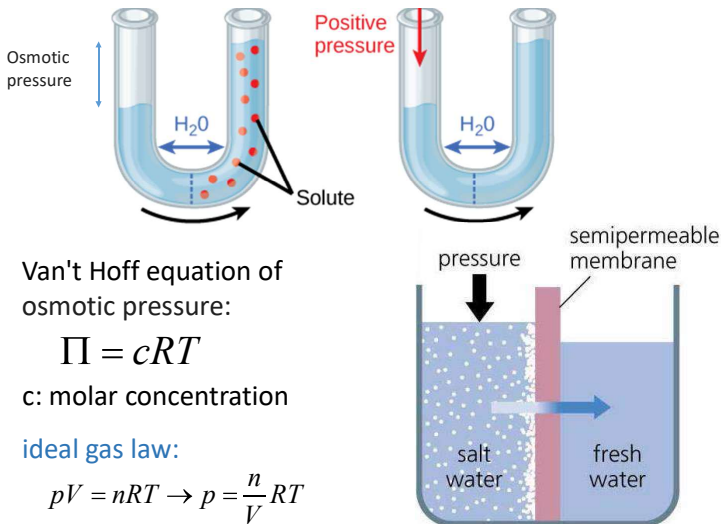
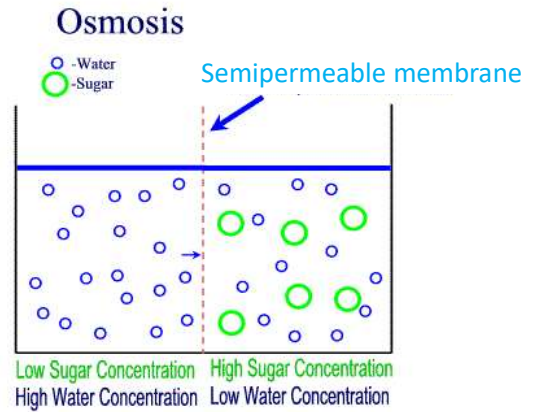
渗透作用: semipermeable membrane

- 组织的边界可看作半透膜, 仅能通过水 (溶剂) 而不能通过盐 (溶质) 分子大小不同
- 渗透速率取决于膜两侧的溶质浓度 渗透使浓度趋同

渗透压摩尔 (osmol): 溶液中以摩尔 (克分子) 表示的不离解的溶质数 1 摩尔 = 6.06x10²³ 个分子, 质量

- 如溶液中含 1 摩尔葡萄糖: 1 osmol
- 如溶液中含 1 摩尔氯化钠: ~ 2 osmol

可假定溶剂渗透速率与膜两侧溶质浓度差成正比



Van't Hoff equation of osmotic pressure:

$$\Pi = cRT$$

c: molar concentration

ideal gas law:

$$pV = nRT \rightarrow p = \frac{n}{V}RT$$

hydrogel



Chemical potential μ driven

Chemical potential μ

Consider a system interacting with a reservoir via **diffusive contact**.

- energy, particles, volume will change.

The internal energy $U = TS - pV + \mu_i N_i$. Then the change in U is,

$$dU = TdS - pdV + \mu_i dN_i$$

where T the temperature, S the entropy. μ_i the Chemical potential of specie i reads

$$\mu_i = \left. \frac{\partial U}{\partial N_i} \right|_{S, V, N_{j \neq i}} \quad j = 1, 2, \dots$$

For chemical solutions, we use a convenient way...

Chemical potential μ

Gibbs free energy

$$G(p, T) = U + pV - TS = H - TS = F + pV$$

The variation in Gibbs free energy G is,

$$dG = dU + Vdp + pdV - SdT - TdS$$

or

$$dG = Vdp - SdT + \mu_i dN_i$$

The definition of Chemical potential of specie i :

$$\mu_i = \left. \frac{\partial G}{\partial N_i} \right|_{p, T, N_{j \neq i}} \quad j = 1, 2, \dots$$

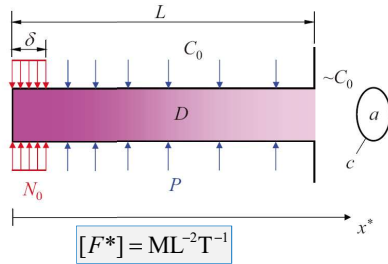
μ_i is the change in G with one newly added particle, at const T & P .

Physical meaning of Chemical potential

- the main driving force for any mass diffusion
- Particles move from high to low chemical potential
- a quantity that is equal for two systems in equilibrium which can exchange particles.
- Diffusion is driven by decrease in Gibbs free energy or chemical potential gradient.
- Diffusion stops when chemical potential of all species everywhere are same.

生理学最感兴趣的量：
溶质从细管开口端逸出的速率

$$O_s^* = \frac{F^*(L)}{v^*(L)} = \frac{\text{溶质量}}{\text{溶液量}_{\text{出口}}}$$



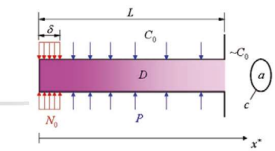
$v^*(x^*)$: 流速 $F^*(x^*)$: 溶质的通量密度
 O_s^* : 逸出渗透压摩尔浓度 emergent osmolarity

逸出渗透压容摩尔, 逸出同渗容摩

流出单位体积溶液中溶质的摩尔数 (浓度量纲)

$$av^*(L)O_s^* = aF^*(L) \rightarrow O_s^* = \frac{F^*(L)}{v^*(L)}$$

4. 函数关系的量纲分析



$$\frac{O_s^*}{C_0} = f(\delta, L, c, D, C_0, P, N_0, a)$$

$\delta^{a_1} L^{a_2} c^{a_3} D^{a_4} C_0^{a_5} P^{a_6} N_0^{a_7} a^{a_8}$ 的量纲:

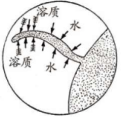
$$\mathcal{L}^{a_1 + a_2 + a_3 + 2a_4 - 3a_5 + 4a_6 - 2a_7 + 2a_8}$$

$$\mathcal{M}^{a_5 - a_6 + a_7} \mathcal{T}^{-a_4 - a_6 - a_7}$$

量	量纲
δ, L, c	\mathcal{L}
D	$\mathcal{L}^2 \mathcal{T}^{-1}$
C_0, O_s^*	$\mathcal{M} \mathcal{L}^{-3}$
P	$\mathcal{M}^{-1} \mathcal{L}^4 \mathcal{T}^{-1}$
N_0	$\mathcal{M} \mathcal{L}^{-2} \mathcal{T}^{-1}$
a	\mathcal{L}^2

3 个方程, 5 个无量纲参数

3. 影响固定梯度流动的因素



忽略细管横截面形状变化、径向效应、管壁对溶质的渗透等次要因素近似假定:

- 1) 细管理想化为直柱体, 截面积 a , 周长 c , 长度 L
- 2) 流过单位面积管壁的水流速率正比于两侧溶质浓度差, 渗透率 P (permeability)
- 3) 细管外浸泡液中 (包括开口处) 溶质浓度为常数 C_0
- 4) 细管中溶液的流速仅与沿细管的距离有关
- 5) 细管中液体存在溶质的扩散, 扩散系数 D
- 6) 溶质的主动输运仅发生在离闭端距离为 δ 以内, 单位面积管壁的输运速率为常数 N_0 $[N_0] = \text{ML}^{-2} \text{T}^{-1}$

$$O_s^* = \frac{F^*}{v^*} \Rightarrow [O_s^*] = \text{ML}^{-2} \text{T}^{-1} / \text{LT}^{-1} = \text{ML}^{-3}$$

concentration

$$J = P \Delta C$$

$$\text{permeability} = \frac{\text{solvent volume}/(\text{area} \times \text{time})}{\text{difference in solute mass per unit volume}}$$

$$[P] = \text{LT}^{-1} / \text{ML}^{-3} = \text{M}^{-1} \text{L}^4 \text{T}^{-1}$$

$$\delta^{a_1} L^{a_2} c^{a_3} D^{a_4} C_0^{a_5} P^{a_6} N_0^{a_7} a^{a_8}$$

$$a_1 + a_2 + a_3 + 2a_4 - 3a_5 + 4a_6 - 2a_7 + 2a_8 = 0$$

$$a_5 - a_6 + a_7 = 0$$

$$a_4 + a_6 + a_7 = 0$$

将 5 个视为参数, 解出其余 3 个
分别取一个参数为 1, 其余为 0

$$\pi_1 = \frac{\delta P C_0}{D}, \pi_2 = \frac{\delta N_0}{D C_0}, \pi_3 = \frac{L}{\delta}, \pi_4 = \frac{c}{\delta}, \pi_5 = \frac{a}{\delta^2}$$

$$\frac{O_s^*}{C_0} = f(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$$

根据一维流假设, 还可进一步简化:

令 $P_1 = cP$, $N_1 = cN_0$, $D_1 = aD$

$$\rightarrow \frac{O_s^*}{C_0} = f(\delta, L, D_1, C_0, P_1, N_1)$$

$$\frac{O_s^*}{C_0} = f_1(\pi_6, \pi_7, \pi_8)$$

$$\pi_6 = \frac{c\delta^2 C_0 P}{aD}, \quad \pi_7 = \frac{c\delta^2 N_0}{aDC_0}, \quad \pi_8 = \frac{L}{\delta}$$

$$= \frac{\pi_1 \pi_4}{\pi_5} = \frac{\pi_2 \pi_4}{\pi_5} = \pi_3$$

或者

$$v = \frac{\pi_7}{\pi_6} = \frac{N_0}{PC_0^2}, \quad \eta = \frac{1}{\pi_7} = \frac{aC_0 D}{cN_0 \delta^2}, \quad \lambda = \pi_8 = \frac{L}{\delta}$$

a, c 仅出现在

渗流: $Pc\Delta x^*$

输运: $N_0 c \Delta x^*$

扩散: Da

5. 建立固定梯度流尺度化模型的可能性

真实细管过小, 无法做实际测量, 应放大

$$\pi_6 = \frac{c\delta^2 C_0 P}{aD}, \quad \pi_7 = \frac{c\delta^2 N_0}{aDC_0}, \quad \pi_8 = \frac{L}{\delta} \quad \text{需保持不变}$$

直径也放大100倍

若 $L' = 100L$, $\delta' = 100\delta \rightarrow \pi_8' = \pi_8$

要求 $\pi_6' = \pi_6 \rightarrow \frac{C_0' P'}{D'} = \frac{1}{100} \frac{C_0 P}{D}$

$$\frac{a'}{c'} = 100 \frac{a}{c}$$

$\pi_7' = \pi_7 \rightarrow \frac{N_0'}{D' C_0'} = \frac{1}{100} \frac{N_0}{DC_0}$

$$\frac{C_0' P'}{D'} = \frac{1}{100} \frac{C_0 P}{D} \quad \frac{N_0'}{D' C_0'} = \frac{1}{100} \frac{N_0}{DC_0}$$

方案 1: $C_0' = C_0, N_0' = \frac{N_0}{100}, P' = \frac{P}{100}, D' = D$
 相对较易实现

方案 2: $C_0' = C_0, N_0' = N_0, P' = P, D' = 100D$

方案 3: $C_0' = C_0, N_0' = \frac{N_0}{10}, P' = \frac{P}{10}, D' = 10D$
 扩散系数较难改变

而无量纲逸出速率 $\frac{O_s'}{C_0'} = \frac{O_s^*}{C_0}$

8.2 一个数学模型及其量纲分析

1. 流体质量守恒

2. 溶质质量守恒

3. 边界条件

4. 无量纲变量的引入

建立问题的控制方程并无量纲化

1. 流体质量守恒

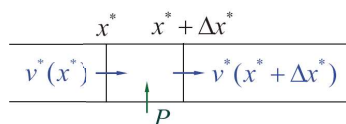
$$a[v^*(x^*) - v^*(x^* + \Delta x^*)] + Pc \int_{x^*}^{x^* + \Delta x^*} [C^*(s) - C_0] ds = 0$$

流入和流出 Δx^* 的水的体积 由侧壁渗入的水的体积
 (单位时间)

其中 $C^* = C^*(x^*)$ 为溶质浓度分布

两边除以 Δx^* 并取极限, 得

$$\frac{dv^*}{dx^*} = Pca^{-1}(C^* - C_0)$$



2. 溶质质量守恒

以 $N^*(x^*)$ 表示侧向主动输运速率

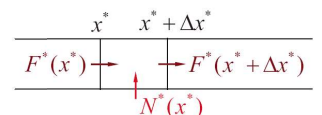
$$N^*(x^*) = \begin{cases} N_0, & 0 \leq x^* \leq \delta \\ 0, & \delta < x^* \leq L \end{cases}$$

$$c \int_{x^*}^{x^* + \Delta x^*} N^*(s) ds + a[F^*(x^*) - F^*(x^* + \Delta x^*)] = 0$$

Δx^* 段主动输入质量 溶质通量密度 (单位时间)

其中 $F^*(x^*)$ 为溶质通量密度

$$\rightarrow cN^*(x^*) - a \frac{dF^*}{dx^*} = 0$$



溶质通量密度 $F^* = F_{\text{conv}}^*(x^*) + F_{\text{diff}}^*(x^*)$
 对流 扩散

$$F_{\text{conv}}^* = C^* v^*, \quad F_{\text{diff}}^* = -D \frac{dC^*}{dx^*}$$

$$\Rightarrow F^* = v^* C^* - D \frac{dC^*}{dx^*}$$

开口端溶质逸出速率还包括扩散的贡献, $O_s^* \neq C_0$

$$O_s^* = \frac{F^*}{v^*} = C^* - \frac{D}{v^*} \frac{dC^*}{dx^*}$$

3. 边界条件

$$F^* = v^* C^* - D \frac{dC^*}{dx^*}$$

$$x^* = 0 \text{ 处: } v^* = 0, F^* = 0 \Rightarrow \frac{dC^*}{dx^*} = 0$$

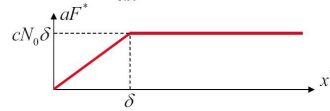
$$x^* = L \text{ 处: } C^* = C_0$$

$$x^* = \delta \text{ 处的连续条件: } [F^*] = 0, [v^*] = 0, [C^*] = 0$$

N^* 间断引起 溶质质量守恒 流体速度连续 浓度连续

$$\Rightarrow \left[\frac{dC^*}{dx^*} \right] = 0 \quad [f(x)] = f(x_+) - f(x_-)$$

$$cN_0(x^*) - a \frac{dF^*}{dx^*} = 0 \Rightarrow aF^* = \begin{cases} cN_0 x^*, & 0 \leq x^* \leq \delta \\ cN_0 \delta, & \delta < x^* \leq L \end{cases}$$



$$F^* = v^* C^* - D \frac{dC^*}{dx^*} \quad aF^* = \begin{cases} cN_0 x^*, & 0 \leq x^* \leq \delta \\ cN_0 \delta, & \delta < x^* \leq L \end{cases}$$

Obviously, we have

$$x = \frac{x^*}{\delta}, \quad C = \frac{C^*}{C_0}$$

最后得关于 v^*, C^* 的微分方程组

$$\begin{cases} \frac{dv^*}{dx^*} = Pca^{-1}(C^* - C_0) \\ v^* C^* - D \frac{dC^*}{dx^*} = \begin{cases} a^{-1}cN_0 x^*, & 0 \leq x^* \leq \delta \\ a^{-1}cN_0 \delta, & \delta < x^* \leq L \end{cases} \\ v^*(0) = 0, C^*(L) = C_0, [v^*]_{x^*=\delta} = 0, [C^*]_{x^*=\delta} = 0 \end{cases} \quad \text{非线性}$$

$$\text{需求解: } O_s^* = \frac{F^*(L)}{v^*(L)} = \frac{cN_0 \delta}{av^*(L)}$$

渗透压容摩尔 emergent osmolarity

dimensionless velocity

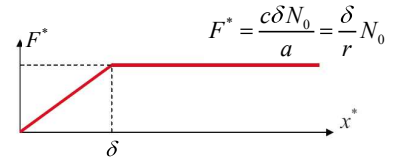
$$F_{\text{convect}}^* = C^* v^*$$



$$V = \left(\frac{F^*}{C^*} \right)_{\text{typical}} = \frac{c\delta N_0}{a C_0} \rightarrow [V] = \frac{LL}{L^2} \frac{ML^{-2}T^{-1}}{ML^{-3}} = LT^{-1}$$



$$v = \frac{v^*}{V} = \frac{v^*}{cN_0 \delta / aC_0}$$



$$x = \frac{x^*}{\delta}, \quad C = \frac{C^*}{C_0}, \quad v = \frac{v^*}{cN_0 \delta / aC_0}$$



$$\begin{cases} C - 1 = v v' \\ v C - \eta C' = g(x) \\ v(0) = 0, C(\lambda) = 1, [v]_{x=1} = [C]_{x=1} = 0 \end{cases}$$

$$\text{where } g(x) = \begin{cases} x, & x \in (0, 1) \\ 1, & x \in (1, \lambda) \end{cases}$$

$$v = \frac{N_0}{PC_0^2}, \quad \eta = \frac{aC_0 D}{cN_0 \delta^2}, \quad \lambda = \frac{L}{\delta}$$

$$O_s = \frac{O_s^*}{C_0} = \frac{cN_0 \delta / aC_0}{v^*(L)} = \frac{1}{v(\lambda)}$$

Let's look at first $F^* = F_{\text{convect}}^* + F_{\text{diffusion}}^* = C^* v^* - D \frac{dC^*}{dx^*}$

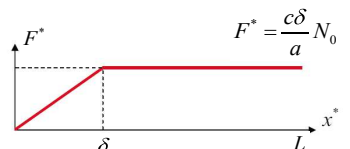
Which one is dominant ?

$$\text{as } x=0, v^*=0, \frac{dC^*}{dx^*}=0 \quad F^* = C^* v^* - D \frac{dC^*}{dx^*} = 0$$

$$\text{as } x \in (0, \delta), F_{\text{convect}}^* \ll F_{\text{diffusion}}^* \quad F^* \approx -D \frac{dC^*}{dx^*}$$

$$\text{as } x \sim \delta, F_{\text{convect}}^* \approx F_{\text{diffusion}}^* \quad F^* \approx 2C^* v^* \approx -2D \frac{dC^*}{dx^*}$$

$$\text{as } x \approx L, F_{\text{convect}}^* \gg F_{\text{diffusion}}^* \quad F^* \approx C^* v^*$$



Try to understand dimensionless group

$$v = \frac{N_0}{PC_0^2}, \quad \eta = \frac{aC_0D}{cN_0\delta^2}, \quad \lambda = \frac{L}{\delta}$$

$$1. \quad \eta = \frac{aC_0D}{cN_0\delta^2}$$

$$\text{as } x \sim \delta, \quad F^* \approx 2C^*v^* \approx -2D \frac{dC^*}{dx^*} \Rightarrow C_0v^* \sim D \frac{C_0}{\delta} \Rightarrow v^*(\delta) \sim \frac{D}{\delta}$$

$$\text{as } x \approx L, \quad F^* \approx \frac{c\delta N_0}{a} = C_0v^* \Rightarrow v^*(L) = \frac{c\delta N_0}{aC_0}$$

$$\eta = \frac{v^*(\delta)}{v^*(L)} = \frac{D}{\delta} \frac{aC_0}{cN_0\delta} = \frac{aC_0D}{cN_0\delta^2}$$

$$2. \quad v = \frac{N_0}{PC_0^2}$$

Water permeated in δ region in unit time

$$a \frac{dv^*}{dx^*} = Pc(C^* - C_0) \Rightarrow av^* \sim \delta cPC_0$$

$$v = \frac{\delta c N_0}{\delta c PC_0 C_0} = \frac{\delta c N_0}{V_{water} C_0} = \frac{m_{solute}}{M_{solute}} \quad [N_0] = ML^{-2}T^{-1}$$

$$[C_0] = ML^{-3}$$

in the end δ region:

$$m_{solute} = \delta c N_0 \quad \text{mass of solute transported actively}$$

$$M_{solute} = V_{water} C_0 \quad \text{mass of solute in permeated water}$$

量纲分析的物理方法和数学方法之比较

8.1 节用物理方法得到 π_6, π_7, π_8 , 8.2 节用数学方法得到 v, η, λ

数学方法:

- 根据已知物理定律通过简化建立现象的控制方程
- 进行量纲分析, 引入无量纲参数

物理方法:

- 确定参数依赖关系需要相当深刻的物理直觉
- 不需要知道控制方程

a, c, N_0, P, D 仅以组合
 cP, cN_0, aD 出现

$$\pi_6 = \frac{c\delta^2 C_0 P}{aD}, \quad \pi_7 = \frac{c\delta^2 N_0}{aDC_0}, \quad \pi_8 = \frac{L}{\delta}$$

➢ 可引入等效半径 $\frac{r}{2} \equiv \frac{a}{c}$

➢ 可以用改变 r 代替改变 D

➢ N_0, P, r 同时变化, 结果不变

压扁细管, r 减小
相当于 D 减小

8.3 求得最终尺度化后的无量纲数学模型

1. 尺度化

2. 一个失败的正则摄动计算

3. 参数间的关系

4. 最后公式

多个参数时可能发生困难

1. 尺度化

J Diamond, W Bossert

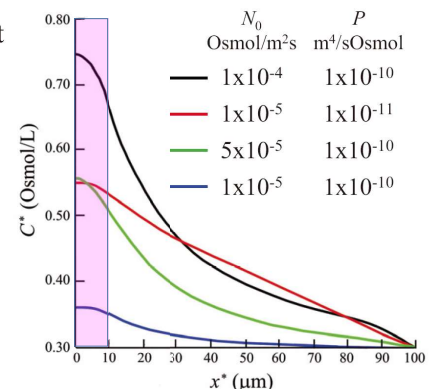
的数值计算结果

(细管中的浓度分布)

$L = 100 \mu\text{m}$, $\delta = 10 \mu\text{m}$
 $r = 0.05 \mu\text{m}$, $D = 10^{-9} \text{m}^2/\text{s}$
 $C_0 = 0.3 \text{Osmol/L}$

浓度与 C_0 相当

$$C = \frac{C^*}{C_0}$$



Concentration scale

速度的尺度

$x^* = L$ 处的通量密度

最大速度在开口端

$$F^*(L) = v^*(L)C^*(L) - D \frac{dC^*}{dx^*}(L) \quad \frac{dC^*}{dx^*} < 0$$

对流 扩散

$$\Rightarrow F^*(L) \geq v^*(L)C^*(L) \quad v^*(L) \leq \frac{F^*(L)}{C^*(L)}$$

边界条件 $C^*(L) = C_0$, $F^*(L) = a^{-1}cN_0\delta \Rightarrow$

$$v^*(L) \leq \frac{cN_0\delta}{aC_0} \Rightarrow \boxed{v^* \leq \frac{cN_0\delta}{aC_0}}$$

扩散作用不会大大超过对流

26

x^* 的尺度: C^*, v^* 变化最激烈的区域为主动输运段

$$\Rightarrow x = \frac{x^*}{\delta}, \quad C = \frac{C^*}{C_0}, \quad v = \frac{v^*}{cN_0\delta/(aC_0)}$$

$$\begin{cases} C-1 = v v' \\ vC - \eta C' = g(x) \\ v(0) = 0, \quad C(\lambda) = 1, \quad [v]_{x=1} = [C]_{x=1} = 0 \end{cases}$$

where $g(x) = \begin{cases} x, & x \in (0, 1) \\ 1, & x \in (1, \lambda) \end{cases}$

$$v = \frac{N_0}{PC_0^2}, \quad \eta = \frac{aC_0D}{cN_0\delta^2}, \quad \lambda = \frac{L}{\delta}, \quad Os = \frac{Os^*}{C_0} = \frac{1}{v(\lambda)}$$

ν : \epsilon 如果有一个小参数, 可以用摄动展开求解

无量纲参数大小的估计

$$v = \frac{N_0}{PC_0^2}, \quad \eta = \frac{aC_0D}{cN_0\delta^2}$$

参数	单位	最小值	典型值	最大值
r	m	1×10^{-8}	5×10^{-8}	1×10^{-6}
L	m	4×10^{-6}	1×10^{-4}	2×10^{-4}
δ	m	4×10^{-7}	1×10^{-5}	2×10^{-5}
D	m^2/s	1×10^{-10}	1×10^{-9}	5×10^{-9}
N_0	Osmol/ m^2s	1×10^{-9}	1×10^{-6}	1×10^{-4}
P	$m^4s/Osmol$	1×10^{-11}	2×10^{-10}	2×10^{-9}
C_0	Osmol/ m^3	—	3×10^{-1}	—
小参数 ν	1	1×10^{-5}	5×10^{-2}	1×10^2
η	1	4×10^{-3}	75	1×10^{10}
κ	1	4×10^{-3}	5	200

28

2. 一个失败的正则摄动计算

$$\begin{cases} C-1 = \nu v', & 0 < x < \lambda \\ vC - \eta C' = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases} \\ v(0) = 0, \quad C(\lambda) = 1, \quad [C]_{x=1} = 0, \quad [v]_{x=1} = 0 \end{cases} \Rightarrow C(x) \approx 1 \quad C^*(x^*) \approx C_0$$

将 v, C 按小参数 ν 展开:

$$C(x) = C^{(0)}(x) + \nu C^{(1)}(x) + \nu^2 C^{(2)}(x) + \dots$$

$$v(x) = v^{(0)}(x) + \nu v^{(1)}(x) + \nu^2 v^{(2)}(x) + \dots$$

$$\begin{cases} C-1 = \nu v', & 0 < x < \lambda \\ vC - \eta C' = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases} \\ v(0) = 0, \quad C(\lambda) = 1 \\ [C]_{x=1} = 0, \quad [v]_{x=1} = 0 \end{cases}$$

$$[C^{(0)} + \nu C^{(1)} + \dots] - 1 - \nu [v^{(0)} + \nu v^{(1)} + \dots]' = 0$$

$$[v^{(0)} + \nu v^{(1)} + \dots][C^{(0)} + \nu C^{(1)} + \dots] - \eta [C^{(0)} + \nu C^{(1)} + \dots]' = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

自然假设 η 与 ν 无关, 得

$$O(1): \quad C^{(0)} = 1, \quad v^{(0)} C^{(0)} - \eta C^{(0)'} = \begin{pmatrix} x \\ 1 \end{pmatrix} \quad \rightarrow$$

$$O(\nu): \quad C^{(1)} = v^{(0)'}, \quad v^{(1)} C^{(0)} + v^{(0)} C^{(1)} - \eta C^{(1)'} = 0$$

满足边界条件的解

$$C^{(0)} = 1, \quad v^{(0)} = \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad C^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^{(1)} = -\begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{不连续!}$$

30

3. 参数间的关系

$$\begin{cases} C-1 = \nu v', & 0 < x < \lambda \\ vC - \eta C' = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases} \end{cases}$$

问题: 无量纲参数 η 也可以为 $\eta_1 = \eta/\nu$, $\eta_2 = \nu\eta$, ...

$$\eta_1: \quad vC - \nu\eta_1 C' = \begin{pmatrix} x \\ 1 \end{pmatrix} \quad \eta_2: \quad \nu vC - \eta_2 C' = \nu \begin{pmatrix} x \\ 1 \end{pmatrix}$$

当 $\nu \rightarrow 0$ 时, 究竟是 η 还是 η_1 或 η_2 保持不变?

办法: ① 尝试各种可能性, ② 依靠物理洞察力

$$\nu \rightarrow 0 \text{ 可能 } N_0 \rightarrow 0 \Rightarrow \eta \rightarrow \infty \quad v = \frac{N_0}{PC_0^2}, \quad \eta = \frac{aC_0D}{cN_0\delta^2}$$

C_0 基本不变

$$\eta_2 = \nu\eta = \frac{aD}{c\delta^2 PC_0} \quad 31$$

4. 最后公式

$$[C^{(0)} + \nu C^{(1)} + \dots] - 1 - \nu[v^{(0)} + \nu v^{(1)} + \dots]' = 0$$

$$\nu[v^{(0)} + \nu v^{(1)} + \dots][C^{(0)} + \nu C^{(1)} + \dots] - \eta_2[C^{(0)} + \nu C^{(1)} + \dots]' = \nu \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$O(1): C^{(0)} = 1, -\eta_2 C^{(0)'} = 0 \rightarrow C^{(0)} = 1$$

$$O(\nu): C^{(1)} = \nu^{(0)'}, \nu^{(0)} C^{(0)} - \eta_2 C^{(1)'} = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

对流 扩散

对流的影响是高阶的

为使公式看来简单一些, 令

$$\eta_2 = \frac{\lambda^2}{\kappa^2} \rightarrow \eta = \frac{\lambda^2}{\nu \kappa^2}, \kappa = \left(\frac{c P C_0 L^2}{a D} \right)^{1/2} \quad \begin{matrix} \eta = 75 \\ \text{典型值 } \eta_2 = 3.75 \\ \kappa = 5 \end{matrix}$$

$$\begin{aligned} C-1 &= \nu v', & 0 < x < \lambda & & v(0) &= 0, & C(\lambda) &= 1 & & \eta = \frac{\lambda^2}{\nu \kappa^2} \\ \nu C - \eta C' &= \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases} & & & [C]_{x=1} &= 0, & [v]_{x=1} &= 0 & & \end{aligned}$$

$$\begin{cases} C-1 = \nu v', & 0 < x < \lambda \\ \nu \kappa^2 \nu C - \lambda^2 C' = \nu \kappa^2 \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases} \\ v(0) = 0, & C(\lambda) = 1, & [C]_{x=1} = 0, & [v]_{x=1} = 0 \end{cases}$$

$$\nu = \frac{N_0}{P C_0^2}, \kappa = \left(\frac{c P C_0 L^2}{a D} \right)^{1/2}, \lambda = \frac{L}{\delta}$$

$$0 < \nu \ll 1, \kappa, \lambda = O(1) \quad \text{求 } O_s = \frac{1}{\nu(\lambda)}$$

8.4 解和解释

1. 解的一级近似

2. 与数值解的比较

3. 解释: 无量纲参数的物理意义

近似解和对解的物理含义的讨论

1. 解的一级近似

$$\begin{cases} C-1 = \nu v' \\ \nu \kappa^2 \nu C - \lambda^2 C' = \nu \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix} \\ v(0) = 0, & C(\lambda) = 1, & [C]_{x=1} = 0, & [v]_{x=1} = 0 \end{cases} \quad 0 < \nu \ll 1, \kappa, \lambda = O(1)$$

$$\begin{aligned} C(x) &= C^{(0)}(x) + \nu C^{(1)}(x) + \nu^2 C^{(2)}(x) + \dots \\ v(x) &= v^{(0)}(x) + \nu v^{(1)}(x) + \nu^2 v^{(2)}(x) + \dots \end{aligned} \rightarrow$$

$$[C^{(0)} + \nu C^{(1)} + \dots] - 1 - \nu[v^{(0)} + \nu v^{(1)} + \dots]' = 0$$

$$\nu \kappa^2 [v^{(0)} + \nu v^{(1)} + \dots][C^{(0)} + \nu C^{(1)} + \dots] - \lambda^2 [C^{(0)} + \nu C^{(1)} + \dots]' = \nu \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$[C^{(0)} + \nu C^{(1)} + \nu^2 C^{(2)} + \dots] - 1 - \nu[v^{(0)} + \nu v^{(1)} + \dots]' = 0$$

$$\nu \kappa^2 [v^{(0)} + \nu v^{(1)} + \dots][C^{(0)} + \nu C^{(1)} + \dots] - \lambda^2 [C^{(0)} + \nu C^{(1)} + \nu^2 C^{(2)} + \dots]' = \nu \kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$O(1): C^{(0)} = 1, \lambda^2 C^{(0)'} = 0 \rightarrow C^{(0)}$$

$$O(\nu): C^{(1)} = v^{(0)'}, \lambda^2 C^{(1)'} = \kappa^2 \left[v^{(0)} C^{(0)} - \begin{pmatrix} x \\ 1 \end{pmatrix} \right] \rightarrow v^{(0)}, C^{(1)}$$

$$O(\nu^2): C^{(2)} = v^{(1)'}, \lambda^2 C^{(2)'} = \kappa^2 [v^{(0)} C^{(1)} + v^{(1)} C^{(0)}] \rightarrow v^{(1)}, C^{(2)}$$

$$\rightarrow C^{(0)} = 1 \quad \rightarrow \lambda^2 v^{(0)''} - \kappa^2 v^{(0)} = -\kappa^2 \begin{pmatrix} x \\ 1 \end{pmatrix}$$

齐次通解为指数函数或双曲函数

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

由边界条件可解出

$$\begin{cases} v^{(0)} = x - K_1 \sinh \kappa \lambda^{-1} x & 0 \leq x \leq 1 \\ C^{(1)} = 1 - K_1 \kappa \lambda^{-1} \cosh \kappa \lambda^{-1} x \end{cases}$$

$$\begin{cases} v^{(0)} = 1 - K_2 \cosh \kappa (1 - \lambda^{-1} x) & 1 \leq x \leq \lambda \\ C^{(1)} = K_2 \kappa \lambda^{-1} \sinh \kappa (1 - \lambda^{-1} x) \end{cases}$$

$$\text{其中 } K_1 = \frac{\lambda \cosh \kappa (1 - \lambda^{-1})}{\kappa \cosh \kappa}, \quad K_2 = \frac{\lambda \sinh \kappa \lambda^{-1}}{\kappa \cosh \kappa}$$

$$\text{由此得一级近似 } O_s = \frac{O_s^*}{C_0} = \frac{1}{\nu(\lambda)} \approx \frac{1}{v^{(0)}(\lambda)} = \frac{1}{1 - K_2}$$

$$O_s = \frac{O_s^*}{C_0} \approx \frac{1}{1-K_2} \quad K_2 = \frac{\lambda \sinh \kappa \lambda^{-1}}{\kappa \cosh \kappa}$$

$$O_s = \frac{1}{1-K_2} = \frac{1}{1 - \frac{\lambda \sinh(\kappa \lambda^{-1})}{\kappa \cosh \kappa}}$$

一般 $\lambda = \frac{L}{\delta} \sim 10$, $\kappa \lambda^{-1} < 1$ 典型值 $\kappa = 5$

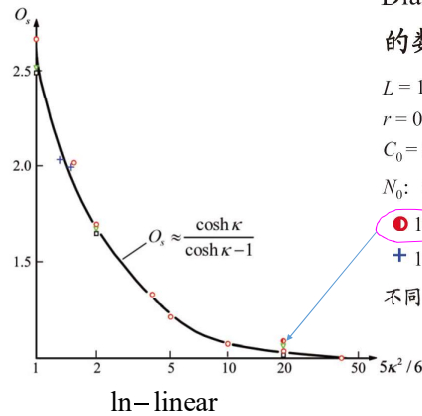
故 $\sinh(\kappa \lambda^{-1}) \approx \kappa \lambda^{-1}$, $K_2 \approx (\cosh \kappa)^{-1}$

$$\text{最后得 } O_s \approx \frac{\cosh \kappa}{\cosh \kappa - 1} = \frac{1 + \frac{\kappa^2}{2} + \dots}{\frac{\kappa^2}{2} + \dots} \approx 2\kappa^{-2} \quad \text{当 } \kappa \text{ 很小时}$$

与 λ 无关

无量纲逸出速率主要取决于无量纲参数 κ

2. 与数值解的比较

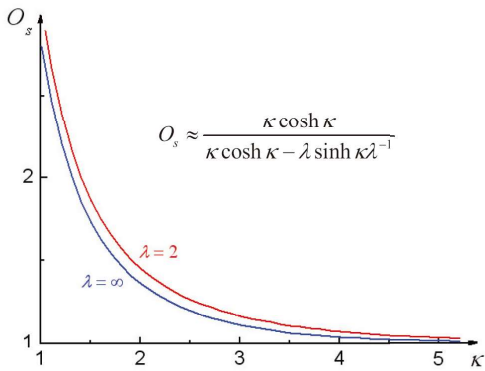


Diamond, Bossert
的数值计算结果

$L = 100 \mu\text{m}$, $\delta = 10 \mu\text{m}$
 $r = 0.05 \mu\text{m}$, $D = 10^{-9} \text{m}^2/\text{s}$
 $C_0 = 0.3 \text{Osmol/L}$

N_0 : (Osmol/m²s)

不同 P (κ)
● 10^{-4} ◻ 10^{-5} ▼ 10^{-6}
+ 10^{-7} ◻ 10^{-8}



$$C-1 = \nu v', \quad 0 < x < \lambda$$

$$\nu C - \eta C' = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases}$$

$$v(0) = 0, \quad C(\lambda) = 1, \quad [C]_{x=\lambda} = 0, \quad [v]_{x=\lambda} = 0$$

数值计算

1. 两点边界条件

$$\eta \nu v'' - \nu^2 v' = \nu + \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < \lambda \end{cases}$$

2. 两个区域非齐次项不同

二阶非线性非齐次DE

方法: 打靶法

Shooting method Bvp4c in Matlab

(假设另一个边界条件, 比较、预估校正)

3. 解释: 无量纲参数的物理意义

量纲分析表明 O_s 取决于 3 个无量纲参数, 而通过近似解发现 O_s 往往仅是 1 个参数 κ 的函数

周长/面积比加倍的效果与渗透率加倍相同

$$\kappa = \left(\frac{cPC_0L^2}{aD} \right)^{1/2}$$

$$F^* = C^* \nu^* - D \frac{dC^*}{dx^*}$$

$$O_s^* = \frac{F^*}{\nu^*} \Big|_L = \frac{F_{conv}^* + F_{diff}^*}{\nu^*} \Big|_L$$

$$a \nu^*(L) O_s^* = a F^*(L) \rightarrow O_s^* = \frac{F^*(L)}{\nu^*(L)}$$

单位时间渗入细管的水体积

$$\frac{1}{2} \kappa^2 = \frac{cPC_0L^2}{2aD} = \frac{[c(\bar{C} - C_0)PL]C_0}{aD(\bar{C} - C_0)/(L/2)} \rightarrow \begin{cases} \text{单位时间出口的对流量} \\ \text{单位时间出口的扩散量} \end{cases}$$

细管中的典型浓度

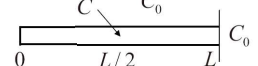
$$\frac{1}{2} \kappa^2 \approx \frac{F_{conv}^*}{F_{diff}^*} \quad O_s \equiv \frac{O_s^*}{C_0} = \frac{F_{conv}^* + F_{diff}^*}{F_{conv}^*} = 1 + \frac{F_{diff}^*}{F_{conv}^*}$$

精确表达式

$$\rightarrow O_s \approx 1 + 2\kappa^{-2}$$

κ 很小时
扩散主导

$$O_s \approx 2\kappa^{-2}$$





$$v = \frac{N_0}{PC_0^2}, \quad \kappa = \left(\frac{cPC_0L^2}{aD} \right)^{1/2}, \quad \lambda = \frac{L}{\delta}$$

➤ 除非 v 很大, O_s 与 v 无关, 即主动输运速率基本上不影响逸出速率

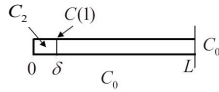
$$\frac{1}{2} \kappa^2 = \frac{cPC_0L^2}{2aD} = \frac{[c(\bar{C} - C_0)PL]C_0}{aD(\bar{C} - C_0)/(L/2)}$$

$N_0 \nearrow, v \nearrow \Rightarrow \bar{C} \nearrow$, 但 κ 不变

$$v = \frac{N_0}{PC_0^2} = \left[\frac{N_0 c \delta}{Pc\delta(C_2 - C_0)C(1)} \right] \left[\frac{C(1)}{C_0} \right] \left[\frac{C_2 - C_0}{C_0} \right]$$

$$1 + \frac{F_{diff}^*}{F_{convect}^*}$$

主动输入溶质量 $F_{diff}^* + F_{conv}^*$
 δ 段水渗入量
 $x=1$ 处的 F_{conv}^*



主动输运相对于渗透的重要性