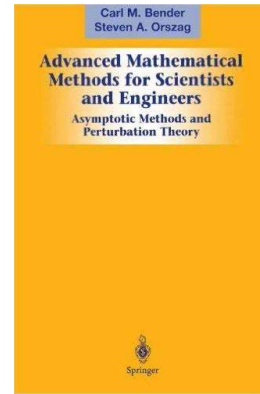


Chapter 7

Regular Perturbation Theory



Ch. 7 正则摄动理论

介绍求解单一尺度问题的基本摄动方法

- 7.1 用级数方法解单摆问题
 - 1. 单摆问题的尺度化
 - 2. 级数方法
- 7.2 用摄动理论解抛体问题
 - 1. 级数方法
 - 2. 参数微分法
 - 3. 逐次逼近法 (迭代法)

摄动理论 (微扰理论):
研究确定摄动的大小和变化规律的理论和方法

The general procedure of perturbation theory:

- identify a small parameter ϵ
- the problem is soluble as $\epsilon = 0$.

$$y(x) = y_0(x) + y_1(x)\epsilon + y_2(x)\epsilon^2 + \dots$$

this series is called **perturbation series**, obtained if

- $y_0(x)$ is soluble as $\epsilon = 0$
- $y_n(x)$ can be obtained using $y_0(x), \dots, y_{n-1}(x)$

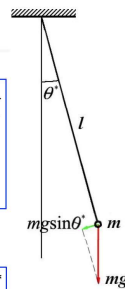
Features:

- a local analysis near $\epsilon = 0$.
- The perturbation series is **local in ϵ** and **global in x** .
- If $\epsilon \ll 1$, $y(x)$ is well approximated by only a few terms.

7.1 应用于单摆问题的级数方法

1. 单摆问题的尺度化

■ 数学模型:
$$\begin{cases} \frac{d^2\theta^*}{dt^{*2}} + \omega_0^2 \sin\theta^* = 0, & t^* > 0, \quad \omega_0 = \sqrt{g/L} \\ \theta^*(0) = a, \quad \frac{d\theta^*}{dt^*}(0) = 0 \end{cases}$$

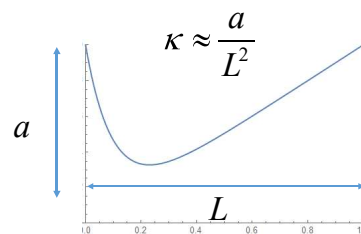


■ 尺度化 (考虑 a 很小)

- 线性化近似: $\sin\theta^* \approx \theta^*$ $\theta^*(t^*) = a \cos \omega_0 t^*$
- 特征尺度: $|\theta^*|_{\max} = a, \quad T = \sqrt{|\theta^*|_{\max} / |d^2\theta^* / dt^{*2}|_{\max}} = 1/\omega_0$
- 尺度化的无量纲变量和方程:

$$t = \omega_0 t^*, \quad \Theta = \frac{\theta^*}{a} \quad \frac{d^2\Theta}{dt^2} + \frac{\sin a\Theta}{a} = 0, \quad \Theta(0) = 1, \quad \frac{d\Theta}{dt}(0) = 0$$

Estimation of the curvature of a function



$$\kappa = \theta''(0) = a\omega_0^2 = \frac{a}{L^2}\omega_0^2 \Rightarrow L = \omega_0^{-1}$$

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0$$



$$\frac{a}{L^2} \sim \omega_0^2 a \Rightarrow L = \omega_0^{-1}$$

$$\frac{d^4\theta}{dt^4} + \omega_0^2 \sin\theta = 0$$



$$\frac{a}{L^4} \sim \omega_0^2 a \Rightarrow L = \omega_0^{-1/2}$$

2. 级数方法

$$\begin{cases} \ddot{\Theta} + a^{-1} \sin a\Theta = 0 \\ \Theta(0) = 1, \dot{\Theta}(0) = 0 \end{cases}$$

1. 假设解可展为小参数 a 的幂级数

$$\Theta = \Theta_0 + \Theta_1 a + \Theta_2 a^2 + \dots = \sum_{i=0}^{\infty} \Theta_i(t) a^i \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

2. 将方程展为 a 的幂级数

$$\begin{aligned} & (\ddot{\Theta}_0 + \ddot{\Theta}_1 a + \ddot{\Theta}_2 a^2 + \dots) + a^{-1} \left[(\Theta_0 a + \Theta_1 a^2 + \Theta_2 a^3 + \dots) - \frac{1}{3!} (\Theta_0 a + \dots)^3 + \dots \right] = 0 \\ & (\ddot{\Theta}_0 + \Theta_0) + (\ddot{\Theta}_1 + \Theta_1) a + \left(\ddot{\Theta}_2 + \Theta_2 - \frac{1}{3!} \Theta_0^3 \right) a^2 + \dots = 0 \end{aligned}$$

3. 令各幂次系数为零, 得到一系列线性微分方程

| | |
|---|---|
| $O(1): \ddot{\Theta}_0 + \Theta_0 = 0$ | $\Theta_0(t) = A_0 \cos t + B_0 \sin t$ |
| $O(a): \ddot{\Theta}_1 + \Theta_1 = 0$ | $\Theta_1(t) = A_1 \cos t + B_1 \sin t$ |
| $O(a^2): \ddot{\Theta}_2 + \Theta_2 = \frac{1}{3!} \Theta_0^3; \dots$ | $\Theta_2(t) = A_2 \cos t + B_2 \sin t$ +特解 |
| | \dots 常数变易法 |

$$\ddot{\Theta}_2 + \Theta_2 = \frac{\Theta_0}{6} = \frac{1}{6} \cos^3 t = \frac{1}{4} (\cos 3t + 3 \cot)$$

$$\dot{\Theta}_2(0) = \Theta_2(0) = 0$$

variation of parameter or
method of undetermined coefficient

$$\ddot{\Theta}_2 + \Theta_2 = \frac{1}{4} \cos 3t \Rightarrow \Theta_{2p} = -\frac{1}{192} \cos 3t$$

$$\ddot{\Theta}_2 + \Theta_2 = \frac{3}{4} \cos t \Rightarrow \Theta_{2p} = \frac{1}{16} t \cos t$$

$$\Theta(t) \approx \cos t + a^2 \left(\frac{1}{192} \cos t - \frac{1}{192} \cos 3t + \frac{1}{16} t \sin t \right)$$

if $t \in [0, \infty)$

$$\frac{a^2}{16} t \sin t \gg \cos t \text{ for larger } t$$

$$\Theta(t) \gg 1$$

Not a valid approximation for large t

Or: not genuine consistent

(续)

$$\begin{cases} \ddot{\Theta} + a^{-1} \sin a\Theta = 0 \\ \Theta(0) = 1, \dot{\Theta}(0) = 0 \end{cases}$$

4. 将初边值条件展为 a 的幂级数

$$\begin{aligned} \Theta(0) = 1 & \Rightarrow \Theta_0(0) = 1, \Theta_1(0) = 0, \Theta_2(0) = 0, \dots \\ \dot{\Theta}(0) = 0 & \Rightarrow \dot{\Theta}_0(0) = 0, \dot{\Theta}_1(0) = 0, \dot{\Theta}_2(0) = 0, \dots \end{aligned}$$

5. 相继求解一系列线性微分方程和初边值条件

$$\Theta_0(t) = \cos t; \quad \Theta_1(t) = 0; \quad \Theta_2(t) = \frac{1}{192} (\cos t - \cos 3t) + \frac{1}{16} t \sin t; \quad \dots$$

$$\begin{aligned} \Theta(t) &= \sum_{i=0}^{\infty} \Theta_i(t) a^i \\ &= \cos t + a^2 \left(\frac{1}{192} \cos t - \frac{1}{192} \cos 3t + \frac{1}{16} t \sin t \right) + O(a^4) \end{aligned}$$

$$\Theta(t) \approx \cos t + a^2 \left(\frac{1}{192} \cos t - \frac{1}{192} \cos 3t + \frac{1}{16} t \sin t \right)$$

$$\text{if } t \in [0, T], \frac{a^2 T}{16} \ll 1$$

$\Theta(t)$ asymptotically and uniformly valid as $a \rightarrow 0$

For the solution of the first order approximation obtained by removing the small parameter

$$\ddot{\Theta} + \frac{\sin a\Theta}{a} = 0 \Rightarrow \ddot{\Theta}_0 + \Theta_0 = 0, \quad \Theta_0 = \cos t$$

apparent and genuine consistency

All odd terms disappear $\Theta_i(t) = 0$ for odd i

$$\ddot{\Theta} + \frac{\sin a\Theta}{a} = 0 \quad \dot{\Theta}(0) = 1, \Theta(0) = 0$$

Let $a \rightarrow -a$, the original ODE remains unchanged

$$\ddot{\Theta} + \frac{\sin a\Theta}{a} = 0 \quad \dot{\Theta}(0) = 1, \Theta(0) = 0$$

This indicates $\Theta(t, a) = \Theta(t, -a)$

i.e., $\Theta(t, a)$ is an even function of a . We can start with

$$\Theta(t) = \sum_{i=0}^{\infty} \Theta_{2i}(t) a^{2i} \quad \text{Physical meaning?}$$

7.2 用摄动理论解抛体问题

1. 级数方法

$$x = \frac{x^*}{V^2 g^{-1}}, \quad t = \frac{t^*}{V g^{-1}}, \quad \varepsilon = \frac{V^2}{gR}$$

$$\ddot{x} = -(1 + \varepsilon x)^{-2}; \quad x(0) = 0, \quad \dot{x}(0) = 1, \quad 0 < \varepsilon \ll 1$$

■ 假设级数解: $x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$

$$(\ddot{x}_0 + 1) + (\ddot{x}_1 + 2x_0 \ddot{x}_0) \varepsilon + (\ddot{x}_2 + 2x_0 \ddot{x}_1 + 2x_1 \ddot{x}_0 + x_0^2 \ddot{x}_0) \varepsilon^2 + \dots = 0$$

■ 各阶控制方程:

$$O(1): \quad \ddot{x}_0 = -1, \quad x_0(0) = 0, \quad \dot{x}_0(0) = 1$$

$$O(\varepsilon): \quad \ddot{x}_1 = -2x_0 \ddot{x}_0, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0$$

$$O(\varepsilon^2): \quad \ddot{x}_2 = -2x_0 \ddot{x}_1 - 2x_1 \ddot{x}_0 - x_0^2 \ddot{x}_0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0$$

...

2. 参数微分法

■ 思想:

• 幂级数展开式 $x(t, \varepsilon) = \sum_{n=0}^{\infty} x_n(t) \varepsilon^n$ 对 ε 的导数:

$$\frac{\partial x(t, \varepsilon)}{\partial \varepsilon} = \sum_{n=1}^{\infty} x_n(t) \cdot n \varepsilon^{n-1}$$

$$\frac{\partial^2 x(t, \varepsilon)}{\partial \varepsilon^2} = \sum_{n=2}^{\infty} x_n(t) n(n-1) \varepsilon^{n-2}$$

$$x^{(m)}(t, \varepsilon) \equiv \frac{\partial^m x(t, \varepsilon)}{\partial \varepsilon^m} = \sum_{n=m}^{\infty} x_n(t) \frac{n!}{(n-m)!} \varepsilon^{n-m}, \quad m = 0, 1, 2, \dots$$

• 若 $\varepsilon = 0$, 有:

$$x^{(m)}(t, 0) = x_n(t) \frac{n!}{(n-m)!} \delta_{mn} = x_m(t) \cdot m!$$

• 待定系数为:

$$x_n(t) = \frac{x^{(n)}(t, 0)}{n!} = \frac{1}{n!} \left[\frac{\partial^n x(t, \varepsilon)}{\partial \varepsilon^n} \right]_{\varepsilon=0}$$

$$\text{Let } y^{(n)}(t) = \left. \frac{\partial^n x(t, \varepsilon)}{\partial \varepsilon^n} \right|_{\varepsilon=0}$$

(续)

$$x(t, \varepsilon) = \sum_{n=0}^{\infty} x_n(t) \varepsilon^n = \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(t) \varepsilon^n$$

■ 将初边值条件对小参数求导, 再令小参数为零

$$x(0) = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} y^{(n)}(0) \varepsilon^n = 0 \Rightarrow y^{(0)}(0) = 0, \quad y^{(1)}(0) = \dots = 0$$

$$\dot{x}(0) = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} \dot{y}^{(n)}(0) \varepsilon^n = 1 \Rightarrow \dot{y}^{(0)}(0) = 1, \quad \dot{y}^{(1)}(0) = \dots = 0$$

$$\ddot{y}^{(0)} = -1, \quad \ddot{y}^{(1)} = 2y^{(0)}, \quad \ddot{y}^{(2)} = 4y^{(1)} - 6(y^{(0)})^2, \dots$$

$$y^{(0)} = \frac{1}{2}t(2-t), \quad y^{(1)} = \frac{1}{12}t^3(4-t), \quad y^{(2)} = -\frac{1}{180}t^4(90-66t+11t^2)$$

$$x = \frac{1}{2}t(2-t) + \frac{1}{12}t^3(4-t)\varepsilon - \frac{1}{360}t^4(90-66t+11t^2)\varepsilon^2 + O(\varepsilon^3)$$

本质上与级数方法相同, 可简化计算

(续)

$$\begin{cases} x_0 = t - \frac{1}{2}t^2 & \rightarrow \dot{x}_1 = 2\left(t - \frac{1}{2}t^2\right) \\ x_1 = \frac{t^3}{3} - \frac{t^4}{12} & \rightarrow \dot{x}_2 = -3t^2 + \frac{11}{3}t^3 - \frac{11}{12}t^4 \\ x_2 = -\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6 \end{cases}$$

■ 各阶系数:

■ 近似解:

$$x = t - \frac{1}{2}t^2 + \left(\frac{t^3}{3} - \frac{t^4}{12}\right)\varepsilon + \left(-\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6\right)\varepsilon^2 + O(\varepsilon^3)$$

■ 到达最高点的时间和高度:

$$\dot{x}(t_m) = 0$$

$$1 - t_m + \left(t_m^2 - \frac{1}{3}t_m^3\right)\varepsilon + \left(-t_m^3 + \frac{11}{12}t_m^4 - \frac{11}{60}t_m^5\right)\varepsilon^2 + O(\varepsilon^3) = 0$$

五次方程

$$t_m = 1 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3)$$

$$t_m = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + O(\varepsilon^3)$$

$$x_m = \frac{1}{2} + \frac{1}{4}\varepsilon + \frac{1}{8}\varepsilon^2 + O(\varepsilon^3)$$

例: 抛体问题

$$\ddot{x} = -(1 + \varepsilon x)^{-2}; \quad x(0) = 0, \quad \dot{x}(0) = 1, \quad 0 < \varepsilon \ll 1$$

■ Step 1: 将方程对小参数求导

$$\frac{\partial \ddot{x}}{\partial \varepsilon} = 2(1 + \varepsilon x)^{-3} \left(\varepsilon \frac{\partial x}{\partial \varepsilon} + x \right)$$

$$x = x(t, \varepsilon)$$

$$x^{(0)}(t, \varepsilon) \equiv x(t, \varepsilon)$$

$$\ddot{x}^{(1)} = 2(1 + \varepsilon x^{(0)})^{-3} (\varepsilon x^{(1)} + x^{(0)})$$

$$x^{(m)}(t, \varepsilon) \equiv \frac{\partial^m x(t, \varepsilon)}{\partial \varepsilon^m}$$

$$\ddot{x}^{(2)} = 2(1 + \varepsilon x^{(0)})^{-3} (\varepsilon x^{(2)} + 2x^{(1)}) - 6(1 + \varepsilon x^{(0)})^{-4} (\varepsilon x^{(1)} + x^{(0)})^2$$

■ Step 2: 在各方程中令小参数为零

• 令 $y^{(n)}(t) \equiv x^{(n)}(t, 0)$

$$\dot{x}^{(0)} = -(1 + \varepsilon x^{(0)})^{-2}$$

$$\ddot{y}^{(0)} = -1, \quad \ddot{y}^{(1)} = 2y^{(0)}, \quad \ddot{y}^{(2)} = 4y^{(1)} - 6(y^{(0)})^2, \dots$$

3. 逐次逼近法 (迭代法)

$$F(z) = G(z)$$

■ 数值迭代:

$$z_k = 2 + 0.01(z_{k-1})^3 \quad | \quad z - 2 = 0.01z^3$$

$$z_0 = 2$$

$$z_0 = 10$$

$$z_1 = 2.08$$

$$z_1 = 12$$

$$F(z_n) = G(z_{n-1})$$

$$z_2 = 2.08999$$

$$z_2 = 19.8$$

$$z_3 = 2.09129$$

$$z_3 = 73.6673$$

$$z_4 = 2.09146$$

$$z_4 = 3999.83$$

■ 带小参数的迭代:

$$z_k = 2 + \varepsilon(z_{k-1})^3$$

$$z - 2 = \varepsilon z^3$$

$$z_0 = 2$$

$$z_1 = 2 + 8\varepsilon$$

$$z_2 = 2 + 8\varepsilon + 96\varepsilon^2 + 384\varepsilon^3 + 512\varepsilon^4$$

格式? 精度?

$$z_3 = 2 + 8\varepsilon + 96\varepsilon^2 + 1536\varepsilon^3 + 14336\varepsilon^4 + \dots + 134217728\varepsilon^{13}$$

用迭代法求含一个未知函数的常微分方程

Step 1: 将微分方程写成

$$f(x, \dot{x}, \ddot{x}, \dots, t) = 0$$

$$F(x, \dot{x}, \ddot{x}, \dots, t) = G(x, \dot{x}, \ddot{x}, \dots, t) \quad G: \text{高阶量}$$

且 $F=0$ 提供所求解的零级近似

Step 2: 逐次解方程 (以 z_n 表示第 n 次近似)

$$F(z_0, \dot{z}_0, \ddot{z}_0, \dots, t) = 0$$

$$F(z_n, \dot{z}_n, \ddot{z}_n, \dots, t) = G(z_{n-1}, \dot{z}_{n-1}, \ddot{z}_{n-1}, \dots, t), \quad n=1, 2, \dots$$

需满足初边值条件, 但有时可把条件放松

example:

$$-\varepsilon x^2 + x - 1 = 0, \quad 0 < x \ll 1$$

$$\downarrow$$

$$x = 1 + \varepsilon x^2$$

$$\downarrow$$

$$x_n = 1 + \varepsilon x_{n-1}^2 \quad x_0 = 1$$

$$\downarrow$$

$$x_1 = 1 + \varepsilon$$

$$x_2 = 1 + \varepsilon(1 + \varepsilon)^2 = 1 + \varepsilon + 2\varepsilon^2 + \varepsilon^3$$

$$x_3 = 1 + \varepsilon + 2\varepsilon^2 + 4\varepsilon^3 + \dots$$

例: 抛体问题

$$\ddot{x} = -(1 + \varepsilon x)^{-2}; \quad x(0) = 0, \dot{x}(0) = 1, \quad 0 < \varepsilon \ll 1$$

方程改写: $\ddot{x} + 1 = 1 - (1 + \varepsilon x)^{-2} = \frac{\varepsilon x(2 + \varepsilon x)}{(1 + \varepsilon x)^2} \quad F(\dots) = G(\dots)$

零级近似: $\ddot{z}_0 + 1 = 0, \quad z_0(0) = 0, \dot{z}_0(0) = 1 \quad z_0 = t - \frac{1}{2}t^2$

迭代式: $\ddot{z}_n + 1 = \frac{\varepsilon z_{n-1}(2 + \varepsilon z_{n-1})}{(1 + \varepsilon z_{n-1})^2}, \quad z_n(0) = 0, \dot{z}_n(0) = 1$

若迭代式的右边可以展成多项式, 那么只要将保留项精确到 $O(\varepsilon^n)$ 即可, 以省去关于高阶项的繁杂、无用的计算

(续) $\ddot{z}_n + 1 = \frac{\varepsilon z_{n-1}(2 + \varepsilon z_{n-1})}{(1 + \varepsilon z_{n-1})^2}, \quad z_n(0) = 0, \dot{z}_n(0) = 1$

一级近似: $\ddot{z}_1 + 1 = \varepsilon z_0(2 + \varepsilon z_0) \sum_{m=1}^{\infty} (-1)^{m-1} m (\varepsilon z_0)^{m-1}$
 $= 2\varepsilon z_0 + \sum_{k=1}^{\infty} (-1)^k (k+2) (\varepsilon z_0)^{k+1}$
 $= 2\varepsilon z_0 + O(\varepsilon^2) \quad z_1(0) = 0, \dot{z}_1(0) = 1$
 $\approx 2\varepsilon \left(t - \frac{1}{2}t^2 \right) \quad z_1 = t - \frac{1}{2}t^2 + \varepsilon \left(\frac{1}{3}t^3 - \frac{1}{12}t^4 \right)$

二级近似: $\ddot{z}_2 + 1 = 2\varepsilon z_1 + \sum_{k=1}^{\infty} (-1)^k (k+2) (\varepsilon z_1)^{k+1}, \quad z_1 = x_0 + x_1\varepsilon$
 $= 2x_0\varepsilon + (2x_1 - 3x_0^2)\varepsilon^2 + O(\varepsilon^3) \approx \dots$

优点: 不需要辨认计算过程的小量, 错误仅推迟得到正确解
 缺点: 计算较繁

评述

- 需要一个有意义的小参数; 需指出精确到的量级
- 形式解: $x = x_0 + x_1\delta_1 + x_2\delta_2 + \dots, \quad 0 < \delta_i(\varepsilon) \ll \delta_{i-1}, \quad i \in \mathbb{N}$
- 当 $\varepsilon = 0$ 时如问题无解、解在整个区域非一致有效、有很多解时, 要用奇异摄动法 (见第九章)
- 摄动法适用于代数/差分/偏微分/积分方程(组)
- 计算过程中运用了各种极限运算交换次序但没有严格证明, 解的存在性依赖于参数
- 当要求的项数增加时, 计算复杂性迅速增加, 但通常计算一项或两项就足够了 (例子见第八章)