PART B

Some Fundamental Procedures Illustrated on Ordinary Differential Equation

Chapter 6

Simplification, Dimensional Analysis, and Scaling

Section 6.1

The Basic Simplification Procedure

Basic simplification procedure:

1. *Identify relatively small terms*

   the smallness is gauged relative to other terms.

2. *Delete small terms and solve the simplified problem.*

3. *Check for consistency.*

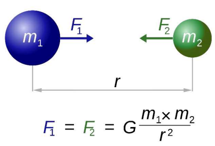
   use the simplified solution to check the deleted are small.

$\sim$:	seems to be approximately equal to
$\approx$:	is approximately equal to

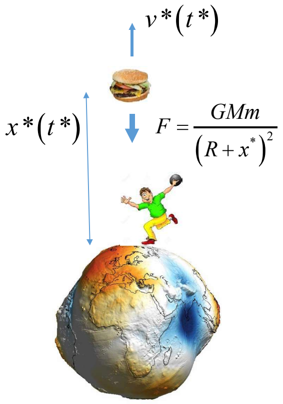
Example 1. The projectile problem.

$$x^* = x^*\left(t^*; V, R, g\right)$$

x^*: distance from earth surface
t^*: time
V: initial velocity
R: radius of earth (6378.1 KM)
g: gravity acceleration
M,m: mass of earth and ball



Newton's law of universal gravitation



The acceleration on the earth surface is g

$$mg = G\frac{M\,m}{R^2} \Rightarrow gR^2 = GM \Rightarrow F = mg\frac{R^2}{\left(R+x^*\right)^2}$$

Therefore, we have the IBV problem

$$\frac{d^2x^*}{dt^{*2}} = -\frac{gR^2}{\left(R+x^*\right)^2} \qquad \frac{dx^*(0)}{dt^*} = V \qquad x^*(0) = 0$$

Assuming $x^* \ll R$ The Simplified IBV problem now reads

$$\frac{d^2x^*}{dt^{*2}} = -g \qquad \frac{dx^*(0)}{dt^*} = V \qquad x^*(0) = 0$$

Solution $x^* = -\frac{1}{2}gt^{*2} + Vt^*$ $\quad \frac{dx^*}{dt^*} = -gt^* + V$

at $t^* = V/g$, the maximum $x^*_{\max} = V^2/2g \ll R$ Genuine consistency

6.1.1 The Basic Simplification Procedure

Example 2. $x+10y=21, \quad 5x+y=7$

solution $x=1, \ y=2$

1. Identify small term $\quad 1x+10y=21, \quad 5x+1y=7$

$1<10$

2. Delete small terms $\quad 1x+10y=21, \quad 5x+1y=7$

Simpilified solution $\quad y \sim 2.1, \quad x \sim 0.98$

3. Check for consistency $\quad 1x+10y=21$

$1\times 0.98+10\times 2.1=21.98$

$0.98 << 21$

genuine consistency

6.1.2 Two chastening examples

Example 3. $0.01x+y=0.1, \quad x+101y=11$

solution $x=-90, \quad y=1$

Simplification $\quad 0.01x+y=0.1, \quad x+101y=11$

Approximated solution $\quad y\sim 0.1, \quad x\sim 11-101*0.1=0.9$

Consistency check $\quad 0.01x+y=0.1, \quad x+101y=11$

$0.01\times 0.9+0.1=0.109 \quad 0.9+101\times 0.1=11$

$0.009 << 0.1$

Almost Perfect Consistency !

But it is wrong! Why?

6.1.2 Two chastening examples

Introducing a small parameter ε, and rewrite as

$$\varepsilon x+y=0.1, \quad x+101y=11$$

the exact solution

$$x(\varepsilon)=\frac{0.9}{1-101\varepsilon} \quad y(\varepsilon)=\frac{0.1-11\varepsilon}{1-101\varepsilon}, \quad \varepsilon\neq 1/101$$

The previous simplified solutions actually are

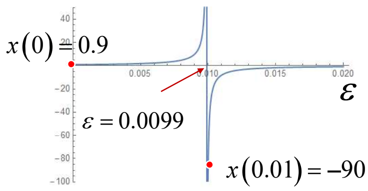
$$x(0)=0.9 \quad y(0)=0.1 \quad \text{as} \quad \varepsilon=0$$

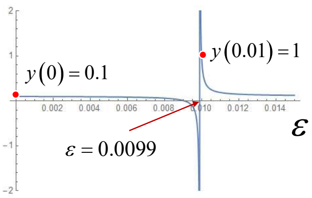
Our problem is whether

$$x(\varepsilon)\approx x(0) \quad y(\varepsilon)\approx y(0) \quad \text{as} \quad \varepsilon=0.01$$

6.1.2 Two chastening examples

$$x(\varepsilon)=\frac{0.9}{1-101\varepsilon} \qquad y(\varepsilon)=\frac{0.1-11\varepsilon}{1-101\varepsilon}$$





$$x(0.01)=-90\neq x(0)=0.9 \qquad y(0.01)=1\neq y(0)=0.1$$

In simplified solution, we see apparent consistency

$$\frac{|0.01x(0)|}{|y(0)|}=0.09 \quad \text{as} \quad \varepsilon=0$$

But in fact, it is not genuine consistency

$$\frac{|0.01x(0.01)|}{|y(0.01)|}=0.9 \quad \text{as} \quad \varepsilon=0.01$$

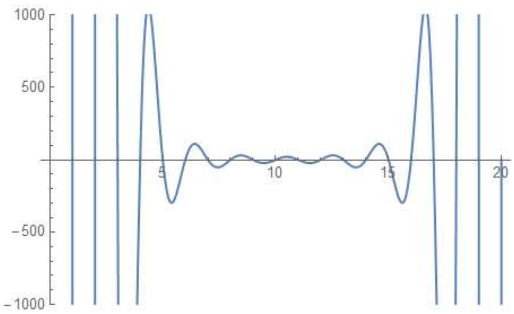
$0.01x+y=0.1, \quad x+101y=11$

6.1.2 Two chastening examples

Example 4.

$$f(x)=(x-1)(x-2)(x-3)\cdots(x-20)=x^{20}-210x^{19}+\cdots$$

We have zeros at the 20 positive integers



6.1.2 Two chastening examples

But if there is a tiny numerical error like

$$f(x)\approx x^{20}-(210+\varepsilon)x^{19}+\cdots \qquad \text{with } \varepsilon=-2^{-23}$$

- the first 10 zeros are almost unaltered.
- the rest 10 zeros are radically altered to five pairs of complex conjugate roots

$$10\pm i0.64, \ 12\pm i1.7, \ 14\pm i2.5, \ 17\pm i2.8, \ 20\pm i19$$

- apparent and genuine consistency verified

$$\varepsilon x(\varepsilon)^{19} \ll 210x(\varepsilon)^{19} \ or \ \varepsilon x(0)^{19} \ll 210x(0)^{19} \quad \text{as } \varepsilon=-2^{-23}$$

- but some roots are still wrong

6.1.3 Conditioning and sensitivity

ill conditioned problem (病态的):
- solution is sensitive to a neglected term T.

The only secure statement:
- No apparent consistency means poor approximation.

To see this, let
- x the true solution
- $\tilde{x}$ the approximate solution with term T neglected.
- $T(\tilde{x})$ small, apparently consistent
- $T(x)$ small, genuinely consistent

If the solution is *not sensitive* to the neglect of *T*,

$$\text{small } T(x) \;\Rightarrow\; \tilde{x} = x \;\Rightarrow\; \text{small } T(\tilde{x})$$

6.1.3 Conditioning and sensitivity

What is large or what is small ?

Consider Temperature: $0^{o}\text{C} = 273.15\text{K}$ (absolute temp.)

$T_{true} = 1^{o}\text{C}$. $T_{approx} = 1.1^{o}\text{C}$. error: 10%

$= 274.15\text{K}$. $= 274.25\text{K}$. error: 0.036%

To decide larger or small,
- scientific grounds
- must be independent of the units employed
- Where is the zero point

6.1.4 Zeros of a function

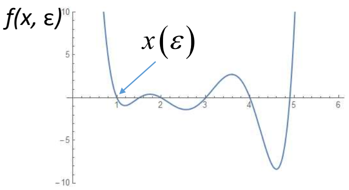
Let x(ε) a zero of *f(x, ε)*, i.e.

$$f\left[x(\varepsilon), \varepsilon\right] = 0$$

with ε a small parameter.

Let ε=0, and $\tilde{x} = x(0)$ is a zero of the simplified equation

$$f\left[\tilde{x}, 0\right] = 0$$



6.1.4 Zeros of a function

In the previous example:

$$f(x,\varepsilon) = x^{20} - (210+\varepsilon)x^{19} + \cdots = 0 \qquad f[x,\varepsilon] = 0$$

$$f(\tilde{x},0) = \tilde{x}^{20} - (210+0)\tilde{x}^{19} + \cdots = 0 \qquad f[\tilde{x},0] = 0$$

$$f(\tilde{x},\varepsilon) = \tilde{x}^{20} - (210+\varepsilon)\tilde{x}^{19} + \cdots = -\varepsilon\tilde{x}^{19} = r \qquad f[\tilde{x},\varepsilon] = r$$

$$f(x,0) = x^{20} - (210+0)x^{19} + \cdots = \varepsilon x^{19} = -g \qquad f[x,0] = -g$$

$$g = r$$

6.1.4 Zeros of a function

Define equation error $= f(x,\varepsilon) - f(x,0)$

Genuine equation error

$$g = f\left[x(\varepsilon),\varepsilon\right] - f\left[x(\varepsilon),0\right]$$

$$= -f\left[x(0),0\right] - \left.\frac{\partial f}{\partial x}\right|_{x(0)}\left[x(\varepsilon) - x(0)\right] + O(\varepsilon^2)$$

$$= -\varepsilon f_x^{(0)} x_\varepsilon(0) + O(\varepsilon^2)$$

Residual equation error

$$r = f\left[x(0),\varepsilon\right] - f\left[x(0),0\right] \;= f\left[x(0),0\right] + \varepsilon\left.\frac{\partial f}{\partial \varepsilon}\right|_{x(0)} + O(\varepsilon^2)$$

$$= \varepsilon f_\varepsilon^{(0)} + O(\varepsilon^2)$$

6.1.4 Zeros of a function

We also have full Taylor expansion

$$f\left[x(\varepsilon),\varepsilon\right] = f\left[x(0),0\right] + \varepsilon\left\{f_x^{(0)} x_\varepsilon(0) + f_\varepsilon^{(0)}\right\} + O(\varepsilon^2)$$

= 0 = 0

To the first order $f_x^{(0)} x_\varepsilon(0) = -f_\varepsilon^{(0)}$ $\qquad g = -\varepsilon f_x^{(0)} x_\varepsilon(0)$

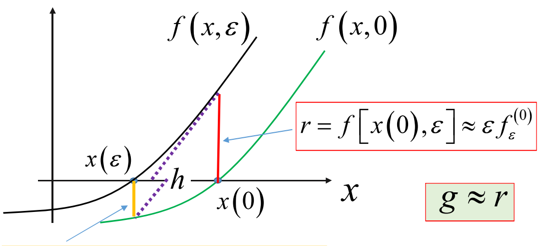
or $g = r$ $\qquad r = \varepsilon f_\varepsilon^{(0)}$

Therefore the error of the solution :

$$h = x(\varepsilon) - x(0) = \varepsilon x_\varepsilon(0) = -\varepsilon\frac{f_\varepsilon^{(0)}}{f_x^{(0)}} = -\frac{r}{f_x^{(0)}} = -\frac{g}{f_x^{(0)}}$$

the error *h* depends on :
- the residual *r* or the genuine *g*
- the condition $f_x^{(0)}$

6.1.4 Zeros of a function



$g = -f[x(\varepsilon),0] \approx -\varepsilon f_x^{(0)} x_\varepsilon(0)$

$$h = x(\varepsilon) - x(0) = \varepsilon x_\varepsilon(0) = -\frac{r}{f_x^{(0)}} \Rightarrow \left(\frac{h}{x}\right) = -\frac{1}{f_x^{(0)}}\left(\frac{r}{x}\right)$$

ill conditioned

if $\left|f_x^{(0)}\right| \ll 1$ ➡ Small residual r , big error in solution h

6.1.4 Zeros of a function

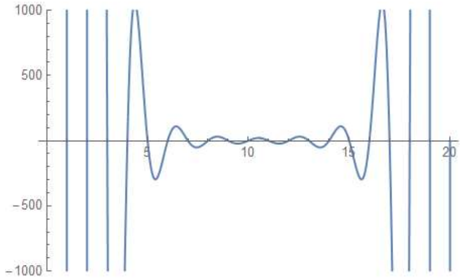
In example 4:

$$f(x) = x^{20} - 210x^{19} + \cdots$$

$$f(x) \approx x^{20} - (210 + \varepsilon)x^{19} + \cdots \qquad \text{with } \varepsilon = -2^{-23}$$

$$\frac{h}{x} = -\frac{1}{f_x^{(0)}}\frac{r}{\tilde{x}} = \frac{1}{f_x^{(0)}}\frac{\varepsilon \tilde{x}^{19}}{\tilde{x}}$$

For later zeros, h/x increases because: $\tilde{x}$>1, $f_x^{(0)}$varies



6.1.4 Zeros of a function

Newton's method (also Newton–Raphson method)

To find the root of $f(x) = 0$

If x_n is an approximate root, we have Taylor expansion

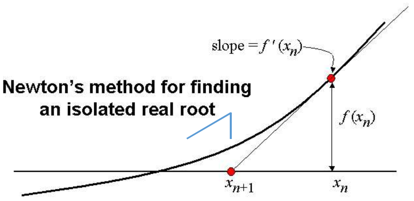
$$f(x) = f(x_n) + f_x(x_n)(x - x_n) + O(x - x_n)^2 \to 0$$

⬇

$$x \approx x_{n+1} = x_n - \frac{f(x_n)}{f_x(x_n)}$$

$f(x_n)$ is the residual

$$h = x(\varepsilon) - x(0) = -\frac{r}{f_x^{(0)}}$$



6.1.5 Second order differential equations

Consider an initial value problem x(t, ε) ,

$$f(t, x, \dot{x}, \ddot{x}, \varepsilon) = 0; \quad x(0) = A, \quad \dot{x}(0) = B, \quad |\varepsilon| \ll 1, \quad \cdot = \frac{d}{dt}$$

with ε a small parameter.

An approximate zero

$$f(t, x(0), \dot{x}(0), \ddot{x}(0), 0) = 0$$

Genuine equation error

$$g \equiv -f(t, x, \dot{x}, \ddot{x}, 0) = -\varepsilon[f_2 x_\varepsilon + f_3 \dot{x}_\varepsilon + f_4 \ddot{x}_\varepsilon]_{\varepsilon=0} + O(\varepsilon^2)$$

Residual equation error

$$r \equiv f(t, x(0), \dot{x}(0), \ddot{x}(0), \varepsilon) = \varepsilon[f_5]_{\varepsilon=0} + O(\varepsilon^2)$$

6.1.5 Second order differential equations

We see

$$f(t, x, \dot{x}, \ddot{x}, \varepsilon) = f(t, x(0), \dot{x}(0), \ddot{x}(0), \varepsilon) + \varepsilon[f_2 x_\varepsilon + f_3 \dot{x}_\varepsilon + f_4 \ddot{x}_\varepsilon + f_5]_{\varepsilon=0} + O(\varepsilon^2)$$

Thus to the first order

$$f_2^{(0)} x_\varepsilon^{(0)} + f_3^{(0)} \dot{x}_\varepsilon^{(0)} + f_4^{(0)} \ddot{x}_\varepsilon^{(0)} = -f_5^{(0)} \qquad \text{or} \qquad g = r$$

Define

$$\mathbf{F}(t) = \left(f_2^{(0)}, f_3^{(0)}, f_4^{(0)}\right)$$

$$\mathbf{h}(t) = (x - x(0), \dot{x} - \dot{x}(0), \ddot{x} - \ddot{x}(0)) = \varepsilon\left(x_\varepsilon^{(0)}, \dot{x}_\varepsilon^{(0)}, \ddot{x}_\varepsilon^{(0)}\right) + O(\varepsilon^2)$$

$$\mathbf{F} \cdot \mathbf{h} = -r$$

So, $|r| = |\mathbf{F} \cdot \mathbf{h}| = |\mathbf{F}||\mathbf{h}||\cos(\mathbf{F}, \mathbf{h})| \le |\mathbf{F}||\mathbf{h}|$ ➡ $|\mathbf{h}| \ge \frac{|r|}{|\mathbf{F}|}$

6.1.5 Second order differential equations

Example: non-dimensionalized projectile problem

$$\ddot{x} + (1 + \varepsilon x)^{-2} = 0; \quad x(0) = 0, \quad \dot{x}(0) = 1$$

the parameter ε = V^2/Rg is small.

In this case

$$f(t, x, 0, \ddot{x}, \varepsilon) = \ddot{x} + (1 + \varepsilon x)^{-2} = 0$$

$$\mathbf{F}(t) = \left(f_2^{(0)}, f_3^{(0)}, f_4^{(0)}\right) = (0, 0, 1)$$

$$\mathbf{h}(t) = \varepsilon\left(x_\varepsilon^{(0)}, \dot{x}_\varepsilon^{(0)}, \ddot{x}_\varepsilon^{(0)}\right)$$

$$r \approx \varepsilon[f_5]_{\varepsilon=0} \approx -2\varepsilon x^{(0)}(t)$$

6.1.5 Second order differential equations

$$\mathbf{F}\cdot\mathbf{h} = -r$$

⬇

$$\varepsilon \ddot{x}_\varepsilon^{(0)} = 2\varepsilon x^{(0)}(t) + O(\varepsilon^2)$$

⬇

$$\left|\varepsilon \ddot{x}_\varepsilon^{(0)}\right| \le 2\varepsilon \max\left|x^{(0)}(t)\right| + O(\varepsilon^2)$$

$$\boxed{\ddot{x} - \ddot{x}^{(0)} = \varepsilon \ddot{x}_\varepsilon^{(0)} + O(\varepsilon^2) \le \varepsilon}$$

因此，$\ddot{x}$ 的误差很小 $\xrightarrow{\text{积分}}$ $\dot{x}, x$ 的误差很小

6.1.5 Second order differential equations

Example: nonlinear pendulum

$$\ddot{\theta} + \varepsilon^{-1/2}\sin\left(\varepsilon^{1/2}\theta\right) = 0;$$

$$\theta(0) = 1, \quad \dot{\theta}(0) = 0, \quad 0 < \varepsilon \ll 1$$

With *Θ(t)* angular displacement. ε is the initial displacement.

Zeroth approximation:

$$\ddot{\theta} + \varepsilon^{-1/2}\sin\left(\varepsilon^{1/2}\theta\right) = 0 \xrightarrow{\varepsilon\to 0} \ddot{\theta}^{(0)} + \theta^{(0)} = 0$$

⬇

$$\theta^{(0)} = \cos(t)$$

6.1.5 Second order differential equations

$$f\left(t,\theta,0,\ddot{\theta},\varepsilon\right) = \ddot{\theta} + \varepsilon^{-1/2}\sin\left(\varepsilon^{1/2}\theta\right) = 0$$

$$\mathbf{F}(t) = \left(f_2^{(0)}, f_3^{(0)}, f_4^{(0)}\right) = (1,0,1)$$

$$\mathbf{h}(t) = \varepsilon\left(\theta_\varepsilon^{(0)}, \dot{\theta}_\varepsilon^{(0)}, \ddot{\theta}_\varepsilon^{(0)}\right)$$

$$r = \varepsilon f_5^{(0)} = \varepsilon \lim_{\varepsilon\to 0}\frac{-\sin\varepsilon^{1/2}\theta^{(0)} + \varepsilon^{1/2}\theta^{(0)}\cos\varepsilon^{1/2}\theta^{(0)}}{2\varepsilon^{3/2}} = \frac{\varepsilon}{6}\left[\theta^{(0)}\right]^3$$

$$\mathbf{F}\cdot\mathbf{h} = -r \Rightarrow \varepsilon\left[\ddot{\theta}_\varepsilon^{(0)} + \theta_\varepsilon^{(0)}\right] = -\frac{\varepsilon}{6}\left[\theta^{(0)}\right]^3$$

We obtained DE on $\theta_\varepsilon^{(0)}$

$$\ddot{\theta}_\varepsilon^{(0)} + \theta_\varepsilon^{(0)} = -\frac{1}{6}\cos^3 t \qquad \theta_\varepsilon^{(0)}(0) = 0, \quad \dot{\theta}_\varepsilon^{(0)}(0) = 0$$

6.1.5 Second order differential equations

Forcing term

$$\begin{cases} \ddot{\theta}_\varepsilon^{(0)} + \theta_\varepsilon^{(0)} = \frac{1}{6}\cos^3 t \\ \theta_\varepsilon^{(0)}(0) = 0, \quad \dot{\theta}_\varepsilon^{(0)}(0) = 0 \end{cases}$$

$$\theta_\varepsilon^{(0)} = A(t)\cos t + B(t)\sin t$$

$$\begin{cases} A(t) = -\frac{1}{6}\int_0^t \sin\xi\cdot\cos^3\xi\, d\xi \\ B(t) = \frac{1}{6}\int_0^t \cos\xi\cdot\cos^3\xi\, d\xi \end{cases}$$

$$\longrightarrow \theta_\varepsilon^{(0)}(t) = \frac{1}{6}\int_0^t (\sin t\cos\xi - \cos t\sin\xi)\cos^3\xi\, d\xi$$

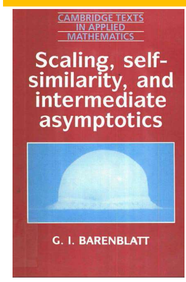
$$\longrightarrow \varepsilon\theta_\varepsilon^{(0)}(t) = -\int_0^t \sin(t-\xi)r(\xi)d\xi \qquad \because |r| \le \frac{\varepsilon}{6}$$

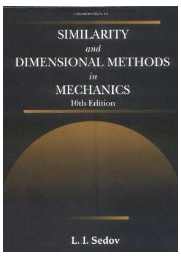
$$\longrightarrow \left|\varepsilon\theta_\varepsilon^{(0)}(t)\right| \le \frac{\varepsilon T}{6}, \quad (0 \le t \le T)$$

Section 6.2

Dimension Analysis

6.2.0 Dimensional analysis





Many of those who have taught dimensional analysis have realized that it has suffered an unfortunate fate.

The idea of dimensional analysis is based is very simple: physical laws do not depend on arbitrarily chosen basic units of measurement.

An important conclusion: the functions that express physical laws must possess a certain fundamental property, which in mathematics is called symmetry.

Dimensional analysis, researchers have been able to obtain remarkably deep results that have sometimes changed entire branches of science.

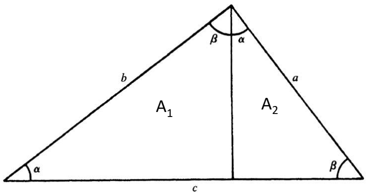
Proof of Pythagorean Theorem

- Physics: angle α and side c determine a right triangle.
- a, b, β are functions of α, c.
- Dimension

  $[\alpha]=[\beta]=1 \quad [a]=[b]=[c]=L \quad [A]=L^2$

with dimensional reasoning,

$$a(\alpha,c)=c\,f_a(\alpha)$$
$$b(\alpha,c)=c\,f_b(\alpha)$$
$$\beta(\alpha,c)=f_\beta(\alpha)$$
$$A(\alpha,c)=c^2\,f(\alpha)$$



Pythagorean Theorem

Partition the right triangle into two right triangles

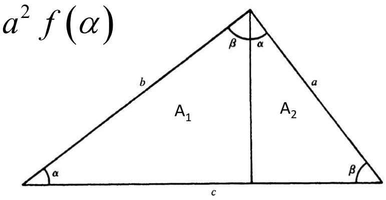
$$A(\alpha,c)=A_1(\alpha,b)+A_2(\alpha,a)$$

⇓

$$c^2 f(\alpha)=b^2 f(\alpha)+a^2 f(\alpha)$$

⇓

$c^2 = a^2 + b^2$



Pythagorean Theorem

Dimensional Analysis refers to the physical nature of the quantity (**Dimension**) and the type of unit used to specify it.

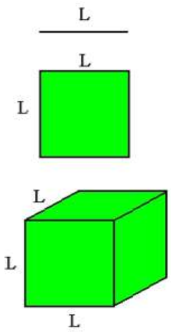
Distance has dimension L.

Area has dimension L^2.

Volume has dimension L^3.

Time has dimension T.

Speed has dimension L/T



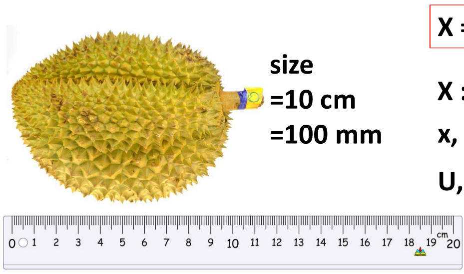
The international system of units (SI)

Fundamental Dimension	Base Unit
length [*L*]	meter (m)
mass [*M*]	kilogram (kg)
time [*T*]	second (s)
electric current [*A*]	ampere (A)
absolute temperature [θ]	kelvin (K)
luminous intensity [*I*]	candela (cd)
amount of substance [*n*]	mole (mol)

Secondary dimension

Velocity	$[v] = L\,T^{-1}$
Acceleration	$[a] = L\,T^{-2}$
Force	$[F] = M\,L\,T^{-2}$
Pressure	$[P] = M\,L^{-1}\,T^{-2}$
Mass Density	$[\rho] = M\,L^{-3}$
………	
………	

[Physical Quantity] = $M^a L^b T^c$



size
=10 cm
=100 mm

X = x U = x' U'

X : physical quantity
x, x' : measurement
U, U' : unit

- The measured value of a quantity depends on the unit of measurement.
- Such dependency is called dimensional.
- Otherwise, called dimensionless。

The projectile problem:

$$\frac{d^2x^*}{dt^{*2}} = -\frac{gR^2}{\left(R+x^*\right)^2} \qquad \frac{dx^*(0)}{dt^*} = V \qquad x^*(0) = 0$$

Solution $x^* = x^*\left(t^*; g, R, V\right)$

x^*: distance from earth surface
t^*: time
V: initial velocity
R: radius of earth (6378.1 KM)
g: gravity acceleration

A two-step procedure for nondimensionalization

Step A. *List all parameters and variables, together with their dimensions*

Variables	Dimension
Dependent variable x^*	$\mathscr{L}$
Independent variable t^*	$\mathscr{T}$
Parameters	
Gravitational acceleration g	$\mathscr{L}\mathscr{T}^{-2}$
Initial speed V	$\mathscr{L}\mathscr{T}^{-1}$
Earth radius R	$\mathscr{L}$

Step B. *Construct new dimensionless variables.*

$$y = \frac{x^*}{L} \qquad \tau = \frac{t^*}{T}$$

$L = R$ intrinsic length $T = RV^{-1}$ intrinsic time

- the two new variables are dimensionless
- their numerical value is the same whatever unit of measurement is used.

  if *R* = 4000 miles, *x** = 8000 miles, then *y* = 2.
  If *R* = 6436 km, *x** = 12,872 km, then *y* = 2.

Remarks:

(i) Intrinsic reference *quantities* are defined to be gauges in a given problem.
(ii) We have different choices of Intrinsic reference *quantities*

	intrinsic length *L*	intrinsic time *T*
1	R	RV^{-1}
2	R	$\sqrt{Rg^{-1}}$
3	V^2g^{-1}	Vg^{-1}

1. $y = \frac{x^*}{R}, \quad \tau = \frac{t^*}{RV^{-1}}$

$$R = Vt \quad \rightarrow t = RV^{-1}$$

$$\frac{d^2x^*}{dt^{*2}} = \frac{R}{R^2V^{-2}}\frac{d^2y}{d\tau^2} = \frac{V^2}{R}\frac{d^2y}{d\tau^2}$$

$$= -\frac{gR^2}{\left(R+x^*\right)^2} = -\frac{gR^2}{\left(R+Ry\right)^2} = -\frac{g}{\left(1+y\right)^2}$$

$$\begin{cases} \varepsilon\dfrac{d^2y}{d\tau^2} = -\dfrac{1}{\left(1+y\right)^2} \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

$$y = y(\tau;\varepsilon)$$

$$\varepsilon = \frac{V^2}{gR} = \frac{h_{\max}}{R} \ll 1$$

2. $z = \frac{x^*}{R}, \quad \tau_1 = \frac{t^*}{\sqrt{Rg^{-1}}}$

$$R = \tfrac{1}{2}gt^2 \quad \rightarrow t \sim \sqrt{Rg^{-1}}$$

$$\frac{d^2x^*}{dt^{*2}} = \frac{R}{Rg^{-1}}\frac{d^2z}{d\tau_1^2} = g\frac{d^2z}{d\tau_1^2}$$

$$= -\frac{gR^2}{\left(R+x^*\right)^2} = -\frac{gR^2}{\left(R+Rz\right)^2} = -\frac{g}{\left(1+z\right)^2}$$

$$\begin{cases} \dfrac{d^2z}{d\tau_1^2} = -\dfrac{1}{\left(1+z\right)^2} \\ z(0) = 0, \quad z'(0) = \sqrt{\varepsilon} \end{cases}$$

$$z = z(\tau_1;\varepsilon)$$

$$\varepsilon = \frac{V^2}{gR} = \frac{h_{\max}}{R} \ll 1$$

3. $x = \frac{x^*}{V^2 g^{-1}}, \quad \tau = \frac{t^*}{V g^{-1}}$

$h_{max} = \frac{1}{2} V^2 g^{-1}$

$t_{max} = V g^{-1}$

$\frac{x^*}{h_{max}} \in [0,1]$

$$\frac{d^2 x^*}{dt^{*2}} = \frac{V^2 g^{-1}}{V^2 g^{-2}} \frac{dx}{d\tau^2} = g \frac{d^2 x}{d\tau^2}$$

$$= -\frac{gR^2}{(R+x^*)^2} = -\frac{gR^2}{(R+Rx)^2} = -\frac{g}{(1+x)^2}$$

$$\begin{cases} \frac{d^2 x}{d\tau^2} = -\frac{1}{(1+\varepsilon x)^2} \\ x(0) = 0, \ x'(0) = 1 \end{cases}$$

$$x = x(\tau; \varepsilon)$$

$$\varepsilon = \frac{V^2}{gR} = \frac{h_{max}}{R} \ll 1$$

- There exists many reference times, e.g.

$$T_1 = RV^{-1} \qquad T_2 = \sqrt{Rg^{-1}} \qquad T_3 = Vg^{-1}$$

- Transition between them, $T_i = h(\varepsilon) T_j$

$$\sqrt{\varepsilon} T_1 = \sqrt{\frac{V^2}{gR}} RV^{-1} = \sqrt{Rg^{-1}} = T_2$$

$$\varepsilon T_1 = \frac{V^2}{gR} RV^{-1} = Vg^{-1} = T_3$$

$$\sqrt{\varepsilon} T_2 = \sqrt{\frac{V^2}{gR}} \sqrt{Rg^{-1}} = Vg^{-1} = T_3$$

$$f(x^*, t^*; g, R, V) = 0$$

$$x^{*\alpha_1} t^{*\alpha_2} g^{\alpha_3} R^{\alpha_4} V^{\alpha_5} \quad \text{dimensionless}$$

$$L^{\alpha_1} T^{\alpha_2} (LT^{-2})^{\alpha_3} L^{\alpha_4} (LT^{-1})^{\alpha_5}$$

$$= L^{\alpha_1+\alpha_3+\alpha_4+\alpha_5} T^{\alpha_2 - 2\alpha_3 - \alpha_5} = 1$$

$$\begin{cases} \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 = 0 \\ \alpha_2 - 2\alpha_3 - \alpha_5 = 0 \end{cases}$$

$$\begin{cases} \alpha_4 = -\alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_5 = \alpha_2 - 2\alpha_3 \end{cases} \qquad (\alpha_1, \alpha_2, \alpha_3)$$

3 degree of freedom

1. First combination

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 0, 0, -1, 0)$

$$x^{*1} t^{*0} g^0 R^{-1} V^0 = x^* R^{-1} \rightarrow \quad x = x^*/R$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 1, 0, -1, 1)$

$$x^{*0} t^{*1} g^0 R^{-1} V^1 = t^* R^{-1} V \rightarrow \quad t = t^*/RV^{-1}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 1, 1, -2)$

$$x^{*0} t^{*0} g^1 R^1 V^{-2} = gRV^{-2} \rightarrow \quad \varepsilon = V^2/gR$$

2. Second combination

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 0, 0, -1, 0)$

$$x^{*1} t^{*0} g^0 R^{-1} V^0 = x^* R^{-1} \rightarrow \quad x = x^*/R$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \left(0, 1, \frac{1}{2}, -\frac{1}{2}, 0\right)$

$$x^{*0} t^{*1} g^{1/2} R^{-1/2} V^0 = t^* g^{1/2} R^{-1/2} \rightarrow \quad t = t^*/\sqrt{g^{-1}R}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 1, 1, -2)$

$$x^{*0} t^{*0} g^1 R^1 V^{-2} = gRV^{-2} \rightarrow \quad \varepsilon = V^2/gR$$

3. Third combination

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 0, 1, 0 - 2)$

$$x^{*1} t^{*0} g^1 R^0 V^{-2} = x^* g V^{-2} \rightarrow \quad x = x^*/V^2 g^{-1}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 1, 1, 0, -1)$

$$x^{*0} t^{*1} g^1 R^0 V^{-1} = t^* g V^{-1} \rightarrow \quad t = t^*/Vg^{-1}$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 1, 1, -2)$

$$x^{*0} t^{*0} g^1 R^1 V^{-2} = gRV^{-2} \rightarrow \quad \varepsilon = V^2/gR$$

6.2.1 Nondimensionalization of a differential equation

Via dimensionless process

$$x^* = x^*\left(t^*; g, R, V\right)$$

$$\Downarrow$$

$$x = x\left(\tau; \varepsilon\right)$$

We see many benefits. And one can say that the time to reach maximum height is

$$\tau_M = f\left(\varepsilon\right)$$

$$\Downarrow$$

$$\boxed{\frac{t_M^*}{RV^{-1}} = f\left(\frac{V^2}{gR}\right)}$$

6.2.2 Nondimensionalization of a functional relationship

- The following relation obtained from dimensionless DE.

$$\frac{t_M^*}{RV^{-1}} = f\left(\frac{V^2}{gR}\right)$$

- Now we try to deduce it without DE

How?

6.2.2 Nondimensionalization of a functional relationship

(1) *Assume that a dimensionless quantity of interest is a* function ϕ *of the dimensional parameters.*

$$\frac{t_M^*}{R^{1/2}g^{-1/2}} = \Phi\left(V, g, R\right)$$

(2) *Construct a product of powers of the parameters in the argument* of ϕ.

$$\pi = V^\alpha g^\beta R^\gamma$$

(3) *Insert the dimensions of the various quantities.*

$$[\pi] = \left(LT^{-1}\right)^\alpha \left(LT^{-2}\right)^\beta \left(L\right)^\gamma = L^{\alpha+\beta+\gamma}T^{-\alpha-2\beta}$$

where, [x] *is the dimension of x*

6.2.2 Nondimensionalization of a functional relationship

(4) *To require a dimensionless* parameter π

$$\alpha + \beta + \gamma = 0, \quad -\alpha - 2\beta = 0$$

(5) *Find the general solution in terms of arbitrary constants c_1, c_2, ... , etc.*

$$\alpha = 2c_1, \quad \beta = c_1, \quad \gamma = c_1$$

(6) *Substitute back into* π.

$$\pi = V^{-2c_1} g^{c_1} R^{c_1} = \left(\frac{gR}{V^2}\right)^{c_1}$$

$$\pi_1 = \frac{gR}{V^2}, \quad \text{with } c_1 = 1$$

6.2.2 Nondimensionalization of a functional relationship

Finally, we have

$$\frac{t_M^*}{R^{1/2}g^{-1/2}} = \Phi\left(V, g, R\right) = \phi\left(\frac{gR}{V^2}\right) \qquad \varepsilon = \frac{V^2}{gR}$$

$$\Downarrow$$

$$t_M^* = R^{1/2}g^{-1/2}\phi\left(\frac{gR}{V^2}\right) \rightarrow = RR^{-1/2}VV^{-1}g^{-1/2}\phi\left(\frac{gR}{V^2}\right)$$

$$= RV^{-1}\sqrt{\frac{V^2}{Rg}}\phi\left(\frac{gR}{V^2}\right)$$

$$= RV^{-1}\sqrt{\varepsilon}\phi\left(1/\varepsilon\right)$$

$$\Updownarrow$$

$$t_M^* = RV^{-1}f\left(\frac{V^2}{gR}\right) \leftarrow = RV^{-1}f\left(\varepsilon\right)$$

6.2.2 Nondimensionalization of a functional relationship

Buckingham Pi Theorem

Historical Note

- The Buckingham Pi Theorem puts the 'method of dimensions' first proposed by Lord Rayleigh in his book *"The Theory of Sound"* (1877) on a solid theoretical basis, and is based on ideas of matrix algebra and concept of the 'rank' of non-square matrices which you may see in math classes.
- Although it is credited to E. Buckingham (1914), in fact, White points out that the theorem has also appeared earlier in independent publications by A. Vaschy (1892) and D. Riabouchinsky (1911).

On Physically Similar Systems; Illustrations of the Use of Dimensional Equations. Buckingham, Phys. Rev. **4**, 345 (1914).

Consider a physical relationship:

$$a = f(a_1, a_2, \cdots a_k \; ; b_1, b_2, \cdots b_m)$$

- a is the quantity under investigating
- Governing parameters $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots b_m$
- Group $a_1, a_2, \ldots, a_k$ have independent dimensions.
- Group $b_1, b_2, \ldots b_m$ is dependent on Group $a_1, a_2, \ldots, a_k$

For example

a_1 length $[a_1]$ = L

a_2 velocity $[a_2]$ = L T^{-1}

a_3 energy $[a_3]$ = M $L^2 T^{-2}$

b_1 acceleration $[b_1]$ = L T^{-2}= $[a_1]^{-1}[a_2]^2 [a_3]^0$

b_2 viscosity $[b_2]$ = M $L^{-1} T^{-1}$= $[a_1]^{-2}[a_2]^{-1} [a_3]^1$

- The dimensions of group $b_1, b_2, \ldots b_m$ is expressed as

$$[b_1] = [a_1]^{p1}[a_2]^{p2}\cdots[a_k]^{pk}$$
$$[b_2] = [a_1]^{q1}[a_2]^{q2}\cdots[a_k]^{qk}$$
$$\cdots\cdots\cdots\cdots$$
$$[b_m] = [a_1]^{s1}[a_2]^{s2}\cdots[a_k]^{sk}$$

- The dimension of a can be expressed by the first group

$$[a] = [a_1]^{m1}[a_2]^{m2}\cdots[a_k]^{mk}$$

Let's change the units,

$$a_1' = \lambda_1 a_1, \quad a_2' = \lambda_2 a_2, \ldots, a_k' = \lambda_k a_k$$

$$[a] = [a_1]^{m1}[a_2]^{m2}\cdots[a_k]^{mk}$$

$$a' = \lambda_1^{m1}\lambda_2^{m2}\cdots\lambda_k^{mk}\cdot a$$

With the same reason,

$$b_1' = \lambda_1^{p1}\lambda_2^{p2}\cdots\lambda_k^{pk}\cdot b_1$$
$$\cdots$$
$$b_m' = \lambda_1^{s1}\lambda_2^{s2}\cdots\lambda_k^{sk}\cdot b_m$$

X = x U = x' U'

x'= λ x **λ=U/U'**

X : physical quantity

x, x' : measurement

U, U' : unit

- The fundamental physical covariance principle claims that all physical laws can be represented in a form equally valid for all observers.
- This principle is valid for observers using different magnitudes of basic units.

We may rewrite

$$a' = f(a_1', a_2', \cdots, a_k' \; ; b_1', b_2', \cdots, b_m')$$
$$= f(\lambda_1 a_1, \cdots, \lambda_k a_k \; ; \lambda_1^{p1}\ldots\lambda_k^{pk} b_1, \cdots, \lambda_1^{s1}\ldots\lambda_k^{sk}\cdot b_m)$$
$$= \lambda_1^{m1}\lambda_2^{m2}\cdots\lambda_k^{mk}\cdot a$$
$$= \lambda_1^{m1}\lambda_2^{m2}\cdots\lambda_k^{mk} f(a_1, a_2, \cdots a_k \; ; b_1, b_2, \cdots b_m)$$

Let's choose $\lambda_1 = \frac{1}{a_1}, \quad \lambda_2 = \frac{1}{a_2}, \ldots, \lambda_k = \frac{1}{a_k}$

then $a' = \frac{a}{a_1^{m1}a_2^{m2}\cdots a_k^{mk}}$

$$= f\left(1, \cdots, 1; \frac{b_1}{a_1^{p1}a_2^{p2}\cdots a_k^{p1}}, \cdots, \frac{b_m}{a_1^{s1}a_2^{s2}\cdots a_k^{s1}}\right)$$

now introduce the dimensionless parameters

$$\Pi = \frac{a}{a_1^{m1}a_2^{m2}\cdots a_k^{mk}},$$

$$\Pi_1 = \frac{b_1}{a_1^{p1}a_2^{p2}\cdots a_k^{pk}}, \cdots, \Pi_m = \frac{b_m}{a_1^{s1}a_2^{s2}\cdots a_k^{sk}}$$

In all,

$$a = f(a_1, a_2, \cdots a_k \; ; b_1, b_2, \cdots b_m)$$

⇩

$$\Pi = f(1, 1, \cdots 1; \Pi_1, \cdots, \Pi_m)$$

⇩

$$\boxed{\Pi = \Phi(\Pi_1, \cdots, \Pi_m)}$$

Buckingham Pi Theorem

Example: in the flow of a fluid through a long cylindrical pipe, the pressure drop per length reads

$$dp/dx = f(U, D, \rho; \mu) \qquad k=3,\ m=1$$

- U the mean fluid velocity
- D the diameter of the pipe
- ρ the fluid density
- μ the fluid viscosity
- The dimensions of U, D, ρ are independent.
- The dimensions of μ and dp/dx can be expressed as

$$[dp/dx] = ML^{-2}T^{-2}$$

$$[U] = LT^{-1} \qquad [D] = L$$

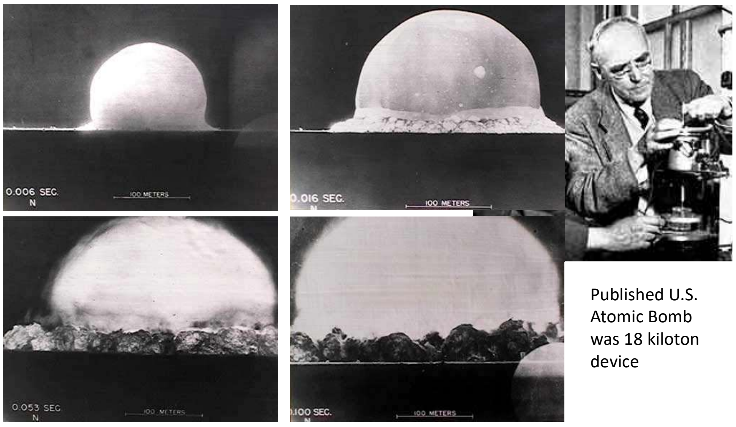
$$[\rho] = ML^{-3} \qquad [\mu] = ML^{-1}T^{-1}$$

$$[\mu] = [U][D][\rho] = LT^{-1}\cdot L\cdot ML^{-3} = ML^{-1}T^{-1}$$

$$[dp/dx] = [U]^2[D]^{-1}[\rho]^1 = L^2T^{-2}\cdot L^{-1}\cdot ML^{-3} = ML^{-2}T^{-2}$$

$$dp/dx = \frac{U^2\rho}{D}\Phi\left(\frac{\mu}{UD\rho}\right)$$

G. I. Taylor's 1947 Analysis



Published U.S. Atomic Bomb was 18 kiloton device

Two assumptions:
1. The energy (E) was released in a small space.
2. The shock wave was spherical.

$$R = R(t, \rho, E)$$

- R: the size of the fire ball (function of t)
- t: time
- E: energy
- ρ: density of the surrounding air

Let's perform a dimensional analysis:

$[R]$=L $[t]$=T $[\rho]=ML^{-3}$ $[E]=ML^2T^{-2}$

- n = 4 physical variables and d = 3 dimensions
- Pi theorem: only one dimensionless group Π_1

$$\Pi_1 = RE^a t^b \rho^c$$

$$1 = [R][E]^a[t]^b[\rho]^c = L\left(ML^2T^{-2}\right)^a T^b\left(ML^{-3}\right)^c$$

$$= L^{1+2a-3c}T^{b-2a}M^{a+c}$$

$$\begin{cases} 1+2a-3c=0 \\ b-2a=0 \\ a+c=0 \end{cases} \Rightarrow a=-1/5,\ b=-2/5,\ c=1/5$$

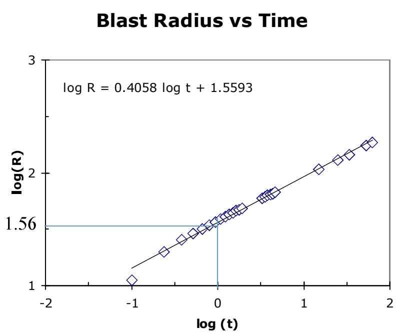
$$\Pi_1 = RE^{-1/5}t^{-2/5}\rho^{1/5} \Rightarrow R = c\,t^{2/5}E^{1/5}\rho^{-1/5}$$

the atomic energy $E = \dfrac{R^5\rho}{c^5t^2}$

$R = (E/\rho)^{1/5}\, t^{2/5}$ Taylor estimated c=1

⬇

$\log R = 0.4 \log t + 0.2 \log(E/\rho)$



$0.2 \log(E/\rho) = 1.56$

$\rho = 1.2$ kg/m^3

$\Downarrow$

$E = 7.9\times10^{13}$ J

= 19.8 kilotons TNT

USTC

3. 相似模型的应用

- 相似性原理：第一定理、第二定理(逆定理)
  - 如两个系统的行为相似，则其对应的无量纲参数相同
  - 如两个系统的无量纲参数相同，则两者的行为相似
- 相似模型：
  - ①尺寸成比例，②无量纲参数相等
- 模型试验被广泛应用于飞机、导弹、高层建筑、城市环境、船舶、水利工程等设计研究中

例：船舶设计（恒速航行船只所需功率）

- 功率消耗：形成水面波动 + 克服水的粘性阻力

$$P = f(U, L, g, \rho, \nu)$$

$N = 6,\ k = 3,\ N\text{-}k = 3$

$$\frac{P}{\rho L^2 U^3} = \phi(\mathrm{Fr}, \mathrm{Re})$$

$$\mathrm{Fr} = \frac{U}{\sqrt{Lg}}$$ Froude 数 波阻

$$\mathrm{Re} = \frac{UL}{\nu}$$ Reynolds 数 粘性阻力

P	功率	ML^2T^{-3}
U	船的速度	LT^{-1}
L	船的长度	L
g	重力加速度	LT^{-2}
ρ	水的密度	ML^{-3}
ν	运动粘性系数	L^2T^{-1}

1:100 模型 $L' = 10^{-2} L$

$\mathrm{Fr} = \frac{U}{\sqrt{Lg}}$ Froude 数相同 ⟶ $U' = 10^{-1} U$

$\mathrm{Re} = \frac{UL}{\nu}$ Reynolds 数相同 ⟶ $\nu' = 10^{-3} \nu$

常温下：
空气 ~ $1.5\times10^{-5}\mathrm{m^2/s}$
水 ~ $1.0\times10^{-6}\mathrm{m^2/s}$
油 ~ $1.5\times10^{-4}\mathrm{m^2/s}$

- 如 L=100m，U=10m/s，$\nu=10^{-6}\mathrm{m^2/s}$ ⟶ $\mathrm{Re} = 10^9$
- 从 $\mathrm{Re} = 10^7$ 和 $\mathrm{Re} = 10^8$ 的 P 外推？

$\mathrm{Re} = 10^9$ 可能突变，从层流转捩为湍流

船舶设计仍然是一门技艺而不是科学

如模型过小，还需考虑表面张力的影响

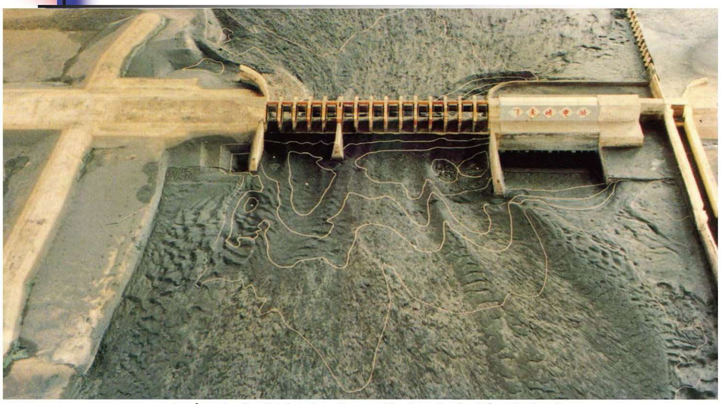
⟶ $\mathrm{Bo} = \frac{\rho g L^2}{T}$ Bond 数

风洞试验：飞行器、高层建筑

http://v.youku.com/v_show/id_XMTA5Mzc5ODA=.html；http://www.sciam.com.cn/html/remenkeji/2011/0212/15202.html

水槽试验

动床模型试验：水利工程、桥墩冲刷、冲淤



飞来峡水利枢纽水工局部动床模型试验

Section 6.3

Scaling

6.3.0 Introduction

Last section shows:

- deleting a small parameter is not as simple as it looks, because of ill conditioning and sensitivity.
- a first approximation can be obtained by ε=0
- But we now see it is dangerous in doing so. For example, consider

$$u(x,\varepsilon) \equiv x + e^{-x/\varepsilon}, \quad 0 < x \le 1, \quad \varepsilon > 0$$

$$u(x,0) = \lim_{\varepsilon \to 0} x + e^{-x/\varepsilon} = x$$

or we can rewrite as

$$\text{Let } x = \varepsilon\xi, \quad v(\xi,\varepsilon) = u(\varepsilon\xi,\varepsilon) = \varepsilon\xi + e^{-\xi}$$

$$v(\xi,0) = \lim_{\varepsilon \to 0} \varepsilon\xi + e^{-\xi} = e^{-\xi}$$

6.3.0 Introduction

And alternatively,

$$\text{Let } x = \varepsilon^2\eta, \quad w(\eta,\varepsilon) = u(\varepsilon^2\eta,\varepsilon) = \varepsilon^2\eta + e^{-\varepsilon\eta}$$

$$w(\eta,0) = \lim_{\varepsilon \to 0} \varepsilon^2\eta + e^{-\varepsilon\eta} = 1$$

Now we have three limits for the first approximation

$$u(x,0) = x, \qquad u(x,0) = e^{-\xi} = e^{-x/\varepsilon}, \qquad u(x,0) = 1$$

Which one is correct?

6.3.0 Introduction

The projectile problem:

1. $y = \dfrac{x^*}{R}, \quad \tau = \dfrac{t^*}{RV^{-1}}$

$$\begin{cases} \varepsilon \dfrac{d^2 y}{d\tau^2} = -\dfrac{1}{(1+y)^2} \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

2. $z = \dfrac{x^*}{R}, \quad \tau = \dfrac{t^*}{\sqrt{Rg^{-1}}}$

$$\begin{cases} \dfrac{d^2 z}{d\tau^2} = -\dfrac{1}{(1+z)^2} \\ z(0) = 0, \quad z'(0) = \sqrt{\varepsilon} \end{cases}$$

3. $x = \dfrac{x^*}{V^2 g^{-1}}, \quad \tau = \dfrac{t^*}{Vg^{-1}}$

$$\begin{cases} \dfrac{d^2 x}{d\tau^2} = -\dfrac{1}{(1+\varepsilon x)^2} \\ x(0) = 0, \quad x'(0) = 1 \end{cases}$$

$$\varepsilon = \frac{V^2}{gR} = \frac{h_{max}}{R} \ll 1$$

How to choose dimensionless variables?

6.3.1 Definition of scaling

The answer is Scaling:

- Scaling is the correct way of nondimensionalization
- The key is to select *intrinsic reference* (scales) *quantities*
- In the dimensional equations, each term is a product of two part:
  - *dimensionless factor: small or larger*
  - *dimensionless variable: O(1)*
  - *We can determine the smallness from the factor.*

6.3.2 Scaling the projectile problem

$$\frac{1}{2}mV^2 = mgh_{max} \to h_{max} = V^2 g^{-1}/2$$

$$V - \frac{1}{2}gt^2_{max} = 0 \to t_{max} = 2Vg^{-1}$$

⇩

$$x = \frac{x^*}{V^2 g^{-1}}, \qquad t = \frac{t^*}{Vg^{-1}}$$

⇩

$$x^* = \frac{V^2}{g}x, \qquad t^* = \frac{V}{g}t$$

$V^2 g^{-1}, Vg^{-1}$:

order of maginitude of x, t

$$\begin{cases} \dfrac{d^2 x}{dt^2} = -(1+\varepsilon x)^{-2} \\ x(0) = 0, \quad x'(0) = 1 \end{cases}$$

$$\varepsilon = \frac{V^2}{gR} = \frac{h_{max}}{R} \ll 1$$

⇩ Simplified properly

$$\begin{cases} \dfrac{d^2 x}{dt^2} = -1 \\ x(0) = 0, \quad x'(0) = 1 \end{cases}$$

6.3.3 Order of magnitude

The order of magnitude of a number A

- is said to be 10^n, if

$$n - \frac{1}{2} < \log_{10}|A| < n + \frac{1}{2}.$$

⇕

$$3 \cdot 10^{n-1} < |A| < 3 \cdot 10^n$$

The order of magnitude of a function f

- is the order of magnitude of $M = |f_{max}|$ over the given region.

6.3.4 Scaling known functions

Consider a first order ordinary differential equation

$$F\left(x^*, \frac{du^*}{dx^*}\right) = 0, \qquad \text{with } x^* \in I$$

u^* *is* velocity with scale U
x^* *is* spatial variable with length scale L

Dimensionless variables are

$$x = \frac{x^*}{L}, \qquad u = \frac{u^*}{U}$$

6.3.4 Scaling known functions

and we have

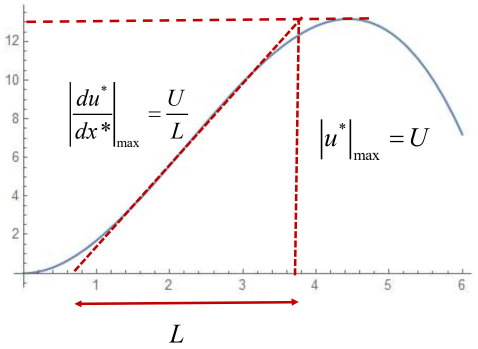
$$u^*(x^*) = U \cdot u(x), \qquad \frac{du^*}{dx^*} = \frac{d(Uu)}{d(Lx)} = \frac{U}{L}\frac{du}{dx}$$

If U and L are appropriate scales

- $\frac{U}{L}$ is a good estimate of the first derivative.
- $U, \frac{U}{L}$ actually are the maximum absolute values

$$U = \max_{x^* \in I} \left|u^*(x^*)\right|$$
$$\frac{U}{L} = \max_{x^* \in I} \left|\frac{du^*}{dx^*}\right| \quad \Rightarrow \quad L = \frac{\left|u^*\right|_{\max}}{\left|du^*/dx^*\right|_{\max}}$$

6.3.4 Scaling known functions



length scale L is an estimate of the shortest distance over which the function undergoes a significant change.

6.3.4 Scaling known functions

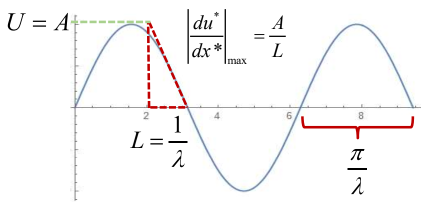
Example 1. Find scales U and L when

$$u^*(x^*) = A \sin \lambda x^*, \qquad -\infty < x^* < \infty, \; A, \lambda > 0$$

$$U = A = \left|u^*\right|_{\max}$$

$$\left|\frac{du^*}{dx^*}\right|_{\max} = \left|A\lambda \cos \lambda u^*\right|_{\max} = A\lambda \rightarrow L = \lambda^{-1}$$

$$\left|\frac{du^*}{dx^*}\right|_{\max} = \frac{A}{L}$$



6.3.4 Scaling known functions

Example 2. Find scales U and L when

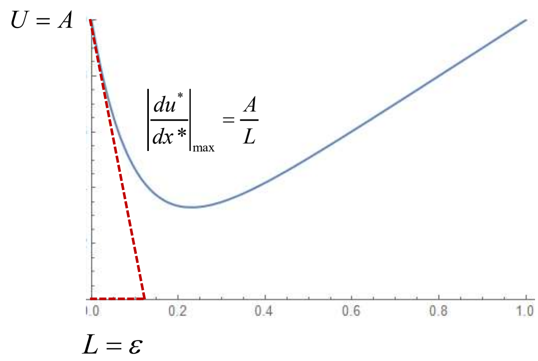
$$u^*(x^*) = A\left[x^* + \exp\left(-\frac{x^*}{\varepsilon}\right)\right],$$
$$\text{with} \quad x^* \in [0,1], \; A > 0, \; 0 < \varepsilon \ll 1$$

$$U = \left|u^*\right|_{\max} \approx A$$

$$\left|\frac{du^*}{dx^*}\right|_{\max} = \frac{A}{L}$$

$$\left|\frac{du^*}{dx^*}\right|_{\max} = A\left|1 + \varepsilon^{-1} \exp\left(-\frac{x^*}{\varepsilon}\right)\right|_{\max} \approx A\varepsilon^{-1} \rightarrow \; L = \varepsilon$$

6.3.4 Scaling known functions



6.3.4 Scaling known functions

Suppose that the problem is governed by

$$F\left(x^*, \frac{du^*}{dx^*}, \cdots \frac{d^N u^*}{dx^{*N}}\right) = 0, \qquad \text{with } x^* \in I$$

velocity scale U

$$U = \max_{x^* \in I} \left|u^*\left(x^*\right)\right| \qquad u^*\left(x^*\right) = U \cdot u\left(\frac{x^*}{L}\right)$$

Length scale L = ?

$$L = \frac{\left|u^*\right|_{\max}}{\left|du^* / dx^*\right|_{\max}}$$

Since now we have multiple derivatives

6.3.4 Scaling known functions

For each of the derivatives, we have length scale $l_{(i)}$,

$$\frac{d^i u^*}{dx^{*i}} = \frac{U}{l_{(i)}^i} \frac{d^i u}{dx^i}, \qquad \left|\frac{d^i u^*}{dx^{*i}}\right|_{\max} = \frac{U}{l_{(i)}^i}$$

$$l_{(i)} = \left[\frac{U}{\left|d^i u^* / dx^{*i}\right|_{\max}}\right]^{1/i}, \qquad i = 1, 2, \cdots N$$

we select *the length scale* L *as*

$$L = \min\left\{l_{(i)}\right\}, \qquad i = 1, 2, \cdots N$$

⬇

$$\left|\frac{d^i u^*}{dx^{*i}}\right|_{\max} = \frac{U}{l_{(i)}^i} \le \frac{U}{L^i}$$

6.3.4 Scaling known functions

Example 3. Find scales *U* and *L* when

$$F\left(x^*, \frac{du^*}{dx^*}, \cdots \frac{d^N u^*}{dx^{*N}}\right) = 0, \qquad \text{with } x^* \in I$$

$$u^*\left(x^*\right) = A \sin \lambda x^*, \qquad -\infty < x^* < \infty, \;\; A, \lambda > 0$$

length scale $l_{(i)}$ for *i*-th derivatives,

$$l_{(i)} = \left[\frac{U}{\left|d^i u^* / dx^{*i}\right|_{\max}}\right]^{1/i} = \left[\frac{A}{A\lambda^i}\right]^{1/i} = \frac{1}{\lambda}$$

⬇

$$L = \min\left\{l_{(i)}\right\} = \frac{1}{\lambda}, \qquad i = 1, 2, \cdots N$$

6.3.4 Scaling known functions

Example 4. Find scales *U* and *L* when

$$F\left(x^*, \frac{du^*}{dx^*}, \cdots \frac{d^N u^*}{dx^{*N}}\right) = 0, \qquad \text{with } x^* \in I$$

$$u^*\left(x^*\right) = M + A \sin \lambda x^*, \qquad -\infty < x^* < \infty, \;\; A, \lambda > 0$$

$$U = M + A$$

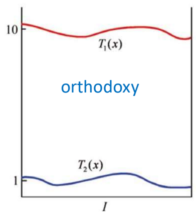
$$l_{(i)} = \left[\frac{U}{\left|d^i u^* / dx^{*i}\right|_{\max}}\right]^{1/i} = \left(\frac{M + A}{A\lambda^i}\right)^{1/i} = \frac{1}{\lambda}\left(1 + \frac{M}{A}\right)^{1/i}$$

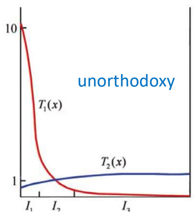
$$L = \min\left\{l_{(i)}\right\} = \frac{1}{\lambda}\left(1 + \frac{M}{A}\right)^{1/N}$$

6.3.5 Orthodoxy

after the process of scaling, we have two issues :

- We have seen neglecting small terms may be wrong
- Another one is the Orthodoxy of each terms.
  - the order of magnitude of a term estimates that term's maximum magnitude.
  - If the absolute value of a term deviates too much from their maximum values, then the order of magnitude may be misleading.





Harmless unorthodoxy:

- terms decay at same rate
- With oscillatory terms of large amplitude, terms of unit order can be deleted

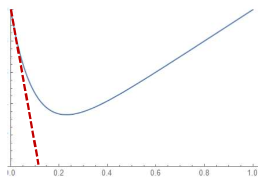
6.3.5 Orthodoxy

Split example of orthodoxy:

$$u^*\left(x^*\right) = A\left[x^* + \exp\left(-\frac{x^*}{\varepsilon}\right)\right], \text{ with } x^* \in [0,1], \;\; A > 0, \;\; 0 < \varepsilon \ll 1$$

$$U = A, \quad L = \varepsilon$$

$$u = u^* / \mathrm{A} = \varepsilon x + e^{-x} \qquad \frac{du}{dx} = \varepsilon - e^{-x}$$



$\text{as } x^* \to 0$

$$L = \varepsilon$$

$\text{as } 3\varepsilon \le x^* \le 1$

$$\left|\frac{du^*}{dx^*}\right|_{\max} \approx A\varepsilon^{-1}e^{-3} \Rightarrow L = \frac{A}{A\varepsilon^{-1}e^{-3}} = \varepsilon e^3 \approx 10\varepsilon$$

$$L = \frac{\left|u^*\right|_{\max}}{\left|du^* / dx^*\right|_{\max}}$$

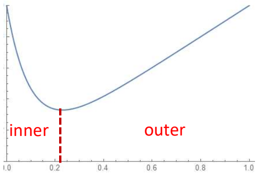
6.3.5 Orthodoxy

To satisfy the orthodoxy requirement,

- split [0, 1] into outer and inner regions
- different length scale in each part.

outer regions: (more than a few ε from x*=0)

$$|u^*|_{\max} = A, \quad \left|\frac{du^*}{dx^*}\right|_{\max} \approx A$$



$$U = A, \quad L \approx 1$$

$$u = \frac{u^*}{A}, \quad x = \frac{x^*}{L} = x^*$$

$$u(x,\varepsilon) = \frac{1}{A} u^*(x^*,\varepsilon) = x + e^{-x/\varepsilon}$$

6.3.5 Orthodoxy

Inner regions: (within a few ε from x*=0)

$$|u^*|_{\max} = A, \quad \left|\frac{du^*}{dx^*}\right|_{\max} \approx A\varepsilon^{-1} \qquad U = A, \quad L = \varepsilon$$

$$v = \frac{u^*}{A}, \quad \xi = \frac{x^*}{\varepsilon} \qquad v(\xi,\varepsilon) = \frac{1}{A} u^*(\varepsilon\xi,\varepsilon) = \varepsilon\xi + e^{-\xi}$$

To obtain a first approximation in the two regions, let ε→ 0,

outer regions: $u(x,\varepsilon) = x + e^{-x/\varepsilon} \to x$

Inner regions: $v(\xi,\varepsilon) = \varepsilon\xi + e^{-\xi} \to e^{-\xi}$ ➡ $u(x,\varepsilon) \approx e^{-x/\varepsilon}$

6.3.5 Orthodoxy

评述：非正统性如何处理？

- 存在非正统性时，不同区域要用不同尺度，寻找不同的近似
- 可能出现两个尺度共存的情况，如 $e^{-\varepsilon x}\sin x$，有时可平均以揭示主要倾向　　多相流，复合材料
- 正确尺度化后，利用方程中的小参数可借助摄动理论得到近似解　　多尺度 —— 奇异摄动
- 满足正统性——只需单一尺度——正则摄动理论
- 非正统性——需不只一个尺度——奇异摄动理论

6.3.6 Scaling unknown functions

5. 未知函数的尺度化

只有了解解的主要性质，才能进行量级估计和尺度化

如何获取量级估算所需的知识？

- 利用相关的实验或观察资料：一种现象往往由两种效应的平衡引起，权衡后可得到尺度化
- 从有关问题的经验中获得启示
- 把所给问题高度简化后求解：如先求零级近似，再尺度化
- 将一些项（如非线性项）忽略后求解，再对所忽略项进行估算，可能的话选择适当的参数范围
- 采用试凑法：假定某种尺度，求解后检验量级
- 利用对典型参数的数值计算结果，获得“感性认识”