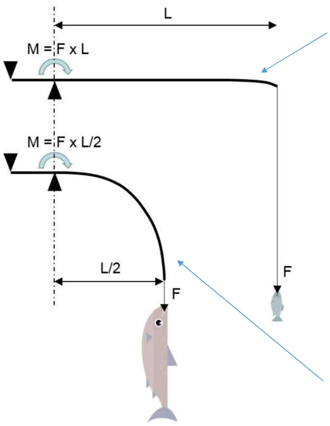
Chapter 4

Superposition, Heat Flow, and Fourier Analysis

4.0 Introduction

Linear or nonlinear



Beam (Linear)

$$B\frac{d^4 y}{dx^4} = 0$$

$$B\frac{d^2 \kappa}{ds^2} = 0$$

curvature

$$\kappa = \frac{d^2 y}{dx^2}$$

Arc length

$s \sim x$

Euler elastica (nonlinear)

$$\frac{d^2 \kappa}{ds^2} + \frac{1}{2}\kappa^3 = 0$$

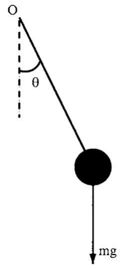
4.0 Introductionv

linear ODE: $\theta \ll 1$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

Non-linear ODE: finite θ

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$$



Pendulum

4.0 Introduction

Consider a function $u(x, y)$

The relationship between u and its derivatives can be described by a differential equation (DE)

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots\right) = 0$$

where the highest-order of a derivative defines the *order* of a DE.

Example:

$$x\,u_x + y\,u_y = 0 \qquad u_{tt} - c^2 u_{xx} = f(x,t)$$

first-order *second-order*

4.0 Introduction

A DE can be put in the operator form

$$L[u] = f(\mathbf{x})$$

- L an differential operator.
- $f(\mathbf{x})$ the inhomogeneous term

linear operator:

$$L[au + bv] = aL[u] + bL[v]$$

- *u , v:* functions
- a , b: constants

Non-linear operator: if it is not linear.

4.0 Introduction

The Linear Superposition Principle

For an *n*-th order, linear, homogeneous differential equation

$$L[u] = 0$$

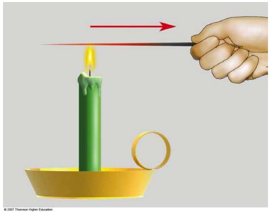
There are *n* linearly independent solutions

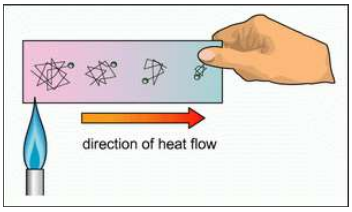
$$L[u_i] = 0, \qquad i = 1, 2, 3, \cdots n$$

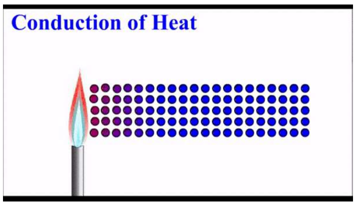
The general solution

$$u = \sum_{i=1}^{n} c_i u_i$$

4.1 Conduction of Heat





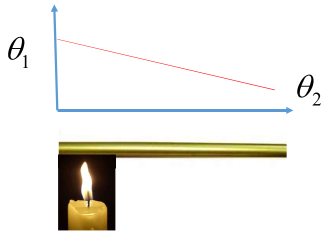


4.1.1 Steady state heat conduction

Steady state t >>1

$$\theta(x) = \theta_1 + \frac{\theta_2 - \theta_1}{L} x$$

$$\frac{d\theta}{dx} = \frac{\theta_2 - \theta_1}{L}$$



heat flux J(x):

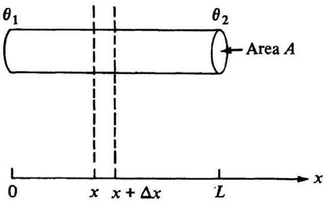
the rate of heat flow per unit area per unit time.

$$J = -k \frac{\partial \theta}{\partial x}$$

where k is the coefficient of heat conductivity

4.1.2 Different equation for 1D heat conduction

Transient state



- *Heat in the segment dx*

  $q = \rho c A \Delta x \theta$
- *Heat change in time dt*

  $$\Delta q = \frac{\partial(\rho c \theta A \Delta x)}{\partial t} \Delta t = \frac{\partial \theta}{\partial t} \rho c A \Delta x$$
- *Heat change due to flux*

$$\frac{\partial \theta}{\partial t} \rho c A \Delta x = \left[J(x,t) - J(x+\Delta x, t) \right] A$$

$$J = -k \frac{\partial \theta}{\partial x}$$

$$\rho c \frac{\partial \theta}{\partial t} = -\frac{\partial J}{\partial x} \Rightarrow \rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial \theta}{\partial x} \right)$$

4.1.2 Different equation for 1D heat conduction

Homogeneous material $\quad \frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}$

coefficient of thermal diffusivity $\quad \kappa = \frac{k}{\rho c}$

TABLE 10.5.1 Values of the Thermal Diffusivity for Some Common Materials

Material	κ (cm^2/sec)
Silver	1.71
Copper	1.14
Aluminum	0.86
Cast iron	0.12
Granite	0.011
Brick	0.0038
Water	0.00144

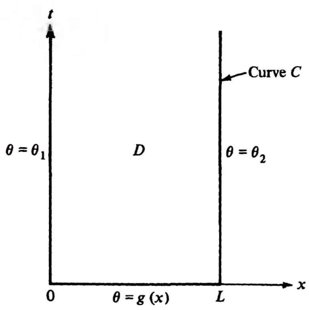
4.1.3 Initial-boundary value problem for 1D heat conduction

To complete the problem, we add Initial boundary conditions as blow

$D: 0 < x < L, \quad 0 < t < \infty.$

Governing equation in 1D

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 < x < L, \quad 0 < t < \infty$$



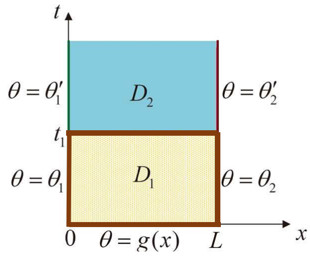
initial conditions

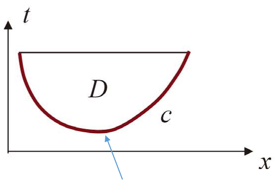
$$\theta(x,0) = g(x), \quad 0 < x < L$$

boundary conditions

$$\theta(0,t) = \theta_1, \quad \theta(L,t) = \theta_2, \quad t > 0$$

4.1.4 Pat, present and future





General boundary

4.1.5 Heat conduction in 3D space

Heat conduction DE in 3D

$$\frac{\partial}{\partial t}\iiint_D \rho c\theta dv = \iint_{\partial D} -\mathbf{J}\cdot\mathbf{n}ds$$

$$= \iint_{\partial D} k\frac{\partial\theta}{\partial n}ds = \iint_{\partial D} k\nabla\theta\cdot\mathbf{n}ds$$

$$= \iiint_D \nabla\cdot\left(k\nabla\theta\right)dv$$

$$\Rightarrow \iiint_D \frac{\partial}{\partial t}\rho c\theta - \nabla\cdot\left(k\nabla\theta\right)dv = 0$$

$$\boxed{\rho c\frac{\partial\theta}{\partial t} = \nabla\cdot\left(k\nabla\theta\right)}$$

$$\mathbf{J} = -k\nabla\theta$$

Gradient operator

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

divergence theorem

$$\iint_{\partial D}\mathbf{F}\cdot\mathbf{n}ds = \iiint_D \nabla\cdot\mathbf{F}dv$$

4.1.5 Heat conduction in 3D space

Homogeneous case:

$$\frac{\partial\theta}{\partial t} = \kappa\nabla^2\theta \qquad \kappa = \frac{k}{\rho c}$$

$$\frac{\partial\theta}{\partial t} = \kappa\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right)$$

Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Inhomogeneous case:

$$\rho c\frac{\partial\theta}{\partial t} = \nabla\cdot\left(k\nabla\theta\right)$$

$$\rho c\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z}\right)$$

4.1.5 Heat conduction in 3D space

Boundary conditions:

1. Dirichlet boundary condition

   $\theta = f$

2. Neumann boundary condition

   $\frac{\partial\theta}{\partial n} = f$ $\qquad$ $\frac{\partial\theta}{\partial n}$ =0 : insulated

   $\mathbf{J} = -k\nabla\theta$

3. Mixed boundary condition

   $a\frac{\partial\theta}{\partial n} + b\theta = f$

   $\frac{\partial\theta}{\partial n} = \theta$: heat flux is determined by temperature

4.1.6 Proof of the uniqueness theorem

For the Initial-boundary value problem in domain *D*

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial x^2} \qquad \begin{aligned}\theta(x,0) &= g(x), \quad x\in\partial D\\ \theta(0,t) &= f(x), \quad t = 0\end{aligned}$$

Let θ_1 , θ_2 be solutions satisfying the IBV problem.
To prove uniqueness, define $\theta = \theta_2 - \theta_1$. Such that

$$\theta(x,0) = 0, \quad x\in\partial D$$

$$\theta(0,t) = 0, \quad t = 0$$

If $\theta = 0$ everywhere, uniqueness proved.

4.1.6 Proof of the uniqueness theorem

$$\rho c\frac{\partial\theta}{\partial t} = \nabla\cdot\left(k\nabla\theta\right) \Rightarrow \boxed{\frac{\rho c}{2}\frac{\partial\theta^2}{\partial t} = \theta\nabla\cdot\left(k\nabla\theta\right)} \qquad \nabla \equiv \mathbf{e}_i\partial_i$$

$$\theta\nabla\cdot\left(k\nabla\theta\right) = \theta\mathbf{e}_i\partial_i\cdot\left(k\mathbf{e}_j\partial_j\theta\right) = \theta\partial_i k\partial_j\theta\mathbf{e}_i\cdot\mathbf{e}_j$$

$$= \theta\partial_i k\partial_j\theta\delta_{ij} = k\theta\partial_i\partial_i\theta = k\partial_i\left(\theta\partial_i\theta\right) - k\partial_i\theta\partial_i\theta$$

$$= k\nabla\cdot\theta\nabla\theta - k\left(\nabla\theta\right)^2$$

$$\iiint_D \frac{\rho c}{2}\frac{\partial\theta^2}{\partial t}dv = \iiint_D k\nabla\cdot\theta\nabla\theta - k\left(\nabla\theta\right)^2 dv$$

$$= \iint_{\partial D} k\theta\nabla\theta\cdot\mathbf{n}ds - \iiint_D k\left(\nabla\theta\right)^2 dv$$

$$= \iint_{\partial D} k\theta\frac{\partial\theta}{\partial n}ds - \varepsilon$$

(the surface integral term is crossed out)

divergence theorem

$$\iint_{\partial D}\mathbf{F}\cdot\mathbf{n}ds = \iiint_D \nabla\cdot\mathbf{F}dv$$

$$\theta(x,0) = 0, \quad x\in\partial D$$

4.1.6 Proof of the uniqueness theorem

$$\iiint_D \frac{\rho c}{2}\frac{\partial\theta^2}{\partial t}dv = -\varepsilon \qquad \varepsilon = \iiint_D k\left(\nabla\theta\right)^2 dv > 0$$

$$\Downarrow$$

$$\int_0^{t_1}\left(\iiint_D \frac{\rho c}{2}\frac{\partial\theta^2}{\partial t}dv\right)dt = -\int_0^{t_1}\varepsilon\, dt$$

$$\Downarrow$$

$$\left(\iiint_D \frac{\rho c}{2}\theta^2 dv\right)_{t_1} = -\int_0^{t_1}\varepsilon\, dt$$

$+ \qquad -$

$$\Downarrow$$

$\theta = 0 \quad \varepsilon = 0$ at any time

Uniqueness proved ☐

4.1.7 The maximum principle

The maximum temperature is found

- on the boundary
- or in initial condition.

4.1.8 Solution by the method of separation of variables

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

***Any** periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

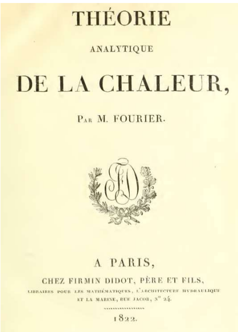
Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

But it's true!

- called Fourier Series

4.1.8 Solution by the method of separation of variables



mathematical theory of heat

Fourier's idea:

superposing an infinite number of appropriate simple solutions to the linear *equation.*

4.1.8 Solution by the method of separation of variables

2. 分离变量法求解

$$\frac{\partial\theta}{\partial t}=\kappa\frac{\partial^2\theta}{\partial x^2}$$

$$\theta(x,0)=g(x),\quad 0<x<L$$

$$\theta(0,t)=\theta_1,\quad \theta(L,t)=\theta_2,\quad t>0$$

- 基本思想:
  - 将非齐次部分分离出去，考察线性方程
  - 线性方程的解可表示为无数简单解的迭加
  - 用分离变量法决定这些简单解
- 方法评述:
  - 这一方法不能得到稳态解 $\theta_s(x)$

    稳态解，满足 D.E.，B.C.
  - 只能用于求解瞬态解（衰减部分的解）

    $v(x,t)=\theta(x,t)-\theta_s(x)$
  - 边界条件必须是齐次的

4.1.8 Solution by the method of separation of variables

D.E. $\dfrac{\partial\theta}{\partial t}=\kappa\dfrac{\partial^2\theta}{\partial x^2},\quad 0<x<L,\ 0<t<\infty$

I.C. $\theta(x,0)=g(x),\quad 0<x<L$

B.C. $\theta(0,t)=\theta_1,\quad \theta(L,t)=\theta_2,\quad t>0$

$$v(x,t)=\theta(x,t)-\theta_s(x)$$

D.E. $\dfrac{\partial v}{\partial t}=\kappa\dfrac{\partial^2 v}{\partial x^2}$

I.C. $v(x,0)=f(x)=g(x)-\theta_s(x),\quad 0<x<L$

B.C. $v(0,t)=v(L,t)=0,\quad t>0$

4.1.8 Solution by the method of separation of variables

Steady solution

$$\frac{d^2\theta_s(x)}{dx^2}=0\qquad \theta_s(0)=\theta_1\qquad \theta_s(L)=\theta_2$$

$$\theta_s(x)=\left(1-\frac{x}{L}\right)\theta_1+\frac{x}{L}\theta_2$$

Transient solution

(A) Assume a product solution

$$v(x,t) = X(x)T(t) \longrightarrow \frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}$$

(B) Substitute into the governing differential equation

$$XT' = \kappa X''T$$

(C) Separate variables

$$\frac{X''}{X} = \frac{T'}{\kappa T} = k \Rightarrow \begin{matrix} X'' = kX \\ T' = \kappa k T \end{matrix}$$

(D) Determine permissible values of the separation constant

$$v(0,t) = v(L,t) = 0 \qquad v(x,t) = X(x)T(t)$$

$$\Downarrow$$

$$X(0)T(t) = 0 \qquad X(L)T(t) = 0$$

$$\Downarrow$$

$$X(0) = X(L) = 0$$

Consider three possibilities $k > 0$, $k = 0$, $k < 0$.

1. $k > 0$ $\quad k = \mu^2, \quad \mu > 0$

$$X'' = \mu^2 X \Rightarrow \lambda_1 = \mu \quad \lambda_2 = -\mu$$

$$X = C_1 e^{\mu x} + C_2 e^{-\mu x} = \tilde{C}_1 \cosh \mu x + \tilde{C}_2 \sinh \mu x$$

$$C_1 = C_2 = 0 \qquad X(0) = X(L) = 0$$

2. $k = 0$

$$X'' = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$$

$$X = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_2 x} = C_1 + C_2 x \qquad C_1 = C_2 = 0$$

3. $k < 0$ $\quad k = -\alpha^2, \quad \alpha > 0$

$$X'' = -\alpha^2 X \Rightarrow \lambda_1 = i\alpha \quad \lambda_2 = -i\alpha$$

$$X = C_1 e^{i\alpha x} + C_2 e^{-i\alpha x}$$

$$= C_1 \cos \alpha x + C_2 \sin \alpha x \qquad X(0) = X(L) = 0$$

$$\Downarrow$$

$$C_1 = 0 \quad C_2 \sin \alpha L = 0$$

$$\Downarrow$$

eigen-value $\quad \alpha_m = \dfrac{m\pi}{L}, \quad m = 1, 2, 3 \cdots$

eigen-function $\quad X_m = B_m \sin \dfrac{m\pi x}{L}$

(E) Solve the remaining equation of T(t)

$$T' = \kappa k T \Rightarrow T' = -\alpha^2 \kappa T \qquad \alpha_m = \frac{m\pi}{L}$$

solution $\quad T_m = \exp\left(-\alpha_m^2 \kappa t\right)$

(F) Superposition of all possible solutions

$$v(x,t) = \sum_{m=1}^{\infty} v_m(x,t) = \sum_{m=1}^{\infty} X_m(x) T_m(t)$$

$$= \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{L} \exp\left(-\frac{m^2 \pi^2 \kappa}{L^2} t\right)$$

initial condition $\quad v(x,0) = f(x)$

$$\Downarrow$$

$$f(x) = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{L}$$

$$\Downarrow$$

$$\int_0^L f(x) \sin \frac{n\pi x}{L} dx = \int_0^L \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx$$

$$= \frac{L}{2} B_m \delta_{mn} = \frac{L}{2} B_n \qquad \int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \frac{L}{2} \delta_{mn}$$

$$\Downarrow$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

4.1.8 Solution by the method of separation of variables

Finally, we get the solution

$$\theta(x,t)=\theta_s(x)+v(x,t)$$

$$=\theta_s(x)+\sum_{m=1}^{\infty}B_m\sin\frac{m\pi x}{L}\exp\left(-\frac{m^2\pi^2\kappa}{L^2}t\right)$$

where the steady solution is $\theta_s(x)=\left(1-\frac{x}{L}\right)\theta_1+\frac{x}{L}\theta_2$

D.E. $\frac{\partial\theta}{\partial t}=\kappa\frac{\partial^2\theta}{\partial x^2},\quad 0<x<L,\ 0<t<\infty$

I.C. $\theta(x,0)=g(x),\quad 0<x<L$

B.C. $\theta(0,t)=\theta_1,\quad \theta(L,t)=\theta_2,\quad t>0$

3.4.1 More on diffusion equation and random walk

$u(k)=A(k)e^{i(kx-\omega t)}$ → $u_t=u_{xx}$

Dispersion relation $\omega=-ik^2$

Phase velocity $c=\mathrm{Re}(\omega)/k=0$

$$u(k)=A(k)\exp\left(ikx-k^2t\right)=A\exp\left(-k^2t\right)\exp(ikx)$$

decay quickly as k>>1

$$k=m\frac{2\pi}{\lambda}=m\frac{\pi}{L}$$

$$u(k)\propto\exp\left(-\frac{m^2\pi^2}{L^2}t\right)\exp\left(i\frac{m\pi x}{L}\right)$$

4.1.8 Solution by the method of separation of variables

Example Θ_0  Θ_0

Θ_b

$$\frac{\partial\theta}{\partial t}=\kappa\frac{\partial^2\theta}{\partial x^2}\qquad \theta(x,0)=g(x)=\Theta_b,\quad 0<x<L$$

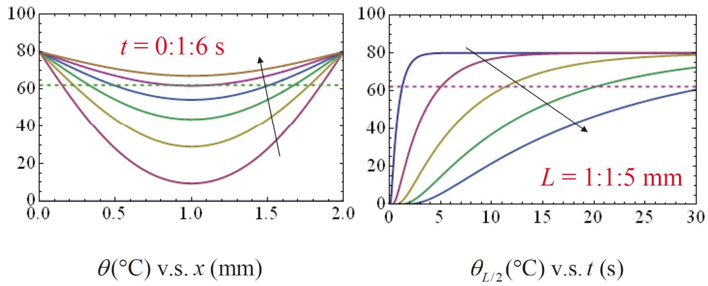
$$\theta(0,t)=\theta(L,t)=\Theta_0,\quad t>0$$

Steady solution $\theta_s(x)=\Theta_0$

Final solution $\theta(x,t)=\Theta_0+\sum_{m=1}^{\infty}B_m\sin\frac{m\pi x}{L}\exp\left(-\frac{m^2\pi^2\kappa}{L^2}t\right)$

where $f(x)=\Theta_b-\Theta_0\qquad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

$\Theta_0=80\qquad \Theta_b=0\qquad \kappa=0.14\,mm^2/s$



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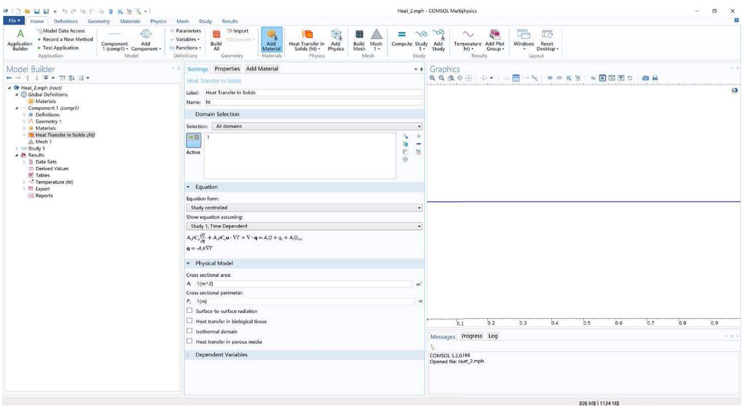


3. 解释和无量纲描述

$\frac{\partial\theta}{\partial t}=\kappa\frac{\partial^2\theta}{\partial x^2}$

$\theta(x,0)=g(x),\quad 0<x<L$

$\theta(0,t)=\theta_1,\quad \theta(L,t)=\theta_2,\quad t>0$

$$\theta(x,t)=\theta_s(x)+\sum_{m=1}^{\infty}B_m\sin\frac{m\pi x}{L}\exp\left(-\frac{m^2\pi^2}{L^2}\kappa t\right)$$

衰减项与 m^2 有关

- 基项 ($m=1$) 比谐波项 ($m>1$) 衰减慢得多，与平衡分布的偏差很快接近正弦形

$$[\kappa]=L^2T^{-1}$$

- 时间尺度 $t_0=L^2/\kappa$，空间尺度 L

无量纲描述

- 引入变量 $\xi=\dfrac{x}{L},\ \tau=\dfrac{t}{t_0}=\dfrac{\kappa t}{L^2}$

$$\Theta(\xi,\tau)=\theta(L\xi,t_0\tau)-\theta_s(L\xi,t_0\tau)$$

比例模型
尺寸： 1:N
x： 放大 N 倍
t： 放大 N^2 倍

- 无量纲方程:

$$\begin{cases}\dfrac{\partial\Theta}{\partial\tau}=\dfrac{\partial^2\Theta}{\partial\xi^2}, & 0<\xi<1,\ 0<\tau<\infty\\ \Theta(\xi,0)=G(\xi) & G(\xi)=g(L\xi)\\ \Theta(0,\tau)=\Theta(1,\tau)=0\end{cases}$$

→ 减少参数

与度量单位无关

$$\Theta(\xi,\tau)=\sum_{m=1}^{\infty}B_m\sin m\pi\xi\cdot e^{-(m\pi)^2\tau}$$

2

扩散到一定距离所需时间的估计

$$\theta(x,t)=\theta_s(x)+\sum_{m=1}^{\infty}B_m\sin\frac{m\pi x}{L}\exp\left(-\frac{m^2\pi^2}{L^2}\kappa t\right)$$

- 扩展得最慢的模式: $\sin\dfrac{\pi x}{L}$
- 热量传播的距离: $d=L/2$ 热衰减问题看成热向 d 处传播
- 衰减因子: $\exp\left(-\dfrac{\pi^2\kappa t}{4d^2}\right)$ $e^{-\frac{\pi^2}{4}}\approx 0.08$
- 扩散所需大致时间: $t=d^2/\kappa$
- 在时间 t 内热扩散的距离近似为 $\sqrt{\kappa t}$

4.2 Fourier's Theorem

$$\frac{\partial v}{\partial t}=\kappa\frac{\partial^2 v}{\partial x^2}$$

B.C. $v(0,t)=v(L,t)$

I.C. $v(x,0)=f(x)$

solution $v(x,t)=\sum_{m=1}^{\infty}B_m\sin\dfrac{m\pi x}{L}\exp\left(-\dfrac{m^2\pi^2\kappa}{L^2}t\right)$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$f(x)=\sum_{m=1}^{\infty}B_m\sin\frac{m\pi x}{L}$$

4.2 Fourier's Theorem

Fourier's Theorem

any periodic (or regularly repeating) wave, however complicated, can be described in terms of an infinite number of sine waves (of various amplitudes and phases) added together

4.2 Fourier's Theorem

Dirichlet conditions: Fourier series converges to $f(t)$ in the interval [-L, L], if

- $f(t)$ is absolutely integrable over the period
- $f(t)$ has a finite number of extrema
- $f(t)$ has a finite number of finite discontinuities

1. Fourier正弦级数的和 $\lim\limits_{N\to\infty}S_N(x)=f(x)$?

- **Fourier 正弦级数:**

$$f(x)=\sum_{m=1}^{\infty}B_m\sin mx,\quad 0<x<\pi \qquad B_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

$$f(x)=\sum_{m=1}^{N}B_m\sin mx+\sum_{m=N+1}^{\infty}B_m\sin mx=S_N(x)+R_N(x)$$

- **级数前 N 项之和:**

$$S_N(x)=\frac{2}{\pi}\int_0^{\pi}K_N(x,\xi)f(\xi)d\xi,\quad 0<x<\pi$$

ξ [xi]

$$K_N(x,\xi)=\sum_{m=1}^{N}\sin(mx)\sin(m\xi),\quad 0<x,\xi<\pi$$

注：为方便，本节取 $L=\pi$，相当于用 L/π 无量纲化，因而不失一般性。

$$\sum_{k=1}^{N}\sin(kx)\sin(k\xi)$$

$$=\sum_{k=1}^{N}\frac{e^{ikx}-e^{-ikx}}{2i}\,\frac{e^{ik\xi}-e^{-ik\xi}}{2i}$$

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i}$$

$$=-\frac{1}{4}\sum_{k=1}^{N}e^{ik(x+\xi)}-e^{ik(x-\xi)}-e^{-ik(x-\xi)}+e^{-ik(x+\xi)}$$

$$x+\xi=X \qquad x-\xi=Y$$

$$=-\frac{1}{4}\sum_{k=1}^{N}e^{ikX}+e^{-ikX}-e^{ikY}-e^{-ikY}$$

$$=-\frac{1}{4}\sum_{k=1}^{N}\begin{array}{l}(\cos kX+i\sin kX)+(\cos kX-i\sin kX)\\-(\cos kY+i\sin kY)-(\cos kY-i\sin kY)\end{array}$$

$$=-\frac{1}{2}\sum_{k=1}^{N}\cos kX-\cos kY$$

consider $\sum_{k=1}^{N}\cos kx$

$$\sin\frac{x}{2}\sum_{k=1}^{N}\cos kx$$

$$=\frac{1}{2}\sum_{k=1}^{N}\sin\frac{(2k+1)x}{2}-\sin\frac{(2k-1)x}{2}$$

$$=\frac{1}{2}\left[\sin\frac{(2k+1)x}{2}-\sin\frac{(2k-1)x}{2}\right]+\left[\sin\frac{(2k-1)x}{2}-\sin\frac{(2k-3)x}{2}\right]+\cdots$$

$$=\frac{1}{2}\sin\frac{(2N+1)x}{2}-\frac{1}{2}\sin\frac{x}{2}$$

$$\boxed{\sum_{k=1}^{N}\cos kx=\frac{\sin(N+1/2)x}{2\sin x/2}-\frac{1}{2}}$$

$$K_N(x,\xi)=\sum_{k=1}^{N}\sin(kx)\sin(k\xi)$$

$$=-\frac{1}{2}\sum_{k=1}^{N}\cos kX-\cos kY$$

$$=-\frac{1}{2}\left(\frac{\sin(N+1/2)X}{2\sin x/2}-\frac{1}{2}-\frac{\sin(N+1/2)Y}{2\sin x/2}-\frac{1}{2}\right)$$

$$=-\frac{1}{2}\left(\frac{\sin(N+1/2)X}{2\sin x/2}-\frac{\sin(N+1/2)Y}{2\sin x/2}\right)$$

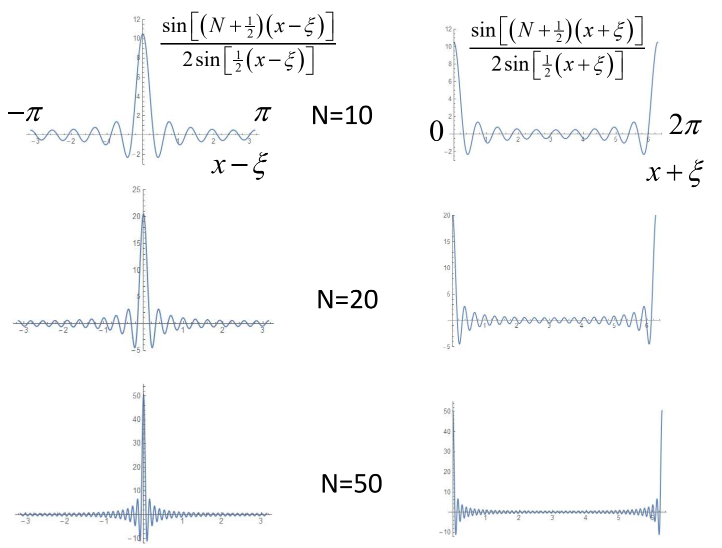
$$=\frac{1}{2}\left(\frac{\sin(N+1/2)(x-\xi)}{2\sin x/2}-\frac{\sin(N+1/2)(x+\xi)}{2\sin x/2}\right)$$

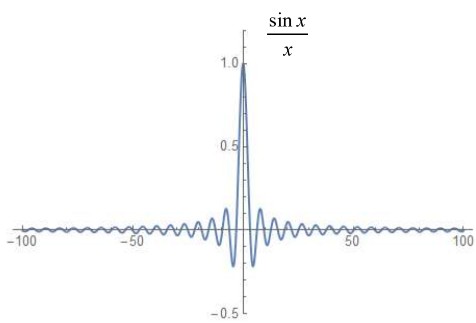
$$S_N(x)=\frac{2}{\pi}\int_0^{\pi}K_N(x,\xi)f(\xi)\,d\xi$$

where Dirichlet's Integral Kernel

$$K_N(x,\xi)=\frac{1}{2}\left\{\frac{\sin\left[\left(N+\frac12\right)(x-\xi)\right]}{2\sin\left[\frac12(x-\xi)\right]}-\frac{\sin\left[\left(N+\frac12\right)(x+\xi)\right]}{2\sin\left[\frac12(x+\xi)\right]}\right\}$$

$x-\xi\to 0$ ⇩ $N+\frac12$ $\qquad$ $x+\xi\to 0,\pi$ ⇩ $N+\frac12$





$$\int_{-\infty}^{\infty}\frac{\sin v}{v}\,dv=\pi$$

$$S_N(x) = S_N^{(1)}(x) + S_N^{(2)}(x) \qquad 0 < x, \xi < \pi$$

where

$$S_N^{(1)}(x,\xi) = \frac{1}{\pi}\int_0^\pi \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{2\sin\left[\frac{1}{2}(x-\xi)\right]} f(\xi)\,d\xi$$

$\Downarrow \; N \to \infty$

$f(x)$

$$S_N^{(2)}(x,\xi) = -\frac{1}{\pi}\int_0^\pi \frac{\sin\left[\left(N+\frac{1}{2}\right)(x+\xi)\right]}{2\sin\left[\frac{1}{2}(x+\xi)\right]} f(\xi)\,d\xi$$

$\Downarrow \; N \to \infty$

0

Lemma 1. If $\phi(\xi)$ is **piecewise smooth** in the closed interval [a,b], then

$$\int_a^b \phi(\xi) \begin{matrix}\sin\\ \cos\end{matrix} \lambda\xi\, d\xi = O(\lambda^{-1})$$

Proof: consider one subdomain $(c, d) \subset [a, b]$

$$\int_c^d \phi(\xi) e^{i\lambda\xi} d\xi = (i\lambda)^{-1}\int_c^d \phi(\xi)\, de^{i\lambda\xi}$$

$$= (i\lambda)^{-1}\phi(\xi)e^{i\lambda\xi}\Big|_c^d - (i\lambda)^{-1}\int_c^d e^{i\lambda\xi}\phi'(\xi)\,d\xi$$

$$= O(\lambda^{-1})$$

Lemma 2. If $\phi(\xi)$ is **piecewise smooth** in the closed interval [a,b], then at a point x_0 where ϕ is continuous,

$$\lim_{\lambda\to\infty}\int_a^b \phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx = \pi\phi(x_0)$$

Proof: consider one subdomain $x_0 \in (c, d) \subset [a, b]$

$$\phi(x) = \phi(x_0) + [\phi(x) - \phi(x_0)] \qquad F(x,x_0) = \frac{\phi(x)-\phi(x_0)}{x-x_0} \to \phi'(x_0)$$

$$I(\lambda) = \phi(x_0)\int_a^b \frac{\sin\lambda(x_0-x)}{x_0-x}dx + \int_a^b F(x,x_0)\sin\lambda(x-x_0)\,dx$$

$$\xrightarrow{\lambda\to\infty} \pi\phi(x_0)$$

$$\int_{-\infty}^{\infty}\frac{\sin v}{v}dv = \pi$$

Lemma-1

Lemma 3. At a point x_0 where $\phi(x)$ has a jump discontinuity, Lemma-2 holds, provided that we define $\phi(x_0)$ to be the average of the left- and right-hand limits of $\phi(\xi)$ as $\xi \to x_0$

$$\phi(x_0) = \frac{1}{2}\left[\phi(x_0-0) + \phi(x_0+0)\right]$$

Proof: break interval (a, b) into (a, x_0) and (x_0, b)

$$\int_{-\infty}^{\infty}\frac{\sin v}{v}dv = \int_{-\infty}^{0} + \int_0^\infty = \pi \quad \Rightarrow \quad \int_{-\infty}^{0}\frac{\sin v}{v}dv = \int_0^\infty \frac{\sin v}{v}dv = \frac{\pi}{2}$$

because $\displaystyle\int_{-\infty}^{0}\frac{\sin v}{v}dv = \int_{\infty}^{0}\frac{\sin(-u)}{(-u)}d(-u) = -\int_{\infty}^{0}\frac{\sin u}{u}du = \int_0^\infty \frac{\sin u}{u}du$

$$\int_a^b \phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx = \int_a^{x_0-\varepsilon}\phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx + \int_{x_0+\varepsilon}^b \phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx$$

$$\int_a^{x_0-\varepsilon}\phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx = \int_a^{x_0-\varepsilon}\phi(x)\frac{\sin\lambda(x-x_0)}{\lambda(x-x_0)}d\lambda(x-x_0)$$

$$= \int_{\lambda(a-x_0)}^{-\lambda\varepsilon}\phi\left(\frac{v}{\lambda}+x_0\right)\frac{\sin v}{v}dv \xrightarrow{\lambda\to\infty,\varepsilon\to 0} \phi(x_0-0)\int_{-\infty}^{0}\frac{\sin v}{v}dv = \frac{\pi}{2}\phi(x_0-0)$$

$$\int_{x_0+\varepsilon}^b \phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx = \int_{x_0+\varepsilon}^b \phi(x)\frac{\sin\lambda(x-x_0)}{\lambda(x-x_0)}d\lambda(x-x_0)$$

$$\xrightarrow{\lambda\to\infty,\varepsilon\to 0} \phi(x_0+0)\int_0^\infty \frac{\sin v}{v}dv = \frac{\pi}{2}\phi(x_0+0)$$

$$\lim_{\lambda\to\infty}\int_a^b \phi(x)\frac{\sin\lambda(x_0-x)}{x_0-x}dx = \frac{\pi}{2}\left[\phi(x_0-0)+\phi(x_0+0)\right] = \pi\phi(x_0)$$

To evaluate $\quad S_N(x) = S_N^{(1)}(x) + S_N^{(2)}(x)$

using Lemma-2

$$S_N^{(1)}(x,\xi) = \frac{1}{\pi}\int_0^\pi \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{2\sin\left[\frac{1}{2}(x-\xi)\right]} f(\xi)\,d\xi$$

$$= \frac{1}{\pi}\int_0^\pi \left\{f(\xi)\frac{x-\xi}{2\sin\left[\frac{1}{2}(x-\xi)\right]}\right\}\frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{x-\xi}d\xi$$

$$= \frac{1}{\pi}\int_0^\pi \phi(\xi)\frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{x-\xi}d\xi$$

$$\xrightarrow{N\to\infty} \frac{1}{\pi}\int_0^\pi \phi(\xi)\delta(x-\xi)\,d\xi = \phi(x)$$

$$0 < x, \xi < \pi$$

$$\delta(x) = \lim_{\lambda\to\infty}\frac{\sin\lambda x}{\pi x}$$

$$\int_{-\infty}^{\infty}\frac{\sin\lambda x}{\pi x}f(x)\,dx \to \int_{-\infty}^{\infty}\delta(x)f(x)\,dx = f(0)$$

using Lemma-1

$$S_N^{(2)}(x,\xi) = -\frac{1}{\pi}\int_0^{\pi}\frac{\sin\left[\left(N+\frac{1}{2}\right)(x+\xi)\right]}{2\sin\left[\frac{1}{2}(x+\xi)\right]}\, f(\xi)\,d\xi$$
$$= \frac{1}{\pi}\int_0^{\pi}\frac{f(\xi)}{2\sin\left[\frac{1}{2}(x+\xi)\right]}\sin\left[\left(N+\tfrac{1}{2}\right)(x+\xi)\right]d\xi$$
$$= \frac{1}{\pi}\int_0^{\pi}\phi(\xi)\sin\left[\left(N+\tfrac{1}{2}\right)(x+\xi)\right]d\xi \xrightarrow{N\to\infty} 0 \qquad 0 < x,\xi < \pi$$

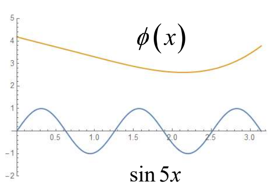
$$\phi(\xi) = \frac{f(\xi)}{2\sin\left[\frac{1}{2}(x+\xi)\right]}$$

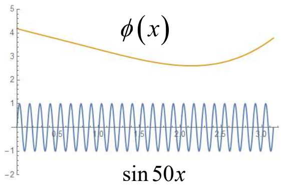
$x+\xi$ is finite and >0

$\phi(\xi)$ is finite

$$\sin\left[N(x+\xi)\right] = \sin\left[N\left(x+\xi+\frac{2\pi}{N}\right)\right] \qquad \text{period} \quad T_\xi = \frac{2\pi}{N} \to 0$$

$\phi(\xi)$ is almost constant in a short period





2. 全范围和半范围 Fourier 级数

- 全范围 Fourier 级数

或周期函数

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx), \qquad -\pi < x < \pi$$

逐段光滑

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

- 复数形式： $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$ $\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi$

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \overline{c_n}$$

The sum of the first N terms

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi$$
$$= \sum_{n=-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}e^{inx}d\xi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\sum_{n=-N}^{N} e^{in(x-\xi)}d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)K_N(x,\xi)\,d\xi$$

where

$$K_N(x,\xi) = \sum_{n=-N}^{N} e^{in(x-\xi)} = \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{2\sin\left[\frac{1}{2}(x-\xi)\right]}$$

半范围 Fourier 级数

$f(x)$ 定义在 $(0,\pi)$ 或周期函数

- Fourier 正弦级数 奇函数

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

- Fourier 余弦级数 偶函数

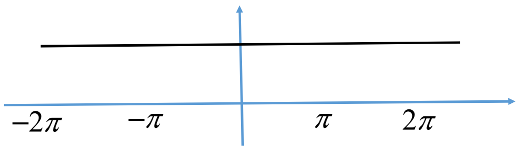
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

- Fourier 级数:
  - ① *f(x)* 定义在一个有限区间或周期函数
  - ② 即使 *f(x)* 连续，边界仍可间断
  - ③ 离散谱

4.3 On the nature of Fourier series

- 1. 一些函数的 Fourier 级数
- 2. Fourier 级数的积分和微分
- 3. Gibbs 现象
- 4. 最小二乘误差近似
- 5. Bessel 不等式和 Parseval 定理
- 6. Parseval 定理的应用

Constant function f(x)=1



- full range series $-\pi < x < \pi$ $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

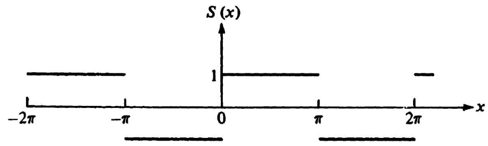
$$f(x)=\frac{a_0}{2}=1 \qquad a_n = b_n = 0 \quad n \ge 1$$

- Half range Cosine series $0 < x < \pi$ $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$

$$f(x)=\frac{a_0}{2}=1 \qquad a_n = 0 \quad n \ge 1$$

- Half range Sine series $0 < x < \pi$ $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$

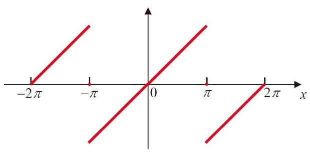
$$f(x)=1=\frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$$



$$S(x)=\begin{cases}1 & -\pi < x < 0 \\ -1 & 0 < x < \pi\end{cases}$$

Square wave function

Linear function f(x)=x



- Full range series $-\pi < x < \pi$

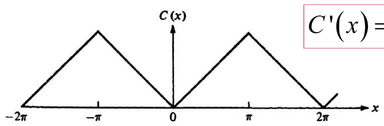
$$f(x)=x=2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)$$

- Half range Sine series $0 < x < \pi$

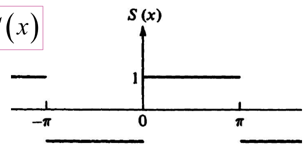
$$f(x)=x=2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right)$$

- Half range Cosine series $0 < x < \pi$

$$f(x)=x=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right)$$



$C'(x)=S(x)$



$$C(x)=|x|, \quad -\pi < x < \pi$$

$$\int_0^x S(x)\,dx = C(x)=\frac{4}{\pi}\left[(1-\cos x)+\frac{1}{3^2}(1-\cos 3x)+\frac{1}{5^2}(1-\cos 5x)+\cdots\right]$$

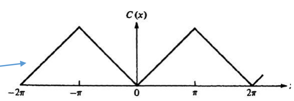
$$C(\pi)=\pi=\frac{8}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right) \quad \Rightarrow \quad \pi^2=8\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

Parseval's theorem $\sum_{n=-\infty}^{\infty}|c_n|^2=\langle f^2\rangle$ $f(x)\frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\cdots\right)$

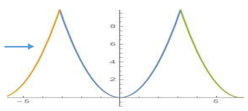
Quadratic function f(x)=x²

- Sine series $0 < x < \pi$

$$C(x)=|x|=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}-\frac{\cos 3x}{3^2}+\cdots\right)$$



$$\int_0^x C(x)\,dx=\frac{x^2}{2}=\frac{\pi}{2}x-\frac{4}{\pi}\left(\frac{\sin x}{1^2}+\frac{\sin 3x}{3^2}+\cdots\right)$$



$$x=\frac{\pi}{2}, \quad \frac{\pi^3}{32}=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots$$

To computer Pi

2. Fourier 级数的积分和微分

- 令 $f(x)$ 为逐段光滑的函数，其 Fourier 级数为

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

- 级数在 $(-\pi, \pi)$ 是收敛的，但非一致收敛，不能保证其能逐项积分和逐项微分，需进一步讨论
- 逐项积分是容许的；不能逐项微分的例子：

$$1=\frac{4}{\pi}\left\{\sin x+\frac{1}{3}\sin 3x+\cdots\right\} \Rightarrow 0 \times \frac{4}{\pi}(\cos x+\cos 3x+\cdots)$$

$$x=2\left\{\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\cdots\right\} \Rightarrow 1 \times 2(\cos x-\cos 2x+\cos 3x-\cdots)$$

Formal integration of $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$

$$F(x)=\int_{-\pi}^{x}f(x)dx\sim\left(\frac{a_0}{2}x+\sum_{n=1}^{\infty}\frac{b_n}{n}\right)\quad+\quad\sum_{n=1}^{\infty}-\frac{b_n}{n}\cos nx+\frac{a_n}{n}\sin nx$$

Because *f*(x) is a piecewise smooth function, *F*(x) is continuous and we have Fourier's series

$$F(x)-\frac{a_0}{2}x=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos nx+B_n\sin nx$$

where

$$A_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[F(x)-\frac{a_0x}{2}\right]dx=2\sum_{n=1}^{\infty}\frac{b_n}{n}$$

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[F(x)-\frac{a_0x}{2}\right]\cos nxdx=-\frac{b_n}{n},\quad n\neq 0$$

$$B_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[F(x)-\frac{a_0x}{2}\right]\sin nxdx=\frac{a_n}{n},\quad n\neq 0$$

Term-by-term integration is permissible

Formal differentiation of $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$

$$f'(x)\sim 0+\sum_{n=1}^{\infty}nb_n\cos nx-na_n\sin nx$$

If *f'*(x) is continuous and we have Fourier's series

$$f'(x)=\frac{a_0'}{2}+\sum_{n=1}^{\infty}a_n'\cos nx+b_n'\sin nx$$

where

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos nxdx=nb_n+\frac{1}{\pi}(-1)^n\left[f(\pi)-f(-\pi)\right]$$

$$b_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\sin nxdx=-na_n$$

Term-by-term differentiation is permissible, only if

- *f*(x) is periodic and has no discontinuity anywhere, including -π or +π
- the resultant series of *f'*(x) converges

$$a_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\cos nxdx$$

$$=\frac{1}{\pi}f(x)\cos nx\Big|_{-\pi}^{\pi}+\frac{n}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}(-1)^n\left[f(\pi)-f(-\pi)\right]+\frac{n}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

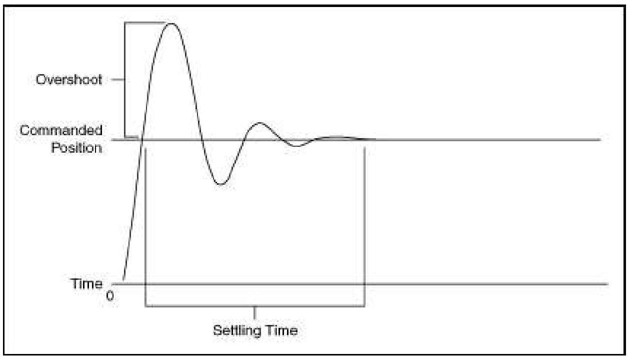
$$=\frac{1}{\pi}(-1)^n\left[f(\pi)-f(-\pi)\right]+nb_n$$

$$b_n'=\frac{1}{\pi}\int_{-\pi}^{\pi}f'(x)\sin nxdx$$

$$=\frac{1}{\pi}f(x)\sin nx\Big|_{-\pi}^{\pi}-\frac{n}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$=-\frac{n}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=-na_n$$

Gibbs phenomena



Over-shoot is a general phenomenon

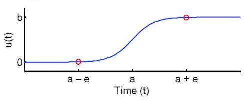
Gibbs phenomena

A function, $v(t)$, has a discontinuity of amplitude b at $t=a$ if

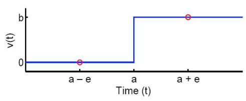
$$\lim_{e\to 0}\left(v(a+e)-v(a-e)\right)=b\neq 0$$

Conversely, $v(t)$, is continuous at $t=a$ if the limit, b, equals zero.

Examples:



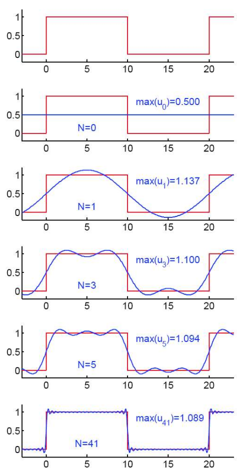
Continuous



Discontinuous

We will see that if a periodic function, $v(t)$, is discontinuous, then its Fourier series behaves in a strange way.

Example: Square wave



$$S(x)=\frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$$

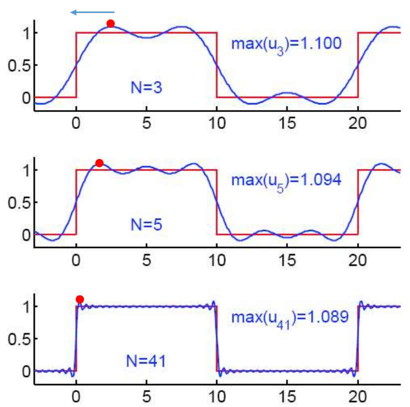
$$=\frac{4}{\pi}\sum_{k=1}\frac{\sin(2k-1)x}{2k-1}$$

as x<<1 and $(2k-1)x<\pi$,

$$\frac{\sin(2k-1)x}{2k-1}>0$$

Otherwise, the oscillatory behavior is evident

The first maximum: Position and Magnitude

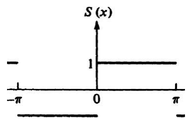


The sum of the first N terms

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi) K_N(x,\xi)\, d\xi \qquad -\pi < x < \pi$$

where

$$K_N(x,\xi) = \sum_{n=-N}^{N} e^{in(x-\xi)} = \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{\sin\left[\frac{1}{2}(x-\xi)\right]}$$



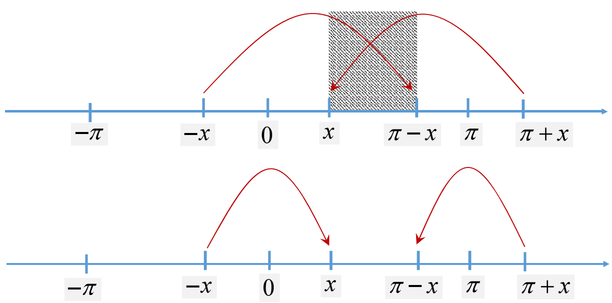
In the case of square wave

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi) K_N(x,\xi)\, d\xi = \frac{1}{2\pi}\left(-\int_{-\pi}^{0} + \int_{0}^{\pi}\right) \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-\xi)\right]}{\sin\left[\frac{1}{2}(x-\xi)\right]} d\xi$$

$$= \frac{1}{2\pi}\left(\int_{x+\pi}^{x} + \int_{-x}^{\pi-x}\right) \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\left[\frac{1}{2}\theta\right]} d\theta \qquad \theta = x-\xi, \quad \theta = \xi - x$$

$$= \frac{1}{2\pi}\left(\int_{-x}^{x} + \int_{\pi+x}^{\pi-x}\right) \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\left[\frac{1}{2}\theta\right]} d\theta$$

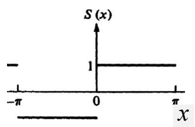
$$\int_{x+\pi}^{x} + \int_{-x}^{\pi-x} \Rightarrow \int_{-x}^{x} + \int_{\pi+x}^{\pi-x}$$



The behavior of the partial sum $S_N(x)$ near $x = 0+$.

As x is small (from 0+)

$$S_N(x) = \frac{1}{2\pi}\left(\int_{-x}^{x} + \int_{\pi+x}^{\pi-x}\right) \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\left[\frac{1}{2}\theta\right]} d\theta$$



since

$$\int_{\pi+x}^{\pi-x} \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\left[\frac{1}{2}\theta\right]} d\theta = O\left(N^{-1}\right)$$

≈ 1

Lemma-1

$$\int_a^b \phi(\xi) \begin{matrix}\sin\\ \cos\end{matrix} \lambda\xi\, d\xi = O\left(\lambda^{-1}\right)$$

$$N + \tfrac{1}{2} = m, \quad m\theta = \eta$$

$$S_N(x) \approx \frac{1}{2\pi}\int_{-x}^{x} \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\left[\frac{1}{2}\theta\right]} d\theta$$

$$= \frac{1}{2\pi}\int_{-mx}^{mx} \frac{\sin\eta}{m\sin(\eta/2m)} d\eta$$

$$= I_N(x)$$

define

$$I_N(x) = \frac{1}{2\pi}\int_{-mx}^{mx} \frac{\sin\eta}{m\sin(\eta/2m)} d\eta$$

$$\int_{-\infty}^{\infty} \frac{\sin v}{v} dv = \pi$$

for any fixed small $x > 0$

$$I_N(x) = \frac{1}{\pi}\int_{-mx}^{mx} \frac{\sin\eta}{\eta} \frac{\eta/2m}{\sin(\eta/2m)} d\eta \xrightarrow{N+\frac{1}{2}=m\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\eta}{\eta} d\eta = 1$$

1

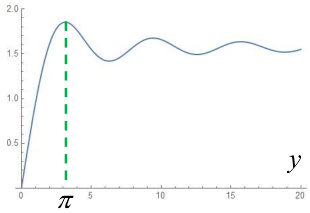
for any fixed $y = mx = \left(N+\frac{1}{2}\right)x, \quad x = O\left(N^{-1}\right)$

$$I_N(x) = \frac{1}{\pi}\int_{-mx}^{mx} \frac{\sin\eta}{\eta} \frac{\eta/2m}{\sin(\eta/2m)} d\eta$$

$$\xrightarrow{N+\frac{1}{2}=m\to\infty} \frac{2}{\pi}\int_{0}^{mx} \frac{\sin\eta}{\eta} d\eta = \frac{2}{\pi} Si(mx)$$

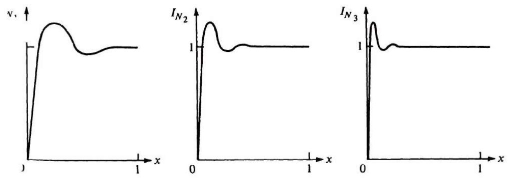
Sine integral function $\quad Si(y) = \int_0^y \frac{\sin\eta}{\eta} d\eta$

$Si(y)$



$$S_{\max} = Si(\pi) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin\eta}{\eta} d\eta \approx 1.179$$

$$mx = \pi \to x = \pi / (N + \tfrac{1}{2})$$



$$S_N(x) \approx I_N(x) = \frac{1}{2\pi}\int_{-mx}^{mx} \frac{\sin\eta}{m\sin(\eta/2m)} d\eta$$

Least Square Approximation and Fourier Series

Approximate f(x) with

$$f(x) \approx \tilde{f}(x) = \sum_{n=0}^{N} \gamma_n \phi_n(x)$$

Fourier Series

$$f(x) = \sum_{n=0}^{N} c_n \phi_n(x)$$

with $\phi_n(x)$ normalized orthogonal function

$$(\phi_n, \phi_m) = \int_a^b w(x)\phi_n(x)\overline{\phi}_m(x)dx = \delta_{mn}$$

$$\gamma_n = (\tilde{f}, \phi_n) = \left(\sum_{m=0}^{N} \gamma_m \phi_m, \phi_n\right) = \sum_{m=0}^{N} \gamma_m (\phi_m, \phi_n) = \sum_{m=0}^{N} \gamma_m \delta_{mn} = \gamma_n$$

$$c_n = (f, \phi_n)$$

We seek for γ_n so as to minimize $M = \dfrac{(f-\tilde{f}, f-\tilde{f})}{\int_a^b w dx}$

$$M\int_a^b w dx = (f-\tilde{f}, f-\tilde{f})$$
$$= (f,f) - (f,\tilde{f}) - (\tilde{f},f) + (\tilde{f},\tilde{f})$$
$$= \langle f^2\rangle_w - 2\left(f, \sum_{m=0}^{N}\gamma_m\phi_m\right) + \left(\sum_{m=0}^{N}\gamma_m\phi_m, \sum_{m=0}^{N}\gamma_m\phi_m\right)$$
$$= \langle f^2\rangle_w - 2\sum_{m=0}^{N}\overline{\gamma}_m(f,\phi_m) + \sum_{m=0}^{N}\sum_{n=0}^{N}\gamma_m\overline{\gamma}_n(\phi_m,\phi_n)$$
$$= \langle f^2\rangle_w - 2\sum_{m=0}^{N}\overline{\gamma}_m c_m + \sum_{m=0}^{N}\|\gamma_m\|^2$$
$$= \langle f^2\rangle_w + \sum_{m=0}^{N}\|\gamma_m - c_m\|^2 - \sum_{m=0}^{N}\|c_m\|^2$$

$$\langle f^2\rangle_w = \int_a^b f^2(x)w(x)dx$$

$$\frac{\partial M}{\partial \gamma_n} = 2\int_a^b (f-\tilde{f})\frac{\partial \tilde{f}}{\partial \gamma_n} w dx \qquad n+1 \quad \text{equations}$$

5. Bessel 不等式和 Parseval 定理

$$\langle f^2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

- 当 $\gamma_n = c_n$ 时，部分和的均方误差：$M = \langle f^2\rangle - \sum_{n=-N}^{N}|c_n|^2 \ge 0$

$$\sum_{n=-N}^{N}|c_n|^2 \le \langle f^2\rangle \qquad \frac{1}{2}a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \le 2\langle f^2\rangle$$

Bessel不等式

- 实域中，有 $c_0 = \frac{a_0}{2}$，$c_n = \frac{1}{2}(a_n - ib_n)$，$c_{-n} = \overline{c}_n$
- 如除了有限个间断点，Fourier 级数在逐点意义上收敛，则

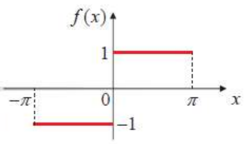
$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \langle f^2\rangle \qquad \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = 2\langle f^2\rangle$$

Parseval 恒等式

即使当 $N \to \infty$ 时，均方误差 M 不为 0，系数级数仍收敛

6. Parseval 定理的应用

- 例1：$f(x) = \text{sign}(x), \quad -\pi < x < \pi$



- Fourier 级数展开：$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$ $\quad \langle f^2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx$

$$f(x) \sim \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

- Parseval 定理：$\frac{1}{2}a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) = 2\langle f^2\rangle$

$$a_n = 0, b_{2m} = 0, b_{2m-1} = \frac{1}{2m-1}, \quad \langle f^2(x)\rangle = 1$$

- 估算 π 值：$\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

项数	π
1	2.8284
2	2.9814
3	**3**.0346
8	**3.1**016
200	**3.14**00
538	**3.141**0