Several Complex Variables (SCV)

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References books.

[H]. Hörmander. An Introduction to Complex Analysis in Several Variables. 3rd ed 1990.

[D1] Demailly. Complex Analytic and Differential Geometry. 2012.

[D2] L^2 Estimates for the $\bar\partial$-Operator on Complex Manifolds 1996

[K] Krantz. Function Theory of Several Complex Variables 2nd.

[R] Range. Holomorphic Functions and Integral Representation in SCV, 1986. (GTM 108)

[T] 涂振汉. 多元复分析. 2015. (武汉大学)

[C] 陈伯勇. Cauchy-Riemann 方程的 L^2 理论. 2022.

本课程主要参考 [H]、[D1]. [D2].

少量作业，不要求交

考试：期末，开闭卷待定.

多复变的起源

· In 1895, Cousin considered the generalization to SCV of the Mittag-Leffler thm & the Weierstrass thm in OCV.

(Recall: Mittag-Leffler thm. 亚纯函数的极点和主部可被预定.
Weierstrass: 全纯函数的零点及阶数可被预定)

Cousin posed what are now called the first & the second Cousin problems.

$\rightsquigarrow$ Sheaf theory (Leray 1940s末).

Cohomology theory of coherent analytic Sheaves.

(Oka. Cartan. Serre. 1950s)

· Stein manifolds (Stein. 1950s)

Cousin I problems: $H^1(X, \mathcal{O})$

... II ......: $H^1(X, \mathcal{O}^*) \leftarrow$ Picard Group.

· In 1906, Hartogs found a new phenomenon (Hartogs phenomenon): $\exists$ domains $\Omega_1 \subsetneq \Omega_2 \subset \mathbb{C}^n$ $(n>1)$, s.t.

$\mathcal{O}(\Omega_1) = \mathcal{O}(\Omega_2)|_{\Omega_1}$ (- marks the beginning of a genius SCV theory)

Rmk. In OCV, no Hartogs phenomenon.
$\Leftarrow$ (ML+W)

$\rightsquigarrow$ holomorphically convex domains, domains of holomorphy & pseudoconvex domains

e.g. (i). $\mathbb{C}\setminus\{0\} = \Omega_1$. $\mathbb{C} = \Omega_2$. $f(z) = \frac{1}{z}$. then

$$f \in \mathcal{O}(\Omega_1), \quad f \notin \mathcal{O}(\Omega_2)\big|_{\Omega_1}.$$

(ii) When $n>1$. $\mathcal{O}(\mathbb{C}^n\setminus\{0\}) = \mathcal{O}(\mathbb{C}^n)\big|_{\mathbb{C}^n\setminus\{0\}}$.

Def. $\Omega \subset \mathbb{C}$ domain. $f : \Omega \to \mathbb{C}$. 称 f 是 holomorphic fns. if $f \in C^1(\Omega)$ & $\frac{\partial f}{\partial \bar{z}_1} = \cdots = \frac{\partial f}{\partial \bar{z}_n} = 0$ on Ω.

Pf of (ii). WLOG, suppose $n=2$. Given $f \in \mathcal{O}(\mathbb{C}^2\setminus\{0\})$.

We define

$$g(z,w) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta, w)}{\zeta - z}\, d\zeta, \quad (z,w) \in \Delta \times \mathbb{C}$$

Δ: Unit disc in $\mathbb{C}$.

$\Rightarrow g \in \mathcal{O}(\Delta \times \mathbb{C})$.

The Cauchy integral formula in OCV $\Rightarrow g = f$ on $\Delta \times \mathbb{C}\setminus\{0\}$ $\xRightarrow{\text{identity principle}}$ $g = f$ on $(\Delta\times\mathbb{C}) \cap (\mathbb{C}^2\setminus\{0\}) = (\Delta\times\mathbb{C})\setminus\{0\}$.

$$\Rightarrow F \triangleq \begin{cases} f & \text{on } \mathbb{C}^2\setminus\{0\} \\ g & \text{on } \Delta\times\mathbb{C}. \end{cases} \quad ※$$

In 1907. Poincaré proved:

Thm. If $n>1$, the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ & the unit polydisc $\Delta^n \triangleq \underbrace{\Delta \times \cdots \times \Delta}_{n \text{个}}$ are not biholomorphically equivalent.

Rmk. (i). $\mathbb{B}^n$ & Δ^n convex, hence are topologically equivalent.

(ii). ~~In SCV~~, Poincaré's thm indicates: The analogue of Riemann mapping thm in SCV does not hold in general.

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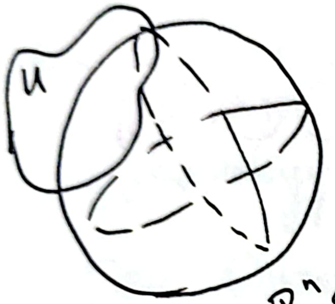
Thm. $U \subset \mathbb{C}^n (n>1)$ domain, $U \cap \partial \mathbb{B}^n \neq \phi$ & $U \cap \mathbb{B}^n$ connected. If $f: U \to \mathbb{C}^n$ holo. nonconstant & $f(U \cap \partial \mathbb{B}^n) \subset \partial \mathbb{B}^n$, then $f \in Aut(\mathbb{B}^n)$.



$\mathbb{B}^n \subset \mathbb{C}^n$

$\rightsquigarrow$ CR geometry (Cartan, 陈 - Moser, Fefferman, Webster, etc…).

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In 1910, Levi introduced what is now called the Levi pseudo convexity → 实凸性的复版本

proved: every domain of holomorphy in $\mathbb{C}^n$ with C^2 bdy is Levi, etc. Posed the following famous problem:

Levi problem: Is every psc domain in $\mathbb{C}^n$ a domain of holomorphy?

Ans. Yes! $n=2 \Rightarrow$ (Oka 1942)

$n \geq 2 \Rightarrow$ (Oka, 1953; Bremermann & Norguet 1954).

Plan of Course.

Ch1. Basis of SCV.

Ch2. (Pluri-) Subharmonic function & pseudoconvexity

Ch3. Hörmander's L^2- estimates for the $\bar\partial$- operator.

Ch4. Ohsawa - Takegoshi L^2- extension thm.

Ch5. L^2- estimates on complete Kähler manifolds.

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Ch1. Basis of SCV.

§1.1. Preliminaries.

Start with some notation. $(1\le j\le n)$.

$$\mathbb{C}^n = \underbrace{\mathbb{C}\times\mathbb{C}\times\cdots\times\mathbb{C}}_{n\geq 2}. \quad z=(z_1,\cdots,z_n).\quad z_j = x_j + iy_j.$$

Define $\dfrac{\partial}{\partial z_j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_j} - i\dfrac{\partial}{\partial y_j}\right)$, $\dfrac{\partial}{\partial \bar z_j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_j} + i\dfrac{\partial}{\partial y_j}\right)$.

$dz_j = dx_j + i dy_j$, $\quad d\bar z_j = dx_j - i dy_j$.

Given $\Omega\subset\mathbb{C}^n$ (domain), $f\in C^1(\Omega;\mathbb{C})$. we define

$$\left.\begin{aligned}\partial f &\triangleq \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \cancel{+ \frac{\partial f}{\partial \bar z_j} d\bar z_j}\\ \bar\partial f &\triangleq \sum_{j=1}^n \frac{\partial f}{\partial \bar z_j} d\bar z_j\end{aligned}\right\} \Rightarrow d = \partial + \bar\partial.$$

anti- $\mathbb{C}$-linear part.

$\mathbb{C}$-linear part

$$df = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j\right) = \sum_{j=1}^n \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar z_j} d\bar z_j\right).$$

Def. A function $f \in C^1(\Omega)$ is called holo if it solves the C-R equation (or $\bar{\partial}$- equation) $\bar{\partial} f = 0$, or equivalently $\frac{\partial f}{\partial \bar{z}_1} = \cdots = \frac{\partial f}{\partial \bar{z}_n} = 0$. denoted by $f \in \mathcal{O}(\Omega)$.

Prop.