A FAMILY OF HIGHER ORDER ISOPERIMETRIC INEQUALITIES

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Abstract In this paper we prove sharp inequalities between the volume and the integral of the $k-th$ mean curvature for $k+1$-convex domains in the Euclidean space for all k. We generalize the recent results of Sun-Yang A. Chang and Yi Wang [CW13] where they prove the case $k=1,2$. The idea is the same as [CW13], but calculations are involved.

1. Introduction

Classical isoperimetric problem is to determine a plane figure of the largest possible area whose boundary has a specified length. The solution to the isoperimetric problem is given by a circle and was known already in ancient Greece. However, the first mathematically rigorous proof of this fact was obtained only in the 19th century [Sch84]. Since then, many other proofs have been found, some of them stunningly simple. Gromov [Gro85] established the inequality by constructing a map from the domain to the unit ball and applying the divergence theorem. Inspired by his idea, many people [McC97], [CENV04], [FMP10] used optimal transport method to establish various sharp geometric inequalities.

People generalized this classical problem to higher dimensions and higher order curvature integrals. We assume Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Then we define higher order curvature integrals as follows. $V_n(\Omega)=vol(\Omega)$, $V_{n-k-1}(\Omega):=\int_{\partial\Omega}\sigma_k(L)d\mu$, for $k=0,\cdots,n-1$, where $d\mu$ is surface area of $\partial\Omega$, L is the second fundamental form on $\partial\Omega$, and $\sigma_k(L)$ will be defined in (2.4). As Alexandrov-Fenchel inequalities [Ale37] state: if Ω is a convex domain in $\mathbb{R}^n$ with smooth boundary, then for $0\leq k\leq n-1$,

$$(\frac{V_{n-k}(\Omega)}{V_{n-k}(B)})^{\frac{1}{n-k}}\leq(\frac{V_{n-1-k}(\Omega)}{V_{n-1-k}(B)})^{\frac{1}{n-k-1}}. \tag{1.1}$$

where B is the n-th unit ball in $\mathbb{R}^n$.

It is natural to ask whether these inequalities are true when Ω is k-convex, i.e. L only lies in Γ_k (defined in (2.3)). Under the additional assumption that Ω is a star-shaped domain in $\mathbb{R}^n$, Guan-Li [GL09] proved inequality (1.1). Inequalities of type (1.1) were also discussed in Trudinger [Tru94]. Castillon [Cas10] used the method of optimal transport to give a new proof of the Michael-Simon inequality, which in particular implies (1.1) for $k=1$ with some constant $C(n)$. In [CW11, CW], Chang and Wang established inequalities of the type (1.1) for all $l\leq k$ with some constant $C(n,k)$ when Ω is $(k+1)$-convex. Recently they [CW13] have established

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the following version of the sharp inequalities in (1.1).

Theorem 1.1. [CW13] Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary. Suppose Ω is 2-convex, i.e. $L \in \Gamma_2$. Then

$$\frac{V_n(\Omega)^{\frac{n-2}{n}}}{V_{n-2}(\Omega)} \leq \frac{V_n(B)^{\frac{n-2}{n}}}{V_{n-2}(B)}. \tag{1.2}$$

The constant in the inequality is sharp and equality holds only when Ω is a ball in $\mathbb{R}^n$.

Theorem 1.2. [CW13] Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary. Suppose Ω is 3-convex, i.e. $L \in \Gamma_3$. Then

$$\frac{V_n(\Omega)^{\frac{n-3}{n}}}{V_{n-3}(\Omega)} \leq \frac{V_n(B)^{\frac{n-3}{n}}}{V_{n-3}(B)}. \tag{1.3}$$

The constant in the inequality is sharp and equality holds only when Ω is a ball in $\mathbb{R}^n$.

In this paper, we can prove the following theorem for general k by using the same method as in [CW13].

Theorem 1.3. Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. For $1 \leq k \leq n-2$, suppose Ω is $k+1$-convex, i.e. $L \in \Gamma_{k+1}$. Then

$$\frac{V_n(\Omega)^{\frac{n-k-1}{n}}}{V_{n-k-1}(\Omega)} \leq \frac{V_n(B)^{\frac{n-k-1}{n}}}{V_{n-k-1}(B)}. \tag{1.4}$$

The constant in the right hand side of the inequality is sharp and equality holds only when Ω is a ball in $\mathbb{R}^n$.

We note that the case $k = 0$ is the higher dimensional classical isoperimetric inequality. And if $k = n-1$, then the inequality (1.4) holds when Ω is $n-1$-convex, i.e. $\partial\Omega$ is convex. This can be seen from Alexandrov-Fenchel inequalities [Ale37].

In this paper, we use the same notation as in [CW13], and sum the repeated indices. The organization of the paper is as follows. After definitions and some facts in section 2, we use the method of optimal transport to reduce Theorem 1.3 to the Proposition. Then we prove the Proposition in section 3.

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2. Preliminaries

2.1 Definitions

Definition 2.1. The k-th elementary symmetric function for $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ is

$$\sigma_k(\lambda) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{2.1}$$

Definition 2.2. The Gårding k-cone is

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}. \tag{2.2}$$

Definition 2.3. A symmetric matrix A lies in Γ_k cone, if its eigenvalues

$$\lambda(A) = (\lambda_1(A), \cdots, \lambda_n(A)) \in \Gamma_k. \tag{2.3}$$

If we denote $\sigma_k(\lambda(A))$ by $\sigma_k(A)$, so

$$\sigma_k(A) = \frac{1}{k!} \delta^{i_1, \cdots, i_k}_{j_1, \cdots, j_k} A_{i_1 j_1} \cdots A_{i_k j_k}. \tag{2.4}$$

Definition 2.4. The Newton transformation tensor is defined as

$$[T_k]_{ij}(A_1, \cdots, A_k) := \frac{1}{k!} \delta^{i, i_1, \cdots, i_k}_{j, j_1, \cdots, j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \tag{2.5}$$

When $A_1 = \cdots = A_k = A$ we denote $[T_k]_{ij}(A_1, \cdots, A_k)$ by $[T_k]_{ij}(A)$ for simplicity. From this definition one can easily show that $\partial_j([T_k]_{ij}(A)) = 0$, provided that A_{ijk} is symmetric with respect to the indices. And this property will be used frequently later.

Definition 2.5. We define polarization of σ_k to be

$$\sigma_k(A_1, \cdots, A_k) := \frac{1}{k!} \delta^{i_1, \cdots, i_k}_{j_1, \cdots, j_k} (A_1)_{i_1 j_1} \cdots (A_k)_{i_k j_k}. \tag{2.6}$$

We note that

$$\sigma_k(A, \cdots, A) = \sigma_k(A), \tag{2.7}$$

and from the definition, it is easy to see that

$$\sigma_{k+1}(A) = \frac{1}{k+1} A_{ij} [T_k]_{ij}(A). \tag{2.8}$$

2.2 Simple facts

Lemma 2.6.

$$\begin{aligned} l[T_{k-1}]_{\alpha\beta}(\overbrace{A, \cdots, A}^{l-1}, B, \cdots, B) A_{\gamma\beta} = &- k[T_k]_{\alpha\gamma}(\overbrace{A, \cdots, A}^{l}, B, \cdots, B) \\ &- (k-l)[T_{k-1}]_{\alpha\beta}(\overbrace{A, \cdots, A}^{l}, B, \cdots, B) B_{\gamma\beta} \\ &+ k\sigma_k(\overbrace{A, \cdots, A}^{l}, B, \cdots, B)\delta_{\alpha\gamma}, \end{aligned} \tag{2.9}$$

and

$$(k-l+2)[T_{k-1}]_{\alpha\beta}(\overbrace{A,\cdots,A}^{l-2},B,\cdots,B)B_{\gamma\beta} = -k[T_k]_{\alpha\gamma}(\overbrace{A,\cdots,A}^{l-2},B,\cdots,B) - (l-2)[T_{k-1}]_{\alpha\beta}(\overbrace{A,\cdots,A}^{l-3},B,\cdots,B)A_{\gamma\beta} + k\sigma_k(\overbrace{A,\cdots,A}^{l-2},B,\cdots,B)\delta_{\alpha\gamma}. \tag{2.10}$$

Proof.

$$\begin{aligned} l[T_{k-1}]_{\alpha\beta}(\overbrace{A,\cdots,A}^{l-1},B,\cdots,B)A_{\gamma\beta} =& \frac{l}{(k-1)!}\delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k}A_{\alpha_2\beta_2}\cdots A_{\alpha_l\beta_l}B_{\alpha_{l+1}\beta_{l+1}}\cdots B_{\alpha_k\beta_k}A_{\gamma\beta_1} \\ =& -\frac{1}{(k-1)!}\delta^{\alpha,\alpha_1,\alpha_2,\cdots,\alpha_k}_{\gamma,\beta_1,\beta_2,\cdots,\beta_k}A_{\alpha_2\beta_2}\cdots A_{\alpha_l\beta_l}B_{\alpha_{l+1}\beta_{l+1}}\cdots B_{\alpha_k\beta_k}A_{\alpha_1\beta_1} \\ & -\frac{k-l}{(k-1)!}\delta^{\alpha_1,\alpha_2,\cdots,\alpha}_{\beta_1,\beta_2,\cdots,\beta_k}A_{\alpha_2\beta_2}\cdots A_{\alpha_l\beta_l}B_{\alpha_{l+1}\beta_{l+1}}\cdots B_{\gamma\beta_k}A_{\alpha_1\beta_1} \\ & +\frac{1}{(k-1)!}\delta^{\alpha_1,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k}\delta_{\alpha\gamma}A_{\alpha_2\beta_2}\cdots A_{\alpha_l\beta_l}B_{\alpha_{l+1}\beta_{l+1}}\cdots B_{\alpha_k\beta_k}A_{\alpha_1\beta_1} \\ =& -k[T_k]_{\alpha\gamma}(\overbrace{A,\cdots,A}^{l},B,\cdots,B) \\ & -(k-l)[T_{k-1}]_{\alpha\beta}(\overbrace{A,\cdots,A}^{l},B,\cdots,B)B_{\gamma\beta} \\ & +k\sigma_k(\overbrace{A,\cdots,A}^{l},B,\cdots,B)\delta_{\alpha\gamma}. \end{aligned}$$

Indeed, we start from the second line, where we divide into three cases. Case one is $\gamma = \alpha_1,\cdots,\alpha_l$, which is the first line. Case two is $\gamma = \alpha_{l+1},\cdots,\alpha_k$, which is the third line. And case three is in the fourth line.

This proves (2.9), and (2.10) follows similarly. ∎

2.3. Optimal Transport

By the result of Brenier [Bre91] on optimal transportation, given a probability measure $f(x)dx$ on Ω, there exists a convex potential function $\phi:\mathbb{R}^n\to\mathbb{R}$, such that $\overline{\nabla}\phi$ is the unique optimal transport map from Ω to $B(0,1)$ (the unit n-ball centered at the origin) which pushes forward the probability measure $f(x)dx$, to the probability measure $g(y)dy = \frac{1}{\omega_n}dy$ on $B(0,1)$, where $\omega_n := V_n(B)$. We denote $\overline{\nabla}\phi$, $\overline{\nabla}^2_{ij}\phi$ the gradient and the Hessian of ϕ with respect to the ambient Euclidean metric, and denote $\nabla\phi$, $\nabla^2_{\alpha\beta}\phi$ (or $\phi_{\alpha\beta}$) the gradient and the Hessian of ϕ with respect to the metric of $\partial\Omega$. For simplicity, we will denote the boundary $\partial\Omega$ by M from now on. Since $\overline{\nabla}\phi$ preserves the measure, we have the equation

$$det(\overline{\nabla}^2\phi)(x) = \frac{f(x)}{g(\overline{\nabla}\phi(x))} = \omega_n f(x). \tag{2.11}$$

Choose $f(x)dx := \frac{1}{vol(\Omega)}dx$ on Ω. $\overline{\nabla}\phi$ is the optimal transport map from Ω to $B(0,1)$. Therefore

$|\overline{\nabla}\phi| \leq 1$. Denote $\psi := \sqrt{1-|\nabla\phi|^2}$, then $\phi_n \leq \psi$, i.e.

$$|\nabla\phi|^2 + \phi_n^2 \leq 1.$$

Note this fact will be used several times later in the argument.
Because ϕ is convex, $\overline{\nabla}^2\phi$ is a positive definite matrix. Therefore by the geometric-arithmetic inequality $(det(\overline{\nabla}^2\phi))^{\frac{k+1}{n}} \leq \frac{1}{C_n^{k+1}}\sigma_{k+1}(\overline{\nabla}^2\phi)$. Thus by integrating over Ω we have

$$\begin{aligned}\int_\Omega (\omega_n f(x))^{\frac{k+1}{n}} dx &= \int_\Omega (det(\overline{\nabla}^2\phi))^{\frac{k+1}{n}} dx \\ &\leq \int_\Omega \frac{1}{C_n^{k+1}}\sigma_{k+1}(\overline{\nabla}^2\phi)dx. \end{aligned} \tag{2.12}$$

Using (2.8) and $\partial_j([T_k]_{ij}(A)) = 0$, hence we have by the divergence theorem that

$$\begin{aligned}\int_\Omega \frac{1}{C_n^{k+1}}\sigma_{k+1}(\overline{\nabla}^2\phi)dx =& \frac{1}{(k+1)C_n^{k+1}}\int_\Omega \overline{\nabla}^2_{ij}\phi[T_k]_{ij}(\overline{\nabla}^2\phi)dx \\ =& \frac{1}{(k+1)C_n^{k+1}}\int_\Omega \partial_j(\phi_i[T_k]_{ij}(\overline{\nabla}^2\phi))dx \\ =& \frac{1}{(k+1)C_n^{k+1}}\int_M [T_k]_{ij}(\overline{\nabla}^2\phi)\phi_i \overrightarrow{n}_j d\mu \end{aligned} \tag{2.13}$$

where $\overrightarrow{n}_j$ is the coordinate of the outward unit normal on M. Then putting $f(x)dx = \frac{1}{vol(\Omega)}dx$ into (2.12), we get

$$vol(\Omega)^{1-\frac{k+1}{n}} \leq (\frac{1}{\omega_n})^{\frac{k+1}{n}} \frac{1}{(k+1)C_n^{k+1}}\int_M [T_k]_{ij}(\overline{\nabla}^2\phi)\phi_i \overrightarrow{n}_j d\mu. \tag{2.14}$$

Now, we reduced Theorem 1.3 to the following Proposition which will be proved in section 3.
Proposition. For $1 \leq k \leq n-2$, suppose $L \in \Gamma_{k+1}$. Then

$$\int_M [T_k]_{ij}(\overline{\nabla}^2\phi)\phi_i \overrightarrow{n}_j d\mu \leq \int_M \sigma_k(L)d\mu. \tag{2.15}$$

Proof of Theorem 1.3: Recall our definition that

$$V_n(\Omega) := vol(\Omega),$$
$$V_{n-k-1}(\Omega) := \int_M \sigma_k(L)d\mu.$$

It follows from (2.14) and (2.15) that

$$\begin{aligned} V_n(\Omega)^{\frac{n-k-1}{n}} =& vol(\Omega)^{1-\frac{k+1}{n}} \\ \leq& (\frac{1}{\omega_n})^{\frac{k+1}{n}} \frac{1}{(k+1)C_n^{k+1}}\int_M [T_k]_{ij}(\overline{\nabla}^2\phi)\phi_i \overrightarrow{n}_j d\mu \\ \leq& (\frac{1}{\omega_n})^{\frac{k+1}{n}} \frac{1}{(k+1)C_n^{k+1}}\int_M \sigma_k(L)d\mu \\ =& (\frac{1}{\omega_n})^{\frac{k+1}{n}} \frac{1}{(k+1)C_n^{k+1}} V_{n-k-1}(\Omega). \end{aligned} \tag{2.16}$$

On the other hand, one can compute that

$$\frac{V_n(B)^{\frac{n-k-1}{n}}}{V_{n-k-1}(B)} = (\frac{1}{\omega_n})^{\frac{k+1}{n}} \frac{1}{(k+1)C_n^{k+1}}, \tag{2.17}$$

here B is unit n-ball, and $\omega_n = vol(B)$.

Combining (2.16) and (2.17), we get a form of Alexandrov-Fenchel inequalities for $1 \leq k \leq n-2$

$$\frac{V_n(\Omega)^{\frac{n-k-1}{n}}}{V_{n-k-1}(\Omega)} \leq \frac{V_n(B)^{\frac{n-k-1}{n}}}{V_{n-k-1}(B)}. \tag{2.18}$$

The equality case is the same as [CW13]. ■

3. Proof of Proposition

We will divide the proof of the Proposition into three steps. First, we integrate by parts in Lemma 3.1 and replace ϕ_n with ψ in Lemma 3.2, then reduce the proof of the Proposition to Lemma 3.3. Secondly, we expand ψ by Taylor series and derive a recursive equality for each individual term in the Taylor series in Lemma 3.4. Inserting these into inequality, we calculate the remaining term (3.10) in Lemma 3.5. Finally, in Lemma 3.6, we write this term into another form by some elementary facts, and prove that this term is in fact nonpositive.

Consider the isometric embedding $i : M \to \mathbb{R}^n$, where $M := \partial\Omega$. For $x \in M$, one can write the Hessian of ϕ in coordinates of tangential derivatives and normal derivatives. Indices i, j, k ranging from 1 to n be the coordinates of $\mathbb{R}^n$, indices α, β, γ ranging from 1 to $n-1$ be the tangential directions, and $\overrightarrow{n}$ be the outward unit normal direction on M. $\overline{\nabla}^2_{\alpha\beta}\phi$ means derivative with respect to the ambient (Euclidean) metric, $\phi_{\alpha\beta}$ with respect to the metric of the surface measure on M. And we denote $\nabla_\alpha(\phi_n)$ by $\phi_{n\alpha}$. Let $L_{\alpha\beta}(x)$ be the second fundamental form at $x \in M$. It is known that

$$\overline{\nabla}^2_{\alpha\beta}\phi = \phi_{\alpha\beta} + L_{\alpha\beta}\phi_n, \overline{\nabla}^2_{\alpha n}\phi = \phi_{n\alpha} - L_{\alpha\beta}\phi_\beta. \tag{3.1}$$

(Note that the sign of L_{ij} differs from that in [Rei77]).

Denote

$$A := \begin{pmatrix} \cdots & \cdots & \cdots & \vdots \\ \cdots & \phi_{\alpha\beta} & \cdots & \phi_{n\alpha} \\ \cdots & \cdots & \cdots & \vdots \\ \cdots & \phi_{n\alpha} & \cdots & \phi_{nn} \end{pmatrix},$$

and

$$B := \begin{pmatrix} \cdots & \cdots & \cdots & \vdots \\ \cdots & L_{\alpha\beta}\phi_n & \cdots & -L_{\alpha\gamma}\phi_\gamma \\ \cdots & \cdots & \cdots & \vdots \\ \cdots & -L_{\beta\gamma}\phi_\gamma & \cdots & 0 \end{pmatrix}.$$

So we can decompose $\overline{\nabla}^2\phi = A + B$.
Because we use index n to represent the normal direction on M,

$$\begin{aligned}\int_M [T_k]_{ij}(\overline{\nabla}^2\phi)\phi_i n_j d\mu &= \int_M [T_k]_{in}(\overline{\nabla}^2\phi)\phi_i d\mu \\ &= \int_M [T_k]_{in}(A+B)\phi_i d\mu \\ &= \int_M \sum_{l=0}^{k} C_k^l [T_k]_{in}(\overbrace{A,\cdots,A}^{l}, B,\cdots,B)\phi_i d\mu.\end{aligned}$$

Define:

$$D_l := \textstyle\int_M C_k^l [T_k]_{in}(\overbrace{A,\cdots,A}^{l}, B,\cdots,B)\phi_i d\mu, \text{ for } l = 0,\cdots,k.$$

Step 1. We apply the divergence theorem to reduce the Proposition to Lemma 3.3.
Lemma 3.1.

$$\begin{aligned}D_l =& \int_M -C_k^l \frac{l(l-1)}{k(k-l+1)}[T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l-2}, L,\cdots,L)\phi_n^{k-l+1}\phi_\alpha\phi_\gamma L_{\gamma\beta} d\mu \\ &+ \int_M C_k^l(\frac{l}{k-l+1}+1)\sigma_k(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l}, L,\cdots,L)\phi_n^{k-l+1} d\mu \\ &+ \int_M C_k^l \frac{k-l}{k}[T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l}, L,\cdots,L)\phi_n^{k-l-1}\phi_\alpha\phi_\gamma L_{\gamma\beta} d\mu. \end{aligned} \tag{3.2}$$

We remark that when $l = 0$ or 1 the first term disappears. And when $l = k$ the last term disappears.
Proof. By definition of $[T_k]_{in}$, A and B,

$$\begin{aligned}D_l =& \int_M C_k^l \frac{1}{k!}\delta^{i,i_1,\cdots,i_k}_{n,j_1,\cdots,j_k} A_{i_1j_1}\cdots A_{i_lj_l}B_{i_{l+1}j_{l+1}}\cdots B_{i_kj_k}\phi_i d\mu, \\ =& \int_M -C_k^l \frac{l}{k!}\delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k}\phi_{n\beta_1}\phi_{\alpha_2\beta_2}\cdots\phi_{\alpha_l\beta_l}L_{\alpha_{l+1}\beta_{l+1}}\cdots L_{\alpha_k\beta_k}\phi_\alpha\phi_n^{k-l} d\mu \\ &+ \int_M C_k^l \frac{k-l}{k!}\delta^{\alpha_1,\cdots,\alpha_{k-1},\alpha}_{\beta_1,\cdots,\beta_{k-1},\beta_k}\phi_{\alpha_1\beta_1}\cdots\phi_{\alpha_l\beta_l}L_{\alpha_{l+1}\beta_{l+1}}\cdots L_{\alpha_{k-1}\beta_{k-1}}L_{\gamma\beta_k}\phi_\gamma\phi_\alpha\phi_n^{k-l-1} d\mu \\ &+ \int_M C_k^l \frac{1}{k!}\delta^{\alpha_1,\cdots,\alpha_k}_{\beta_1,\cdots,\beta_k}\phi_{\alpha_1\beta_1}\cdots\phi_{\alpha_l\beta_l}L_{\alpha_{l+1}\beta_{l+1}}\cdots L_{\alpha_k\beta_k}\phi_n^{k-l+1} d\mu.\end{aligned}$$

Since $\phi_{n\beta_1}\phi_n^{k-l} = \frac{(\phi_n^{k-l+1})_{\beta_1}}{k-l+1}$, we have

$$\begin{aligned}D_l =& \int_M -C_k^l \frac{l}{k!(k-l+1)}\delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k}(\phi_n^{k-l+1})_{\beta_1}\phi_{\alpha_2\beta_2}\cdots\phi_{\alpha_l\beta_l}L_{\alpha_{l+1}\beta_{l+1}}\cdots L_{\alpha_k\beta_k}\phi_\alpha d\mu \\ &+ \int_M C_k^l \frac{k-l}{k!}\delta^{\alpha_1,\cdots,\alpha_{k-1},\alpha}_{\beta_1,\cdots,\beta_{k-1},\beta_k}\phi_{\alpha_1\beta_1}\cdots\phi_{\alpha_l\beta_l}L_{\alpha_{l+1}\beta_{l+1}}\cdots L_{\alpha_{k-1}\beta_{k-1}}L_{\gamma\beta_k}\phi_\gamma\phi_\alpha\phi_n^{k-l-1} d\mu \\ &+ \int_M C_k^l \frac{1}{k!}\delta^{\alpha_1,\cdots,\alpha_k}_{\beta_1,\cdots,\beta_k}\phi_{\alpha_1\beta_1}\cdots\phi_{\alpha_l\beta_l}L_{\alpha_{l+1}\beta_{l+1}}\cdots L_{\alpha_k\beta_k}\phi_n^{k-l+1} d\mu.\end{aligned}$$

Then integrate by parts,

$$
\begin{aligned}
D_l =& \int_M C_k^l \frac{l(l-1)}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} \phi_{\alpha_2\beta_2\beta_1} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_\alpha d\mu \\
&+ \int_M C_k^l \frac{l(k-l)}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} \phi_{\alpha_2\beta_2} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}\beta_1} \cdots L_{\alpha_k\beta_k} \phi_\alpha d\mu \\
&+ \int_M C_k^l \frac{l}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} \phi_{\alpha_2\beta_2} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_{\alpha\beta_1} d\mu \\
&+ \int_M C_k^l \frac{k-l}{k!} \delta^{\alpha_1,\cdots,\alpha_{k-1},\alpha}_{\beta_1,\cdots,\beta_{k-1},\beta_k} \phi_{\alpha_1\beta_1} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_{k-1}\beta_{k-1}} L_{\gamma\beta_k} \phi_\gamma \phi_\alpha \phi_n^{k-l-1} d\mu \\
&+ \int_M C_k^l \frac{1}{k!} \delta^{\alpha_1,\cdots,\alpha_k}_{\beta_1,\cdots,\beta_k} \phi_{\alpha_1\beta_1} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_n^{k-l+1} d\mu.
\end{aligned}
$$

We analyze the above five terms as follows:

Because of Ricci identity $\phi_{\alpha_2\beta_2\beta_1} - \phi_{\alpha_2\beta_1\beta_2} = \phi_\gamma R_{\gamma\alpha_2\beta_2\beta_1} = \phi_\gamma(L_{\gamma\beta_2}L_{\alpha_2\beta_1} - L_{\gamma\beta_1}L_{\alpha_2\beta_2})$, the first term of D_l becomes

$$
\begin{aligned}
&\int_M C_k^l \frac{l(l-1)}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} \phi_{\alpha_2\beta_2\beta_1} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_\alpha d\mu \\
=&\frac{1}{2} \int_M C_k^l \frac{l(l-1)}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} (\phi_{\alpha_2\beta_2\beta_1} - \phi_{\alpha_2\beta_1\beta_2}) \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_\alpha d\mu \\
=&\frac{1}{2} \int_M C_k^l \frac{l(l-1)}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} \phi_\gamma (L_{\gamma\beta_2}L_{\alpha_2\beta_1} - L_{\gamma\beta_1}L_{\alpha_2\beta_2}) \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_\alpha d\mu \\
=& \int_M -C_k^l \frac{l(l-1)}{k!(k-l+1)} \delta^{\alpha,\alpha_2,\cdots,\alpha_k}_{\beta_1,\beta_2,\cdots,\beta_k} \phi_n^{k-l+1} \phi_\gamma L_{\gamma\beta_1} L_{\alpha_2\beta_2} \cdots \phi_{\alpha_l\beta_l} L_{\alpha_{l+1}\beta_{l+1}} \cdots L_{\alpha_k\beta_k} \phi_\alpha d\mu, \qquad (3.3)
\end{aligned}
$$

Due to Codazzi equation $L_{\alpha_{l+1}\beta_{l+1}\beta_1} = L_{\alpha_{l+1}\beta_1\beta_{l+1}}$, the second term of D_l is zero!

The third term and the fifth term of D_l can be summed together.

Finally, we write D_l into more readable terms by definition

$$
\begin{aligned}
D_l =& \int_M -C_k^l \frac{l(l-1)}{k(k-l+1)} [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l-2}, L, \cdots, L) \phi_n^{k-l+1} \phi_\alpha \phi_\gamma L_{\gamma\beta} d\mu \\
&+ \int_M C_k^l \left(\frac{l}{k-l+1} + 1\right) \sigma_k(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L) \phi_n^{k-l+1} d\mu \\
&+ \int_M C_k^l \frac{k-l}{k} [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L) \phi_n^{k-l-1} \phi_\alpha \phi_\gamma L_{\gamma\beta} d\mu.
\end{aligned}
$$

It is easy to check the case $l = 0$ or 1, and $l = k$. ■

So

$$
\begin{aligned}
D :=& \sum_{l=0}^k D_l \\
=& \sum_{l=0}^k \int_M -C_k^l \frac{l(l-1)}{k(k-l+1)} [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l-2}, L, \cdots, L) \phi_n^{k-l+1} \phi_\alpha \phi_\gamma L_{\gamma\beta} d\mu
\end{aligned}
$$

$$+\sum_{l=0}^{k}\int_M C_k^l(\frac{l}{k-l+1}+1)\sigma_k(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_n^{k-l+1}d\mu$$

$$+\sum_{l=0}^{k}\int_M C_k^l\frac{k-l}{k}[T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_n^{k-l-1}\phi_\alpha\phi_\gamma L_{\gamma\beta}d\mu.$$

We observe that the first term and the third term can be canceled each other by every two terms, due to the fact $C_k^{l+2}\frac{(l+2)(l+1)}{k(k-l-2+1)}=C_k^l\frac{k-l}{k}, l=0,\cdots,k-2$. Thus

$$\begin{aligned}D=&\sum_{l=0}^{k}\int_M C_k^l(\frac{l}{k-l+1}+1)\sigma_k(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_n^{k-l+1}d\mu\\&+\int_M[T_{k-1}]_{\alpha\beta}(\nabla^2\phi)\phi_\alpha\phi_\gamma L_{\gamma\beta}d\mu.\end{aligned}$$

Recall that $|\nabla\phi|^2+\phi_n^2\le 1$, i.e. $\phi_n\le\psi$. We now replace ϕ_n with ψ:

Lemma 3.2.

$$D\le\sum_{l=0}^{k}\int_M C_{k+1}^l\sigma_k(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\psi^{k-l+1}d\mu+\int_M[T_{k-1}]_{\alpha\beta}(\nabla^2\phi)\phi_\alpha\phi_\gamma L_{\gamma\beta}d\mu.$$

Proof. Subtracting the right hand from the left hand, then changing $\nabla^2_{\alpha\beta}\phi$ into $\overline{\nabla}^2_{\alpha\beta}\phi-L_{\alpha\beta}\phi_n$, we obtain

$$\begin{aligned}&\sum_{l=0}^{k}\int_M C_{k+1}^l\sigma_k(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)(\phi_n^{k-l+1}-\psi^{k-l+1})d\mu\\
=&\sum_{l=0}^{k}C_{k+1}^l\int_M\sum_{i=0}^{l}C_l^i\sigma_k(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i},L,\cdots,L)((-\phi_n)^{l-i}\phi_n^{k-l+1}-(-\phi_n)^{l-i}\psi^{k-l+1})d\mu\\
=&\int_M\sum_{i=0}^{k}\sum_{l=i}^{k}C_{k+1}^iC_{k+1-i}^{l-i}((-\phi_n)^{l-i}\phi_n^{k-l+1}-(-\phi_n)^{l-i}\psi^{k-l+1})\sigma_k(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i},L,\cdots,L)d\mu.\end{aligned}$$

Here in the second equality, we change orders of summation and use identity $C_{k+1}^lC_l^i=C_{k+1}^iC_{k+1-i}^{l-i}$. In order to sum over index l, we plus $l=k+1$ term. (Note this term is zero.)

$$\begin{aligned}=&\int_M\sum_{i=0}^{k}\sum_{l=i}^{k+1}C_{k+1}^iC_{k+1-i}^{l-i}((-\phi_n)^{l-i}\phi_n^{k-l+1}-(-\phi_n)^{l-i}\psi^{k-l+1})\sigma_k(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i},L,\cdots,L)d\mu\\
=&\int_M\sum_{i=0}^{k}C_{k+1}^i((\phi_n-\phi_n)^{k-i+1}-(\psi-\phi_n)^{k-i+1})\sigma_k(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i},L,\cdots,L)d\mu\\
\le&0.\end{aligned}$$

■

Then in order to prove the Proposition we need only to prove the following lemma.

Lemma 3.3.

$$\begin{aligned} E :=& \sum_{l=0}^{k} \int_M C_{k+1}^l \sigma_k(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L)\psi^{k-l+1} d\mu \\ &+ \int_M [T_{k-1}]_{\alpha\beta}(\nabla^2\phi)\phi_\alpha\phi_\gamma L_{\gamma\beta} d\mu. \\ \leq& \int_M \sigma_k(L) d\mu. \end{aligned}$$

Define:

$$E_l := \textstyle\int_M C_{k+1}^l \sigma_k(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L)\psi^{k-l+1} d\mu, \text{ for } l = 0, \cdots, k.$$

And

$$A_{m,l} := \textstyle\int_M \sigma_k(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L)|\nabla\phi|^{2m} d\mu, \text{ for } l = 0, \cdots, k; m = 0, 1, 2, \cdots.$$

Step 2. Before we prove Lemma 3.3, we need first to deal with $A_{m,l}$, then calculate E_l.
Lemma 3.4.

$$\begin{aligned} A_{m,l} =& \frac{l(l-1)}{k(2m+l)} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l-2}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m} d\mu \\ &+ \frac{2m}{2m+l} \int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L)\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu \\ &+ \frac{2m(k-l)}{k(2m+l)} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu. \end{aligned}$$

And obviously

$$A_{0,0} = \int_M \sigma_k(L) d\mu.$$

We remark that when $l = 0$ and $m \neq 0$; or $l = 1$ and $m \neq 0$ the first term disappears. When $m = 0$ and $l \neq 0$ the last two terms disappear. And when $l = k$ the last term disappears.
Proof. When $l \neq 0, 1$, and $m \neq 0$,

$$\begin{aligned} A_{m,l} &= \int_M \frac{1}{k!} \delta_{\beta_1, \cdots, \beta_k}^{\alpha_1, \cdots, \alpha_k} \phi_{\alpha_1\beta_1} \cdots \phi_{\alpha_l\beta_l} \cdots L_{\alpha_k\beta_k} |\nabla\phi|^{2m} d\mu \\ &= \frac{1}{k!} \delta_{\beta_1, \cdots, \beta_k}^{\alpha_1, \cdots, \alpha_k} \Big(\int_M -(l-1)\phi_{\alpha_1}\phi_{\alpha_2\beta_2\beta_1} \cdots \phi_{\alpha_l\beta_l} \cdots L_{\alpha_k\beta_k} |\nabla\phi|^{2m} \\ &\qquad - 2m\phi_{\alpha_1}\phi_\gamma\phi_{\gamma\beta_1}\phi_{\alpha_2\beta_2} \cdots \phi_{\alpha_l\beta_l} \cdots L_{\alpha_k\beta_k} |\nabla\phi|^{2m-2} d\mu \Big). \end{aligned}$$

Here in the second equality we integrate by parts, and use Codazzi equation as before.

Using Ricci identity as in (3.3), we continue with

$$\begin{aligned}
A_{m,l} = \frac{1}{k!}\delta^{\alpha_1,\cdots,\alpha_k}_{\beta_1,\cdots,\beta_k}\Big(\int_M &(l-1)\phi_{\alpha_1}\phi_\gamma L_{\gamma\beta_1}L_{\alpha_2\beta_2}\cdots\phi_{\alpha_l\beta_l}\cdots L_{\alpha_k\beta_k}|\nabla\phi|^{2m} \\
&-2m\phi_{\alpha_1}\phi_\gamma\phi_{\gamma\beta_1}\phi_{\alpha_2\beta_2}\cdots\phi_{\alpha_l\beta_l}\cdots L_{\alpha_k\beta_k}|\nabla\phi|^{2m-2}d\mu\Big) \\
= \frac{l-1}{k}\int_M [T_{k-1}]_{\alpha\beta}(&\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l-2},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m}d\mu \qquad (3.4) \\
-\frac{2m}{k}\int_M [T_{k-1}]_{\alpha\beta}(&\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l-1},L,\cdots,L)\phi_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu. \qquad (3.5)
\end{aligned}$$

Then we use (2.9) to get

$$\begin{aligned}
A_{m,l} =& \frac{l-1}{k}\int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l-2},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m}d\mu \qquad (3.6) \\
&+\frac{2m}{l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu \\
&+\frac{2m(k-l)}{lk}\int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu \qquad (3.7) \\
&-\frac{2m}{l}A_{m,l}.
\end{aligned}$$

Thus

$$\begin{aligned}
A_{m,l} =& \frac{l(l-1)}{k(2m+l))}\int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l-2},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m}d\mu \\
&+\frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu \\
&+\frac{2m(k-l)}{k(2m+l)}\int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu.
\end{aligned}$$

There may be minor difference in the special cases. And we point out as following:
When $l=1$ and $m\neq 0$, the line (3.4) and (3.6) disappear.
When $l=0$ and $m\neq 0$, using (2.10) we have.

$$\begin{aligned}
A_{m,0} &= \int_M \sigma_k(L)|\nabla\phi|^{2m}d\mu \\
&= \int_M [T_k]_{\alpha\gamma}(L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu \\
&\quad + \int_M [T_{k-1}]_{\alpha\beta}(L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu
\end{aligned}$$

When $m=0$ and $l\neq 0$ the line (3.5) disappears, so the following terms which include m are all zero.
When $l=k$ the line (3.7) becomes zero.

So these special cases are as stated in the remark. ■

We know

$$\psi^{k-l+1}=(1-|\nabla\phi|^2)^{\frac{k-l+1}{2}}=1+\sum_{m=1}^{\infty}\frac{(-1)^m(\frac{k-l+1}{2})(\frac{k-l+1}{2}-1)\cdots(\frac{k-l+1}{2}-m+1)}{m!}|\nabla\phi|^{2m}, \tag{3.8}$$

provided $k-l+1\geq 1$ and $|\nabla\phi|\leq 1$.

For convenience we denote $s:=|\nabla\phi|$ and $b_{m,l}:=\frac{(-1)^m(\frac{k-l+1}{2})(\frac{k-l+1}{2}-1)\cdots(\frac{k-l+1}{2}-m+1)}{m!}$, $b_{0,l}:=1$. We write (3.8) into

$$\psi^{k-l+1}=\sum_{m=0}^{\infty}b_{m,l}s^{2m}. \tag{3.9}$$

Remark: Due to the Raabe's criterion (Suppose $a_n>0$, $r>1$. If $n(\frac{a_n}{a_{n+1}}-1)\geq r$, for n big, then $\sum_{m=0}^{\infty}a_n$ converges.), we have $\sum_{n=0}^{\infty}|b_{m,l}|<+\infty$, provided $\frac{k-l+1}{2}>0$. Because we are working on the range of $|\nabla\phi|(x)\leq 1$, (3.9) is uniform convergence. This insures us to take derivative and integral of each term in the Taylor expansion below.

Next, we apply $A_{m,l}$ in E_l, and calculate the term $E-\int_M\sigma_k(L)d\mu$.

Lemma 3.5.

$$\begin{aligned}E-\int_M\sigma_k(L)d\mu&=\sum_{l=0}^{k}E_l+\int_M[T_{k-1}]_{\alpha\beta}(\nabla^2\phi)\phi_\alpha\phi_\gamma L_{\gamma\beta}d\mu-\int_M\sigma_k(L)d\mu.\\&=\sum_{l=0}^{k}\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{2m}{2m+l}\int_M[T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu.\end{aligned}$$

Proof. We deal with two cases.
For $l=1,\cdots,k$, we have

$$\begin{aligned}E_l&=\sum_{m=0}^{\infty}\int_M C_{k+1}^{l}b_{m,l}\sigma_k(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)|\nabla\phi|^{2m}d\mu\\&=\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}A_{m,l}\\&=\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{l(l-1)}{k(2m+l))}\int_M[T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l-2},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m}d\mu\\&\quad+\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{2m}{2m+l}\int_M[T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu\\&\quad+\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{2m(k-l)}{k(2m+l)}\int_M[T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)L_{\gamma\beta}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu.\end{aligned}$$

For $l = 0$, we have

$$
\begin{aligned}
E_0 =& \int_M \sigma_k(L) d\mu \\
&+ \sum_{m=0}^{\infty} C_{k+1}^l b_{m,l} \frac{2m}{2m+l} \int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L)\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu \\
&+ \sum_{m=0}^{\infty} C_{k+1}^l b_{m,l} \frac{2m(k-l)}{k(2m+l)} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu.
\end{aligned}
$$

From the above two cases, we get

$$
\begin{aligned}
&\sum_{l=0}^{k} E_l - \int_M \sigma_k(L) d\mu \\
=& \sum_{l=2}^{k} \sum_{m=0}^{\infty} C_{k+1}^l b_{m,l} \frac{l(l-1)}{k(2m+l))} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l-2}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m} d\mu \\
&+ \sum_{l=0}^{k} \sum_{m=0}^{\infty} C_{k+1}^l b_{m,l} \frac{2m}{2m+l} \int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L)\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu \\
&+ \sum_{l=0}^{k-1} \sum_{m=0}^{\infty} C_{k+1}^l b_{m,l} \frac{2m(k-l)}{k(2m+l)} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu.
\end{aligned}
$$

We observe that, for $l = 0, \cdots, k-2, m = 0, 1, 2, \cdots$,

$$
C_{k+1}^{l+2} b_{m,l+2} \frac{(l+2)(l+1)}{k(2m+l+2)} + C_{k+1}^l b_{m+1,l} \frac{2(m+1)(k-l)}{k(2m+2+l)} = 0.
$$

So

$$
\begin{aligned}
&\sum_{l=2}^{k} \sum_{m=0}^{\infty} C_{k+1}^l b_{m,l} \frac{l(l-1)}{k(2m+l))} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l-2}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m} d\mu \\
&+ \sum_{l=0}^{k-2} \sum_{m=1}^{\infty} C_{k+1}^l b_{m,l} \frac{2m(k-l)}{k(2m+l)} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{l}, L, \cdots, L) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu \\
&= 0!
\end{aligned}
$$

And in the case $l = k-1$, because $b_{1,k-1} = -1$, $b_{m,k-1} = 0$, for $m \geq 2$

$$
\begin{aligned}
&\sum_{m=0}^{\infty} C_{k+1}^{k-1} b_{m,k-1} \frac{2m(k-k+1)}{k(2m+k-1)} \int_M [T_{k-1}]_{\alpha\beta}(\overbrace{\nabla^2\phi, \cdots, \nabla^2\phi}^{k-1}) L_{\gamma\beta}\phi_\alpha\phi_\gamma |\nabla\phi|^{2m-2} d\mu \\
=& -\int_M [T_{k-1}]_{\alpha\beta}(\nabla^2\phi)\phi_\alpha\phi_\gamma L_{\gamma\beta} d\mu.
\end{aligned}
$$

Then

$$E-\int_M \sigma_k(L)d\mu=\sum_{l=0}^{k}E_l+\int_M [T_{k-1}]_{\alpha\beta}(\nabla^2\phi)\phi_\alpha\phi_\gamma L_{\gamma\beta}d\mu-\int_M\sigma_k(L)d\mu.$$
$$=\sum_{l=0}^{k}\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu$$

■

Step 3. If we prove the above term is nonpositive, then Lemma 3.3 is proved.
Lemma 3.6. Suppose $L\in\Gamma_{k+1}$, then

$$\sum_{l=0}^{k}\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu\le 0. \quad (3.10)$$

Proof. First, we substitute $\nabla^2_{\alpha\beta}\phi$ with $\overline{\nabla}^2_{\alpha\beta}\phi-L_{\alpha\beta}\phi_n$, then change orders of summation.

$$\sum_{l=0}^{k}\sum_{m=0}^{\infty}C_{k+1}^{l}b_{m,l}\frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\nabla^2\phi,\cdots,\nabla^2\phi}^{l},L,\cdots,L)\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu$$
$$=\sum_{l=0}^{k}\sum_{m=0}^{\infty}\sum_{i=0}^{l}C_{k+1}^{l}b_{m,l}C_l^i\frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i},L,\cdots,L)(-\phi_n)^{l-i}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu$$
$$=\sum_{i=0}^{k}\sum_{m=0}^{\infty}\sum_{l=i}^{k}C_{k+1}^{l}b_{m,l}C_l^i\frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i},L,\cdots,L)(-\phi_n)^{l-i}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu$$

Recall that

$$(1-s^2)^{\frac{k-l+1}{2}}=\sum_{m=0}^{\infty}b_{m,l}s^{2m}, \quad (3.11)$$

here $s=|\nabla\phi|$.
We multiply s^{l-1} on both sides of (3.11) and integrate over $[0,s]$ to get

$$\int_0^s t^{l-1}(1-t^2)^{\frac{k-l+1}{2}}dt=\sum_{m=0}^{\infty}\frac{1}{2m+l}b_{m,l}s^{2m+l}. \quad (3.12)$$

Then using (3.12) and integration by parts, we obtain

$$\sum_{m=0}^{\infty}\frac{2m}{2m+l}b_{m,l}s^{2m-2}$$
$$=\sum_{m=0}^{\infty}(1-\frac{l}{2m+l})b_{m,l}s^{2m-2}$$
$$=\frac{1}{s^2}\psi^{k-l+1}-\frac{l}{s^{l+2}}\int_0^s t^{l-1}(1-t^2)^{\frac{k-l+1}{2}}dt.$$
$$=-\frac{k-l+1}{s^{l+2}}\int_0^s t^{l+1}(1-t^2)^{\frac{k-l-1}{2}}dt. \quad (3.13)$$

Using (3.13), we get

$$\sum_{i=0}^{k}\sum_{m=0}^{\infty}\sum_{l=i}^{k} C_{k+1}^{l} b_{m,l} C_l^i \frac{2m}{2m+l}\int_M [T_k]_{\alpha\gamma}(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i}, L,\cdots,L)(-\phi_n)^{l-i}\phi_\alpha\phi_\gamma|\nabla\phi|^{2m-2}d\mu \quad (3.14)$$

$$=-\sum_{i=0}^{k}\int_M\sum_{l=i}^{k} C_{k+1}^{l} C_l^i \frac{k-l+1}{s^{l+2}}\int_0^s t^{l+1}(1-t^2)^{\frac{k-l-1}{2}}(-\phi_n)^{l-i}dt[T_k]_{\alpha\gamma}(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i}, L,\cdots,L)\phi_\alpha\phi_\gamma d\mu.$$

We next claim that $\sum_{l=i}^{k} C_{k+1}^{l} C_l^i \frac{k-l+1}{s^{l+2}}\int_0^s t^{l+1}(1-t^2)^{\frac{k-l-1}{2}}(-\phi_n)^{l-i}dt$ is nonnegative.
Since $C_{k+1}^{l} C_l^i (k-l+1) = C_{k+1}^{i} C_{k-i}^{l-i}(k-i+1)$, we obtain

$$\begin{aligned}
&\sum_{l=i}^{k} C_{k+1}^{l} C_l^i \frac{k-l+1}{s^{l+2}}\int_0^s t^{l+1}(1-t^2)^{\frac{k-l-1}{2}}(-\phi_n)^{l-i}dt\\
=&\sum_{l=i}^{k} C_{k+1}^{i}\frac{k-i+1}{s^{i+2}}\sum_{l=i}^{k}\int_0^s C_{k-i}^{l-i}\frac{(-\phi_n)^{l-i}}{s^{l-i}}(1-t^2)^{\frac{k-l-1}{2}}t^{l+1}dt\\
=&C_{k+1}^{i}\frac{k-i+1}{s^{i+2}}\int_0^s(\psi(t)-\frac{\phi_n t}{s})^{k-i}\psi(t)^{-1}t^{i+1}dt,
\end{aligned} \quad (3.15)$$

here $\psi(t)=\sqrt{1-t^2}$, for $t\in[0,s]$.
Recalling that $\phi_n\le\psi$, we have $\psi(t)-\frac{\phi_n t}{s}\ge\psi-\frac{\phi_n t}{s}\ge 0$.
Then it's obvious that (3.15) is nonnegative.

Under the assumption that $L\in\Gamma_{k+1}$ and positive definiteness of $\overline{\nabla}^2\phi$, one has

$$[T_k]_{\alpha\gamma}(\overbrace{\overline{\nabla}^2\phi,\cdots,\overline{\nabla}^2\phi}^{i}, L,\cdots,L)\ge 0.$$

This can be seen from Gårding's inequality [Går59]. So (3.14) is less than Zero ! ■

This completes the proof of the Proposition!

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