Quantum Phase Transitions in Random Spin Systems

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Abstract

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A number of condensed matter systems undergo a phase transition at zero temperature as some external parameter (such as pressure, magnetic field, or amount of dirt) is varied. Quantum effects are often crucial to the physics of this phenomenon - hence the name "quantum phase transitions". This thesis is concerned with a study of such zero temperature phase transitions in the presence of static randomness (due to impurities or other frozen defects in the system). Experimentally accessible quantum phase transitions often occur in the presence of strong randomness, and are very poorly understood. Theoretically, the description of such phenomena involving competition between various kinds of potential energy of interactions, quantum effects, and randomness presents a challenging problem, where there are as yet few reliable techniques. This thesis studies simple quantum statistical models with randomness as a useful starting point to obtain insight into more complex, realistic systems. Progress is reported in understanding various simple but non-trivial models of random quantum magnetic systems in the vicinity of a quantum phase transition. The results show that the effects of randommess may be quite dramatic, and lead to a phenomenology that is strikingly different from that of pure systems.

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Contents

Chapter 1

Introduction

Phase transitions in condensed matter have been of major interest to physicists for many decades. Most research has focused on the properties of transitions that occur at some finite temperature as an external parameter is varied and there now exists a detailed, quantitative understanding of these. An important feature of these finite temperature phase transitions is that a complete description of them can be achieved based entirely on the principles of classical statistical mechanics. While quantum mechanics may be important in determining the existence and properties of the various phases, it is absolutely correct to ignore it to understand the most interesting properties of the transition. In recent years however, a number of systems have been studied where the phase transition occurs at zero temperature. Quantum mechanics plays an important role in the physics of these transitions, often being responsible for their very occurence. For this reason, such zero temperature phase transitions are often called quantum phase transitions.

There exist many experimental systems where such transitions are seen. For instance, there are transitions from an insulator to a metal[1] or a superconductor[2], and various magnetic-nonmagnetic transitions in heavy fermion compounds[3], high-T_c cuprates[4], and so on most of which are poorly understood. Such many body systems on the brink of a ground state instability show an interesting phenomenology at finite temperature quite different from more conventional ones. On the theoretical side, there seem to exist many significant differences from the treatment of ordinary critical phenomena, especially in metallic and/or disordered systems, which make them interesting.

Most experimental systems are disordered. At sufficiently low temperatures, effects of both randomness and quantum mechanics must be taken into account to provide a proper description. These effects are often particularly important in the vicinity of a $T = 0$ critical point separating two different phases. Such phenomena where randomness, quantum mechanics, and interactions all play an important role represent a class of problems where there are as yet few reliable techniques and are only beginning to be understood.

This thesis is concerned with the study of simple quantum statistical mechanical models with disorder. It is hoped that this will be a useful starting point to obtain insight into the properties of more complex, realistic quantum systems undergoing a $T = 0$ phase transition in the presence of quenched randomness. Even the models considered here turn out to be considerably complicated, and as yet are far from being thoroughly understood. Nevertheless, there are some situations in which it has been possible to make progress and obtain reliable information on universal aspects of the phase transition. These limited available results reveal that the critical properties of random quantum phase transitions may be markedly different from those of pure systems.

The outline of the thesis is as follows: In Chapter 2, we give a brief review of the salient aspects of critical phenomena associated with $T = 0$ transitions in pure quantum systems. This will also be used as an opportunity to introduce the models whose random versions will be studied in later chapters. Chapter 3 is devoted to some general considerations on quantum transitions in random systems. We mention various stray results that are known, present some general arguments to illustrate some of the peculiar properties of random quantum systems as opposed to pure ones, and finally demonstrate that conventional techniques for dealing with critical phenomena (such as the ϵ-expansion) are currently insufficient for the random quantum transitions of interest in this thesis.

The one shining exception to the almost complete absence of reliably understood models of random quantum transitions is the random version of the one dimensional Ising model in a transverse field - perhaps the simplest random quantum system.

This model (or mathematically equivalent versions) have been studied extensively by McCoy and Wu[5], Shankar and Murthy[6], and in particular, by D.S. Fisher[7]. The properties were found to be very unusual as compared to phase transitions in pure quantum systems. One of the main points to be made in this thesis is that there exist a variety of other models, both in one dimension and in higher dimensions, that share many of the unusual properties of the random transverse field Ising chain.

In Chapter 4, we consider the critical properties of a number of random quantum transitions in models (such as the Potts models) with discrete symmetry in $d = 1$. We show that the techniques used by Fisher to solve the Ising problem can also be used to obtain the exact critical properties of all these other models. The results demonstrate that the peculiar properties found for the Ising model also hold for the other models, and hence are not just artifacts of Ising symmetry. Quite remarkably, it turns out that all the computable universal properties of all the models are *identical* to those of the Ising model. This is in striking contrast to the non-random situation where the properties of the transition (and, in certain cases, even whether it is first or second order) depend crucially on the particular symmetry of the model. We call this feature "superuniversality" as it goes well beyond the usual universality expected near second order phase transitions.

In Chapter 5, we address the important question of whether the novel features of the $d = 1$ results can survive in higher dimensions. We show that it is, in principle, possible that they do, by explicitly demonstrating a particular model which displays much of the unusual physics found in $d = 1$. The model we consider is that of an Ising model in a transverse field in a diluted lattice. It turns out that there is a non-trivial quantum transition at the percolation threshold of the lattice whose properties are determined largely by the geometrical properties of the percolating clusters, about which much is known. This enables us to make definitive statements about the critical properties of this transition. To our knowledge, these are the first reliable calculations of the detailed critical properties of a random quantum transition in finite dimension $d > 1$.

Chapter 6 is devoted to a discussion of the situation where the randomness couples directly to the order parameter (random field models). A complete understanding of this problem is lacking, but we provide general scaling hypotheses in close analogy with the corresponding classical problem (which also is not very well understood yet). In particular, we point out various similarities between quantum phase transitions in these random field systems, and the ones considered earlier in the thesis.

In Chapter 7, we discuss the critical properties of quantum spin glass models with infinite range interactions with the q-state Potts model as an example. The critical properties can be found exactly, and hence these models are useful. It turns out, however, that the transition is quite conventional and does not have any of the unusual features found in the models studied in previous chapters. It is not clear yet whether these infinite range models are useful starting points to understand spin glass transitions in realistic models with short range interactions in finite dimensions.

Finally, in Chapter 8, we summarize the principal conclusions of our work, speculate on their more general validity, and comment on their relationship with the work of various other authors.

Chapter 2

Quantum Phase Transitions in Pure Systems

Scaling and renormalization group theory provide the general framework for understanding phase transitions. In the absence of randomness, it has often been possible to implement renormalization group ideas and obtain a fairly general theory of a quantum phase transition. An important and powerful approach in this context, which we will elaborate on below, relies on the possibility of mapping a d-dimensional quantum system at zero temperature to a $d+1$-dimensional *classical* statistical mechanical system at its finite temperature. One can then often use all the knowledge and technology collected on classical statistical mechanical problems to understand the quantum problem. However, it is important to realize that there are some fundamental differences between classical and quantum phase transitions. As is well-known, in classical statistical mechanics, static and dynamic properties can be treated independant of each other. This is however not true in quantum statistical mechanics. Thus it is necessary to treat fluctuations in space and time on an equal footing even to understand thermodynamic properties of a quantum system. (See Ref.[8, 9] for a very clear discussion of this point). Therefore, unlike classical phase transitions, the static and dynamic critical behaviour are intertwined at a quantum phase transition. The extra dimension in the classical system to which a quantum system may formally be mapped indeed reflects the need to include temporal fluctuations to describe the properties of the latter. As in general, the character of the fluctuations in time may be quite different from the spatial ones, the corresponding classical system may well be considerably anisotropic with respect to the extra dimension.

In this chapter, we will outline the general theory of quantum phase transitions in pure systems. We will be brief as there already exist several excellent reviews[8, 9]. For concreteness, we will for the most part base the discussion on a single model - the Ising model in a transverse field. This is the prototypical example of a system undergoing a quantum transition, and many of its properties are quite generic to quantum transitions in pure systems. We will also introduce and discuss the properties of other quantum models whose random versions will be studied in later chapters.

2.1 The transverse field Ising model

Consider the system defined by the Hamiltonian

$$H = \underbrace{-J \sum_{<i,j>} \sigma_i^z \sigma_j^z}_{H_0} \underbrace{- h \sum_i \sigma_i^x}_{H_1} \tag{2.1}$$

where the $\vec{\sigma}_i$ are Pauli spin matrices placed on the sites i of a d-dimensional hypercubic lattice. When $h = 0$, this is just the usual Ising Hamiltonian. In the presence of a non-zero h, the two terms in the Hamiltonian do not commute, and we get an Ising system with non-trivial quantum fluctuations. The first term may be considered as a potential energy of interaction between the Ising spins which tends to point spins at neighbouring sites along the same direction (we have assumed $J > 0$). The second term, on the other hand, corresponds to a kinetic energy which tends to flip a spin pointing in the $+z$ direction to $-z$, and vice versa. Clearly the Hamiltonian is invariant under a Z_2 symmetry: $\sigma_i^z \to -\sigma_i^z$.

First, let us consider the ground states of the Hamiltonian in some extreme limits. If $J \gg h$, the first term in Eqn.2.1 dominates, and we may think of the system perturbatively from the $h = 0$ limit. Exactly at $h = 0$, the Hamiltonian is trivially diagoanlized. There are two degenerate ground states- with all spins either up or down. The lowest excitation about either ground state corresponds to a single spin flip and costs energy $\sim J$. For $h \neq 0$, but small, perturbation theory converges, and we get a ground state with broken symmetry and LRO in σ_z. The lowest-lying excitations can be roughly thought of as a band of delocalized single spin flips. The lowest excitation gap decreases as h increases but is nevertheless non-zero so long as the ground state is ordered.

Similarly, in the opposite limit, when $J \ll h$, we may think of the system perturbatively from the $J = 0$ point. At $J = 0$, different sites decouple, and at each site the spin points along the $+x$ direction. So there is no broken symmetry and long range order(LRO). The lowest excitations about this simple ground state correspond to flipping a single spin to the $-x$ direction, and cost an energy $2h$. For $J \neq 0$ but $\ll h$, perturbation theory converges and we have a paramagnetic ground state with no broken symmetry and LRO. The low-lying excitations can now be roughly thought of as corresponding to a band of delocalized $-x$ spins. The lowest excitation gap, while smaller than $2h$, is still non-zero.

Clearly these two kinds of ground states in these two extreme limits must be separated by a phase transition. As we will see, this transition is second order. Close to this transition, the length scales characterizing the decay of spatial correlations in either phase become very large and ultimately diverge at the critical point. Likewise the energy gap in either phase becomes very small, vanishing exactly at the critical point. Many properties of the system will be universal (*i.e* independant of microscopic details) in this region, and show scaling behaviour characteristic of critical points.

It is useful at this point to introduce the mapping to a classical $(d+1)$-dimensional Ising model. Consider the partition function at a temperature $T = \frac{1}{\beta}$:

$$Z = Tr(e^{-\beta H}) \tag{2.2}$$

The operator $e^{-\beta H}$ can be regarded as the evolution operator from (imaginary) time 0 to β. Using standard techniques, we may write Z as a sum over various possible (imaginary time) histories of the system:

$$Z = Tr(\underbrace{e^{-\epsilon H} e^{-\epsilon H} \ldots e^{-\epsilon H}}_{N factors}) \qquad (\epsilon = \frac{\beta}{N})$$

$$= \sum_{\sigma_i^z(\tau)} \langle \sigma_i^z(0)|e^{-\epsilon H}|\sigma_i^z(N-1)\rangle\langle\sigma_i^z(N-1)|e^{-\epsilon H}|\sigma_i^z(N-2)\rangle$$
$$\ldots\ldots \langle\sigma_i^z(1)|e^{-\epsilon H}|\sigma_i^z(0)\rangle$$

where we have introduced at each time step τ, a complete set of states $\sigma_i^z(\tau)$. We take N large so that ϵ is very small. Then we may write

$$e^{-\epsilon H} = e^{-\epsilon(H_0+H_1)} \simeq e^{-\epsilon H_0}e^{-\epsilon H_1}$$

Consider any matrix element in the product above:

$$\begin{aligned}\langle\sigma_i^z(\tau+1)|e^{-\epsilon H}|\sigma_i^z(\tau)\rangle &\simeq \langle\sigma_i^z(\tau+1)|e^{-\epsilon H_0}e^{-\epsilon H_1}|\sigma_i^z(\tau)\rangle \\ &\sim e^{\epsilon J\sum_{<ij>}\sigma_i^z(\tau+1)\sigma_j^z(\tau+1)+h^*\sum_i\sigma_i^z(\tau+1)\sigma_i^z(\tau)}\end{aligned}$$

where $\tanh(h^*) = e^{-2\epsilon h}$. At $T = 0$, clearly we may interpret the partition function expressed as a sum over products of such matrix elements as the partition function of a classical Ising model in $(d+1)$-dimensions with bond strengths in the d-spatial directions equal to ϵJ, and the bond strength in the extra "time" direction equal to h^*. By universality arguments, we then expect that the global properties of the phase diagram and the phase transitions of the quantum model at $T = 0$ are the same as those of an isotropic classical $(d+1)$-dimensional Ising model at finite temperatures. In particular, universal critical properties of the quantum model close to its $T = 0$ phase transition can be directly obtained from the properties of the isotropic classical model near its finite temperature transition[1].

2.1.1 Scaling and renormalization group theory

In this section we will discuss the properties of the transverse field Ising model near its zero temperature phase transition from the point of view of scaling and renormalization group theory. Whenever appropriate, we will also point out qualitative

[1]We caution that this feature is a very special property of the Ising model and some other simple models considered later in this chapter. In general, the anisotropy between the spatial and time directions cannot be ignored and leads to differences in the way spatial and temporal correlations scale

features that may be special to this model, and comment on the more general situation. On approaching the critical point (from either side), there is a correlation length ξ which diverges as

$$\xi \sim |r - r_c|^{-\nu} \tag{2.3}$$

where $r = \frac{h}{J}$ is the control parameter that needs to be tuned to place the system at the critical point. The mapping to the classical model described in the previous section implies that the exponent ν is the same as that of the classical Ising model in one higher dimension. There also is an energy scale Δ that vanishes as

$$\Delta \sim |r - r_c|^{z\nu} \tag{2.4}$$

thereby defining the exponent z. In the particular case of the Ising model being considered here, this energy scale is just the energy gap in either phase. More generally in cases where one or both phases do not have an energy gap, this energy just corresponds to a scale which distinguishes critical fluctuations from those characterizing either phase. Equivalently, $\frac{1}{\Delta}$ is the time scale below which the temporal fluctuations crossover to those characteristic of the critical point. For the transverse field Ising model, the mapping to the classical Ising model implies that the correlation length along the time direction diverges in exactly the same manner as the spatial correlation length and so

$$\Delta \sim \frac{1}{\xi}$$

Thus, the exponent $z = 1$ for the Ising model. In the general case however, no such simple argument is possible for the value of z. The important qualitative point to note is the fact that Δ vanishes as *some* power of ξ^{-1}.

The ordered phase is characterized by a non-vanishing expectation value of the magnetization $m = \frac{1}{N}\sum_i \langle \sigma_i^z \rangle$. On approaching the transition this vanishes with an exponent β which also is the same as that of the classical Ising model in one higher dimension. Similarly the susceptibility to a uniform magnetic field B along the z-direction diverges (from either side) with an exponent γ which again is that of the classical $(d+1)$-dimensional Ising model. Right at the critical point, the magnetization has a power-law dependance on the field B (for small B):

$$m \sim B^{\frac{1}{\delta}}$$

with $\delta = \frac{\gamma+\beta}{\beta}$. In general, for small fields B, and deviations $(r-r_c)$, the magnetization satisfies the universal scaling form

$$m(r-r_c, B) \sim B^{\frac{\beta}{\gamma+\beta}} F\left(\frac{(r-r_c)}{B^{\frac{1}{\gamma+\beta}}}\right) \tag{2.5}$$

Another important quantity characterizing the ordered phase is its stiffness to changes in the boundary conditions. Consider the system that is finite, of linear size L. Now consider the energy cost to change the boundary conditions from periodic to antiperiodic in one of the hypercubic directions. In the ordered phase, this is essentially the cost of creating a domain wall and so scales as L^{d-1}. In the paramagnetic phase, there is a finite correlation length and so this energy cost is exponentially small for large L. Thus the quantity $\Sigma = lim_{L\to\infty} \frac{(E_{anti-per} - E_{per})}{L^{d-1}}$ is finite in the ordered phase but is zero in the paramagnetic phase. On approaching the transition, Σ vanishes with some exponent ζ. This can be related to ν by the following hyperscaling argument. For large finite L, near the transition, the singular part of the energy density with either periodic or antiperiodic boundary conditions satisfies the hyperscaling relation

$$\frac{E_{per./anti-per}}{L^d} \sim \frac{1}{\xi^{d+z}} \Psi_{per./anti-per}(\frac{L}{\xi})$$

That the exponent in the prefactor is $(d+z)$ instead of d can be seen easily from the mapping to the classical problem in $d+1$ dimensions, and is a consequence of the need to include dynamic fluctuations even to understand the thermodynamics. Thus the difference

$$E_{anti-per} - E_{per} \sim \frac{L^d}{\xi^{d+z}} \left[\Psi_{anti-per}(\frac{L}{\xi}) - \Psi_{per}(\frac{L}{\xi})\right]$$

In the ordered phase, as $L \to \infty$, the left hand side $\sim \Sigma L^{d-1}$. Requiring that the right side have this dependance on L for large L, we find

$$\Sigma \sim \frac{1}{\xi^{d+z-1}} \tag{2.6}$$

Putting $z = 1$ for the Ising model, we get the exponent equality

$$\zeta = \nu d \tag{2.7}$$

Scaling forms can also be written down for various correlation functions in the vicinity of the transition. Most importantly, the order parameter correlation function in imaginary time, defined in the usual way,

$$G(x, \tau) = \langle T_\tau(\sigma_i^z(x, \tau), \sigma_i^z(0, 0))\rangle \tag{2.8}$$

satisfies (for large x, τ)

$$G(x, \tau) = \frac{1}{x^{d+z-2+\eta}} g\left(\frac{x}{\xi}, \tau\Delta\right) \tag{2.9}$$

The exponent η is related to the ones introduced earlier by $\gamma = \nu(2 - \eta)$.

As usual, the scaling and universality of the critical properties can be understood from the point of view of the renormalization group(RG). As has already been mentioned, the quantum nature of the problem implies that it is necessary to include both space and time dependant fluctuations to understand even the thermodynamics. (The formulation as a $(d+1)$-dimensional classical problem does this naturally). Thus, we imagine some RG scheme in which we integrate out fluctuations at short length and time scales to get an effective action for the remaining degrees of freedom, and then rescale lengths and times to get back a theory with the same microscopic cutoffs as before. If we parametrize the length rescaling factor by s, *i.e* $x \to x' = \frac{x}{s}$, then it is natural to rescale time such that $\tau \to \tau' = \frac{\tau}{s^z}$, reflecting the possible anisotropy between space and time. Under this RG, the critical point is described by an unstable fixed point with non-trivial quantum fluctuations. The deviation $|r - r_c|$ is a relevant perturbation and carries the system into the stable ordered or disordered fixed points. As usual the eigenvalue of this perturbation under the linearized RG transformations determines the exponent ν. Turning on a finite B-field is another relevant perturbation, and the corresponding eigenvalue determines the other independant exponent.

Finite temperature properties of the system in the vicinity of the quantum critical point can be undertood from the following observations. Consider the mapping to the classical $(d + 1)$-dimensional problem. A finite temperature in the quantum problem corresponds to a finite size $\beta = \frac{1}{T}$ along the imaginary time direction in the classical problem. Thus finite temperature scaling forms for physical observables can be inferred from the theory of finite size scaling at classical critical points. In particular, it is clear that a finite temperature is a relevant perturbation with eigenvalue exactly equal to the exponent z. Scaling forms at finite temperature can then be written down using quite standard arguments. These and other interesting details of the finite temperature properties near quantum critical points can be found in the references[9, 8].

2.2 Other models

In this section, we will introduce some other simple models showing quantum phase transitions, and describe some of their properties in the absence of randomness. This will set the stage for a discussion of their random versions in later chapters.

2.2.1 Discrete Symmetry

The Ising model has a discrete symmetry group Z_2. It is natural to consider generalizations where the symmetry group is still discrete, but different from Z_2. Interesting, well-studied examples in the classical context are the q-state Potts and clock models. Here we will consider their quantum versions.

The models are defined in terms of a variable that can assume q possible states (which we denote $|0\rangle, |1\rangle, \cdots, |q-1\rangle$) on the sites of a d-dimensional lattice. The *classical* Potts (clock) interaction in the presence of a uniform external "magnetic" field H along the '0' direction is

$$\mathcal{H}_{P,int} = -J\sum_{\langle i,j\rangle}\delta_{n_i n_j} - 2H\sum_i(\delta_{n_i,0} - \frac{1}{q}) \tag{2.10}$$

$$\mathcal{H}_{C,int} = -J\sum_{\langle i,j\rangle}2\cos(\frac{2\pi}{q}(n_i - n_j)) - 2H\sum_i\cos\frac{2\pi n_i}{q} \tag{2.11}$$

where P and C stand for Potts and Clock respectively. We introduce quantum fluctuations into these models by adding at each site a "transverse field" term that attempts to change the state of the variable at that site. Thus we consider the quantum Hamiltonians

$$\mathcal{H}_P = -h\sum_i \left(\sum_{n_i,n'_i=0}^{q-1} \frac{1}{q} |n_i\rangle\langle n'_i| \right) + \mathcal{H}_{P,int} \tag{2.12}$$

$$\mathcal{H}_C = -h \left(\sum_{n_i=0}^{q-1} (|n_i\rangle\langle n_i+1| + h.c) \right) + \mathcal{H}_{C,int} \tag{2.13}$$

(We identify $|n_i + q\rangle = |n_i\rangle$). Note that at $H = 0$, the Hamiltonian $\mathcal{H}_P$ is invariant under a global permutation $|n\rangle \longrightarrow |n'\rangle$ of the states at each site. For $\mathcal{H}_C$, the symmetry is a global cyclic rotation $|n\rangle \longrightarrow |n+1\rangle$. Clearly for $q = 2$, both these models reduce to the transverse field Ising model. For general q, just as in the Ising case, the "transverse field" term plays the role of a kinetic energy that opposes the tendency to order due to the interaction term. Also as in the Ising case, the d-dimensional q-state quantum Potts (clock) model Eqn. 2.12 (2.13) at zero temperature may be regarded as equivalent to a $d+1$-dimensional q-state classical Potts (clock) model[12].

The mapping to the classical $d+1$-dimensional pure problem provides a rather complete picture of the possible phases and the transitions between them. For instance (at zero H), the ferromagnetic (*i.e* $J > 0$, $h > 0$) quantum Potts chain has a first order transition for $q > 4$, and a second order transition for $q \leq 4$ (for which all the exponents are known exactly and depend on the value of q)[32, 13]. The ferromagnetic clock chains, on the other hand, have, for $q > 4$, a quasi-long-range ordered (QLRO) phase sandwiched between a truely long-range ordered phase and a disordered phase[11] . For $q \leq 4$, the quasi-long-range ordered phase disappears and is replaced by an ordinary second order phase transition for which again all the exponents are known exactly[32, 11]. In dimension $d > 1$, the quantum Potts models, for any $q > 2$, have first order transitions. Later we will see that randomness drastically modifies this picture.

In Appendix A, we rewrite the Potts and Clock Hamiltonians in terms of a set of operators R_{ix}, R_{iz} defined at each site of the lattice. This will be quite useful in later chapters to do calculations.

2.2.2 Continuous Symmetry

We now consider a class of models with a continuous symmetry group - the O(N) quantum rotor models. These are defined in terms of an N-component unit vector $\hat{n}_i$ at each site i of a d-dimensional lattice. The Hamiltonian is

$$H = \frac{g}{2} \sum_i L_{i\mu}^2 - J \sum_{<ij>} \hat{n}_i . \hat{n}_j \tag{2.14}$$

where the $L_{i\mu}$ are the μ'th generator of rotations of $\hat{n}_i$, and so satisfy the appropriate commutation relations. The components of $\hat{n}_i$ commute with each other. Operators at two different sites also commute with each other. The first term is the kinetic energy of a single particle moving on the surface of a sphere in N-dimensions, while the second term is a potential energy tending to orient the $\hat{n}$ vectors at two adjacent sites in the same direction. It is easy to see by using the same procedure as in the Ising case that this quantum model at $T = 0$ is equivalent to the classical O(N)-model at its finite temperature.

For large $\frac{g}{J}$, in any dimension, or for any $\frac{g}{J}$, in $d = 1$, the ground state is paramagnetic with a finite excitation gap. If $d > 1$, then for small $\frac{g}{J}$, the ground state spontaneously breaks the O(N) symmetry, and aquires LRO in $\hat{n}$. The lowest lying excitations in this case are linear dispersing spin waves, and hence gapless. These two phases are separated by a second order transition at a critical value of $\frac{g}{J}$. As in the Ising case, the universal properties at both $T = 0$ and $T \neq 0$ in the vicinity of this transition can be obtained from those of the classical O(N) model.

2.3 Discussion

In this chapter, we have outlined the theory of quantum phase transitions in pure systems, using the transverse field Ising model as the primary example. An important distinguishing feature of quantum transitions is the intertwining of the static and dynamic aspects of the critical phenomena. This implies that to understand the critical behaviour, we need to consider fluctuations in both space and time. In the vicinity of the phase transition, there is a cross-over from fluctuations characteristic of either phase to those associated with the critical point at a large length scale ξ and a large time scale $\frac{1}{\Delta}$. Both these scales diverge at the critical point, and generally

$$\Delta \sim \frac{1}{\xi^z}$$

thereby defining the exponent z. In the special examples that we discussed in ths chapter, $z = 1$, but in general this is not required to be so.

In many cases, it is possible to make progress in understanding a quantum transition by mapping it to the critical point of a classical statistial mechanical model in one higher dimension, the extra dimension reflecting the need to include time-dependant fluctuations in the quantum problem. This classical model could in general be considerably anisotropic with respect to the extra "time" dimension (corresponding to the possibility of $z \neq 1$). Finite temperature in the quantum problem corresponds to a finite size in the "time" direction in the classical problem. Thus finite temperature scaling forms near the quantum transition can be written down using standard ideas from the theory of finite-size scaling near classical critical points.

In the remaining chapters, we will try to see to what extent these ideas developed in the context of quantum transitions in pure systems carry over to systems with quenched randomness. Is the general scaling structure preserved, perhaps with different exponents and scaling functions, or does randomness alter the physics in more fundamental ways? At the time of writing this thesis, there is no general understanding of quantum transitions in random systems, and so we will address this question by studying various specific models.