Integer quantum Hall transition: An alternative approach and exact results

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We introduce and analyze a class of model systems to study transitions in the integer quantum Hall effect (IQHE). Even without disorder our model exhibits an IQHE transition as a control parameter is varied. We find that the transition is in the two-dimensional Ising universality class and compute all associated exponents and critical transport properties. The fixed point has time-reversal, particle-hole, and parity invariance. We then consider the effect of quenched disorder on the IQHE transition and find the following. (i) Randomness in the control parameter (which breaks all the above symmetries) translates into bond randomness in the Ising model and is hence marginally irrelevant. The transition may equally well be viewed as a quantum percolation of edge states localized on equipotentials. The absence of random-phase factors for the edge states is responsible for the nongeneric (Ising) critical properties. (ii) For a random magnetic field (which preserves particle-hole symmetry in every realization) the model exhibits an exactly solvable fixed line, described in terms of a product of a Luttinger liquid and an $SU(n)$ spin chain. While exponents vary continuously along the fixed line, the longitudinal conductivity is constant due to a general conformal sum rule for Kac-Moody algebras (derived here), and is computed exactly. We also obtain a closed expression for the extended zero-energy wave function for every realization of disorder and compute its exact multifractal spectrum $f(\alpha)$ and the exponents of all participation ratios. One point on the fixed line corresponds to a recently proposed model by Gade and Wegner. (iii) The model in the presence of a random on-site potential scales to a strong disorder regime, which is argued to be described by a symplectic nonlinear-sigma-model fixed point. (iv) We find a plausible global phase diagram in which all forms of disorder are simultaneously considered. In this generic case, the presence of random-phase factors in the edge-state description indicates that the transition is described by a Chalker-Coddington model, with a so far analytically inaccessible fixed point.

I. INTRODUCTION

The discovery of the quantum Hall effect was one of the most surprising and startling events in condensed-matter physics.[1] While a theoretical framework for understanding the remarkable quantization in both the integer and fractional cases was obtained shortly after the discovery, there remained some unresolved issues concerning the important role of quenched disorder.[2] Indeed, it was initially unclear how to avoid localization in such a $2d$ disordered system. In 1983 Levine, Libby, and Pruisken[3] argued that it was necessary to include an important topological term in the usual nonlinear-sigma-model description of $2d$ localization, and that this term was responsible for delocalization in the middle of a Landau band. Despite the extreme appeal of this approach, it was not possible to carry out an explicit calculation of the critical properties associated with this delocalization transition. Indeed, to this day a calculable theory for the transition in the integer quantum Hall effect has remained elusive.

The difficulty in describing the integer quantum Hall transition is due in large part to the fact that the phases on both sides of the transition are localized insulators. Such localized phases are notoriously difficult to model theoretically, making it nearly impossible to approach the transition by expanding around the neighboring phases. This should be contrasted with the Anderson metal-insulator transition, where an expansion about the metallic phase for dimensions just above two can be used to access the localization transition.

An alternate approach to the integer quantum Hall transition, which we adopt here, involves first constructing a model which exhibits such a transition in the absence of any quenched disorder. Then upon including disorder, one can try to expand about the critical point in the pure system, rather than expanding about one of the nearby phases. To this end we introduce and analyze a

lattice model of spinless fermions which exhibits an integer Hall transition as a control parameter is varied even in the absence of disorder. We show that the critical properties of this pure Hall transition are described by a single massless Dirac fermion. This enables us to extract detailed critical properties, including the critical conductivities.

We then include and analyze various types of disorder. The disorder is characterized by the symmetries of the pure fixed point which they break. We find a rich array of fixed points and lines in the presence of disorder. Critical exponents and transport properties are obtained exactly for a number of the cases. Unfortunately, we have not been able so far to access the generic fixed point which describes the integer Hall transition when all types of randomness are simultaneously present.

Our starting point is a system of spinless nonrelativistic fermions on a square lattice with nearest-neighbor and diagonal hopping amplitudes, and with half a unit of magnetic flux per square plaquette. As shown in Sec. II, the low-energy physics of the pure system is given by a pair of Dirac fermions. The masses of these fermions are unequal, and split by a staggered chemical potential which serves as the control parameter used to tune through the integer Hall transition.[4] As the latter is varied, one of the fermion masses changes sign at a critical point, and there is a jump of e^2/h in the Hall conductance. The correlation length exponent at this pure Hall transition is $\nu=1$, as might be expected given the equivalence between a massless Dirac fermion in $(2+1)$ dimensions and the critical point of a classical two-dimensional (2D) Ising model. In Sec. III we study transport properties near the pure Hall transition, and find that the longitudinal conductivity is universal at the transition, given by $\sigma_{xx}=(\pi/8)(e^2/h)$. However, the density of states vanishes at the transition. The fixed-point Hamiltonian is shown to be invariant under parity, time reversal, and particle-hole transformations.

We then add to the pure system various forms of quenched disorder, and discuss their symmetry properties in Sec. IV. In Sec. V we study the model in detail in the presence of a random Dirac mass $M(x,y)$ which breaks all the above-mentioned symmetries. For a smoothly varying M we show that the eigenstates are confined to constant M contours with the Dirac spin parallel to the contours. Upon encircling a closed loop, the spin picks up a phase of 2π and hence a *minus* sign. This is equivalent to a spinless particle experiencing half a flux quantum as it goes around the contour. As the control parameter, the constant part of M, is varied, these isolated contours will percolate. We use renormalization group RG methods to see how this happens and find, not surprisingly, that the system flows to the pure Ising fixed point. This result is to be expected since we show that the theory describing averaged properties in our model with random M can be mapped into the classical 2D random bond Ising model. The control parameter M in our model corresponds to the temperature variable in the Ising model. Randomness in the latter is known[6] to be irrelevant (in the RG sense) with calculable logarithmic corrections. While our analysis is valid for small M, other evidence suggests that any quantum effects on the classical percolation fixed point drive us away from classical percolation[7] to the Ising point.[8] We also propose a possible way of accessing experimentally the transition with random Dirac mass.

Next, in Sec. VI, we study the effect of a random vector potential $\mathbf{A}$. The corresponding random magnetic field B has a Gaussian distribution with variance

$$\overline{B(\mathbf{k})B(\mathbf{k}')}=(2\pi)^2\delta^2(\mathbf{k}+\mathbf{k}')\Delta_A k^2 \tag{1}$$

($\mathbf{k}$ is the 2D momentum). Note that $B(\mathbf{k}=0)$ has no variance, i.e., the total flux in each realization is fixed at zero. We derive the zero-energy wave function for every realization and study in detail its multifractal scaling characteristics[9–12] as well as the scaling of all participation ratios. We work out the exact multifractal $f(\alpha)$ spectrum, which in this case is precisely a parabola (its curvature and maximum depending on Δ_A) reflecting log-normal character. We then turn to properties that do not simply follow from the zero-energy wave function. We show that the averaged properties are described by a fixed line, parametrized by Δ_A. The density of states, which vanished linearly in the nonrandom case as we approached zero energy, now vanishes as a power, that decreases as Δ_A increases; when $\Delta_A=\pi$, the density of states is constant, and starts diverging for larger values of Δ_A. We show that the longitudinal conductivity, defined as a product of the diffusion constant and density of states (see, e.g., Ref. 13), is constant along the fixed line. This result is established by means of a nonperturbative result of conformal field theory, similar in spirit to Zamolodchikov's c theorem,[14] using an underlying SU(2) symmetry between fields with positive and negative frequencies. Averaged properties on the fixed line are described by a tensor product of a Luttinger liquid and a critical SU(n) [or SU($2n$)] spin chain. The replica limit $n\to 0$ does not introduce unphysical results here, due to gluing conditions between charge and spin excitations.

In Sec. VII we show that a random scalar potential is a relevant perturbation at the pure system's critical point, and we suggest, based on universality, that the flows take one into the 2D symplectic localization fixed point.[15]

In Sec. VIII we consider the general situation where all types of randomness are simultaneously present. We find that the presence of a random Dirac mass term destabilizes the fixed line of the random vector potential model. Physically, the edge states which move on closed contours of constant Dirac mass (discussed above) will now acquire random-phase factors, as in the Chalker-Coddington model. We thus expect that in this case the RG flows will terminate at the generic Hall fixed point with $\nu\simeq\frac{7}{3}$ (Refs. 16–18). We propose a global phase diagram, when all three types of impurities are present simultaneously.

In Sec. IX we discuss the relationship between our work and other recent analytic and numerical studies of the integer quantum Hall transition.

II. HALL TRANSITION IN A PURE SYSTEM

In this section we will consider a pure system which exhibits a quantum Hall transition as a control parameter is varied. The low-energy physics of this system is described by the free Dirac equation. This allows us to characterize the system in detail: calculate its density of states, the conductivities σ_{xy},σ_{xx}, etc. Subsequently we will couple the system to disordering interactions of various symmetries. Let us begin with the derivation of the Dirac equation.

A. Derivation of the Dirac equation

Consider a system of noninteracting spinless fermions on a $2d$ square lattice with the following situation (Fig. 1).

(i) There is an external magnetic field with one half of a magnetic flux quanta per square plaquette.

(ii) There is a nearest-neighbor hopping, with amplitude $t=1$.

(iii) There is a second-neighbor (diagonal) hopping with amplitude $t'/4$.

(iv) There is a staggered potential $\mu(-1)^{x+y}$, where x and y are the integer coordinates on the lattice.

In Fig. 1 we see the lattice divided into repeating unit cells marked off by squares, each enclosing four sites numbered 1–4. The vector potential due to the external magnetic field produces a phase factor of -1 as we go around a square plaquette. By a gauge choice we have arranged to have all the phase factor come from the bond connecting sites 3 and 4 in each cell. The phase factor for diagonal hopping is i along the arrows, and $-i$ if we go against them. The solid circles mark the sublattice where the alternating staggered potential is positive.

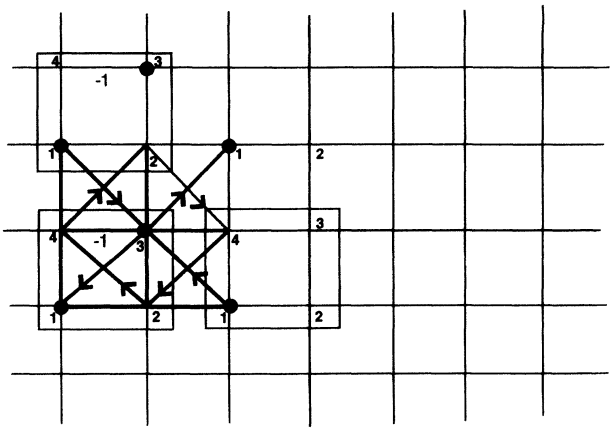

FIG. 1. A schematic representation of the spinless fermion model. The fermions hop on a square lattice with a nearest-neighbor hopping amplitude $t=1$, except where $a=-1$ is shown. The diagonal (next-nearest neighbor) hopping amplitude is t' in magnitude; the phase being $+i$ $(-i)$ if along (against) the arrows shown. There is a staggered chemical potential of size μ, which is positive on the sites with a dark circle and negative on the others.

The Hamiltonian can be expressed as a sum over sites and bonds in the usual way, with each site and bond covered once. We reexpress the Hamiltonian in terms of four fermion fields $\psi_1\ldots\psi_4$, which live on the four sites of each unit cell. In a Fourier representation the Hamiltonian then takes the form

$$
\begin{aligned}
H=\int\frac{d^2k}{4\pi^2}&[\mu(\psi_1^\dagger\psi_1-\psi_2^\dagger\psi_2+\psi_3^\dagger\psi_3-\psi_4^\dagger\psi_4]\\
&+[\psi_1^\dagger\psi_2-\psi_3^\dagger\psi_4+\psi_1^\dagger\psi_4+\psi_2^\dagger\psi_3+\psi_1^\dagger\psi_4 e^{-ik_y}+\psi_2^\dagger\psi_3 e^{-ik_y}-\psi_3^\dagger\psi_4 e^{ik_x}+\psi_1^\dagger\psi_2 e^{-ik_x}+\text{H.c.}]\\
&+\frac{it'}{4}[\psi_1^\dagger\psi_3+\psi_4^\dagger\psi_2+\psi_3^\dagger\psi_1 e^{ik_y}+\psi_2^\dagger\psi_4 e^{-ik_y}+\psi_4^\dagger\psi_2 e^{ik_y}e^{-ik_x}+\psi_1^\dagger\psi_3 e^{-ik_y-ik_x}+\psi_3^\dagger\psi_1 e^{ik_x}+\psi_2^\dagger\psi_4 e^{ik_x}+\text{H.c.}]\,. \qquad (2)
\end{aligned}
$$

We may write this all down in matrix form as

$$
H=\int\frac{d^2k}{4\pi^2}\Psi^\dagger(k)H(k)\Psi(k)\,, \qquad (3)
$$

where Ψ is the four-component field, $\Psi^\dagger$ is its adjoint, and $H(k)$ is the matrix

$$
H(k)=\begin{bmatrix}
\mu & (1+e^{-ik_x}) & \frac{it'}{4}(1-e^{-ik_y})(1-e^{-ik_x}) & (1+e^{-ik_y})\\
(1+e^{ik_x}) & -\mu & (1+e^{-ik_y}) & -\frac{it'}{4}(1-e^{-ik_y})(1-e^{ik_x})\\
-\frac{it'}{4}(1-e^{ik_y})(1-e^{ik_x}) & (1+e^{ik_y}) & \mu & -(1+e^{ik_x})\\
(1+e^{ik_y}) & \frac{it'}{4}(1-e^{ik_y})(1-e^{ik_x}) & -(1+e^{-ik_x}) & -\mu
\end{bmatrix}.
$$

If we consider this matrix at $\mu=t'=0$, it is readily seen that it vanishes at $\mathbf{k}=(\pi,\pi)$. Near this point lies the low-energy sector on which we focus. To this end let us introduce a lattice spacing a and write

$$\mathbf{k}=(\pi,\pi)+a\mathbf{p} , \tag{4}$$

$$t'=aT' , \tag{5}$$

$$\mu=am , \tag{6}$$

and expand $H(k)$ to first order in a, and divide both sides by a to obtain the following continuum Hamiltonian:

$$H_c=\frac{H}{a} \tag{7}$$

$$=\begin{vmatrix} m & ip_x & iT' & ip_y \\ -ip_x & -m & ip_y & -iT' \\ iT' & -ip_y & m & ip_x \\ -ip_y & iT' & -ip_x & -m \end{vmatrix} \tag{8}$$

$$=\begin{vmatrix} m\sigma_z-p_x\sigma_y & iT'\sigma_z+i\sigma_x p_y \\ -iT'\sigma_z-i\sigma_x p_y & m\sigma_z-p_x\sigma_y \end{vmatrix} \tag{9}$$

$$=m\sigma_z\otimes I-p_x\sigma_y\otimes I-T'\sigma_z\otimes\tau_y-p_y\sigma_x\tau_y , \tag{10}$$

where σ and τ are two copies of the standard Pauli matrices. We remark that in the absence of diagonal hopping t', the original lattice Hamiltonian of Eq. (2) is time-reversal invariant, since there is half a flux quantum per plaquette. The time-reversal operation is implemented on the continuum Hamiltonian in Eq. (10) by

$$H_c\rightarrow\sigma_z H_c^*\sigma_z . \tag{11}$$

Notice that under this operation $T'\rightarrow -T'$ and $m\rightarrow m$, so that this Hamiltonian is time reversal invariant in the absence of diagonal hopping, as it must be.

In the last equation (10), let us make the change $p_x\leftrightarrow -p_y$ to obtain finally

$$H_c=m\sigma_z\otimes I+p_y\sigma_y\otimes I-T'\sigma_z\otimes\tau_y+p_x\sigma_x\tau_y . \tag{12}$$

Hereafter we will drop the subscript on H_c. Since τ_y commutes with all the other operators, it behaves like a c number. If we set it equal to its eigenvalues ±1, we obtain in the corresponding subspaces

$$H_\pm=\boldsymbol{\sigma}\cdot\mathbf{p}+\sigma_z(m\mp T') , \tag{13}$$

where $\boldsymbol{\sigma}$ and $\mathbf{p}$ will always stand for vectors in the x-y plane. The above Hamiltonians describe two massive Dirac fermions.

The eigenvalues of $H_\pm$ give the single-particle levels

$$E_{\tau_y=\pm1}=\pm\sqrt{p^2+(m\mp T')^2} . \tag{14}$$

The ground state will consist of a filled Fermi sea of all the negative-energy levels. This corresponds to half-filling, i.e., a spinless particle for every other site on the lattice.

The low-energy physics of interest to us is dominated by the sector of τ_y which has the smallest gap (the "lightest particle"). Let us choose

$$m-T'=M , \tag{15}$$

with M very small. Then the relevant sector is $\tau_y=1$, while the other sector, with gap $m+T'$, will not contribute to physical properties such as the diagonal conductivity, density of states, etc. It will, however, contribute to the Hall conductivity σ_{xy} in a very interesting way as we shall see shortly. Till then, let us focus on the "light particle" with small M.

B. Phase diagram of the pure system

In order to obtain the phase diagram it is convenient (though hardly necessary) to pass from the operator formalism for the Dirac Hamiltonian of the light species,

$$H=\int\frac{d^2p}{4\pi^2}\Psi^\dagger(\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p}+\sigma_z M)\Psi(\mathbf{p}) , \tag{16}$$

to a path integral formulation. We use fermionic coherent states.[19] The zero-temperature theory is then given by the following imaginary-time functional integral:

$$Z=\int[d\bar\psi(\omega,\mathbf{p})d\psi(\omega,\mathbf{p})]e^{S_0}\equiv\prod_\omega Z_\omega , \tag{17}$$

$$S_0=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{d^2p}{4\pi^2}[\bar\psi(\omega,\mathbf{p})i\omega\psi(\omega,\mathbf{p})-H(\bar\psi,\psi)] . \tag{18}$$

In the above, ψ and $\bar\psi$ are completely independent Grassmann variables. In particular $\bar\psi$ is not the adjoint of the two-component spinor ψ. This means that they may be subject to completely independent transformations. Notice that Z factorizes into a product over frequencies of Z_ω. This will be true even in the presence of a static random potential that we will introduce later [in contrast to the factorization over momenta implicit in (16)]. This is clear provided there are no genuine interactions between the fermions, since we can then describe them in terms of exact energy eigenstates in the random potential.

In this paper we will sometimes, but not always, exploit this factorization. If the question involves a single frequency, we will then ignore all modes not at that frequency since they will be just dead weight in the calculation, canceling between numerator and denominator. (To average over an ensemble, we will first calculate at the given frequency using Z_ω, and then average the result.) On the other hand, if we need an integral over all frequencies, it is necessary to retain the full partition function Z. In the former case, we will be dealing with a Dirac theory in two dimensions, with ω being simply a parameter.

Let us now compute some of the properties of this system, starting with the density of states at energy E. This is given by

$$\rho(E)=-\lim_{\omega\rightarrow0}\frac{1}{\pi}\mathrm{Im}\,\mathrm{Tr}_{\mathrm{p,spin}}G(i\omega-E,\mathbf{p}) , \tag{19}$$

where the Green's function for the light fermion is

$$G(i\omega-E,\mathbf{p})=\frac{1}{i\omega-E-\boldsymbol{\sigma}\cdot\mathbf{p}-M\sigma_z} . \tag{20}$$

Notice that only a single frequency ω comes into play,

and the answer is given as a two-dimensional integral. Rationalizing the denominator, we obtain

$$\rho(E)=-\lim_{\omega\to 0}\frac{1}{\pi}\mathrm{Im}\int\frac{d^2p}{4\pi^2}\frac{\mathrm{Tr}(i\omega-E+\boldsymbol{\sigma}\cdot\mathbf{p}+\sigma_z M)}{(i\omega-E)^2-p^2-M^2}\ , \tag{21}$$

where Tr is now only over the "spin space." Dropping terms of order ω^2 in the denominator and of order ω in the numerator, and all traceless objects, we end up with

$$\rho(E)=\frac{|E|}{\pi}\theta(E^2-M^2)\ . \tag{22}$$

This result can be readily checked from the dispersion relation $E=\pm\sqrt{p^2+M^2}$ by taking $E>0$ and computing

$$\rho(E)=\int\frac{d^2p}{4\pi^2}\delta(E-\sqrt{p^2+M^2})\ . \tag{23}$$

[Note that in (23) there is no factor of 2 for "spin," since in this problem the "spin" is just an index that keeps track of the sign of the energy: At each momentum p there is only one state for a given sign of E.]

Observe that the density of states vanishes at zero energy, even when the Dirac mass is zero, $M=0$. This will be significant in what follows.

Let us now consider a quantity which forces us to deal with all frequencies: σ_{xy}. To this end we apply a static electric field in the y direction, and measure the current in the x direction. For the Dirac field the current and charge density are given by

$$\begin{aligned}\mathbf{J}&=\bar{\psi}\boldsymbol{\sigma}\psi\ ,\\ \rho&=\bar{\psi}\psi\ .\end{aligned} \tag{24}$$

Thus from Wick's theorem we have

$$\sigma_{xy}=\frac{j_x}{E_y}=\frac{i}{q_y}\int\mathrm{Tr}\left[\sigma_x\frac{1}{i\omega-E-\boldsymbol{\sigma}\cdot\mathbf{p}-M\sigma_z}\frac{1}{i\omega-E-\boldsymbol{\sigma}\cdot\mathbf{p}-q_y\sigma_y-M\sigma_z}\right]\frac{d^2p\,d\omega}{8\pi^3} \tag{25}$$

$$=\frac{i}{q_y}\int\mathrm{Tr}\frac{\sigma_x M\sigma_z\sigma_y q_y}{[(i\omega-E)^2-p^2-M^2]^2}\frac{d^2p\,d\omega}{8\pi^3}\ . \tag{26}$$

In order to reach the last line we had to (i) rationalize the denominators, (ii) set $q_y=0$ in the denominator due to the explicit power of q_y in the numerator, and (iii) drop all terms in the numerator that would not survive the trace operation or integration over p_x. Continuing on, we find

$$\sigma_{xy}=\frac{M}{4\pi^3}\int\frac{d^2p\,d\omega}{[(i\omega-E)^2-p^2-M^2]^2} \tag{27}$$

$$=\frac{M}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{-1}{[\omega-i(E+|M|)][\omega-i(E-|M|)]} \tag{28}$$

$$=\frac{-M}{4\pi|M|}\theta(|M|-E)\ . \tag{29}$$

Repeating the calculation for negative E reveals that E should be replaced by $|E|$ in the above. If we measure the conductivity in standard units, we must put back the factor $e^2/\hbar$ which has been set equal to unity. In these units the Hall conductivity at $E=0$ is

$$\sigma_{xy}=-\frac{\mathrm{sgn}M}{2}\frac{e^2}{h}\ . \tag{30}$$

The situation is depicted in Fig. 2, which shows that the Hall conductivity σ_{xy} is nonzero only in the bandgaps, $|E|<M$. Consider now specializing to $E=0$, which puts the Fermi level right between the upper and lower Dirac bands, and varying M from some positive value to a negative value, moving along the y axis in Fig. 2. We see a jump from $-\frac{1}{2}$ to $\frac{1}{2}$ in $\sigma_{xy}/(e^2/h)$. Including the contribution from the other heavy Dirac field, whose contribution does not change as the light mass changes sign, implies that the total physical Hall conductivity jumps from 0 to 1 in these units. Thus, at $E=0$, as M is tuned through zero, the system undergoes an integer Hall transition. This reveals our main reason for introducing the Dirac system. It manages to have an integer jump in σ_{xy} (in units of e^2/h) without any need for topological terms.

There are, however, striking similarities between the topological term in the sigma model of Pruisken[20] and the Dirac mass term here. To see this consider first the symmetries of the massless Dirac Hamiltonian

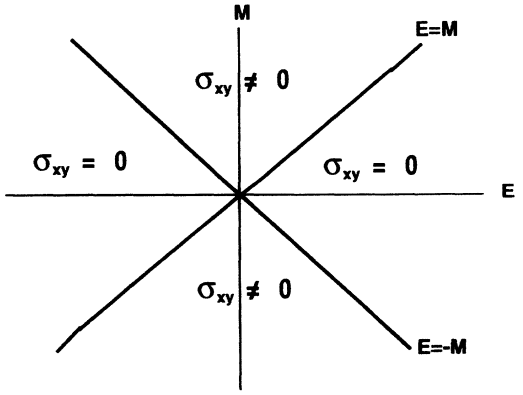

FIG. 2. The phase diagram of the pure system. The density of states is nonzero only in the regions where $|E|>|M|$. The Hall conductivity jumps by an integer upon crossing the E axis at $E=0$.

$$\mathbf{H}_0=\sigma\cdot\mathbf{p}\ . \tag{31}$$

The following can be said of $\mathbf{H}_0$ [we chose, for convenience, a (nonstandard) basis of the Pauli matrices, where σ_x and σ_y are both real and symmetric. This implies that σ_z is purely imaginary and antisymmetric]:

(i) It is invariant under the discrete symmetry transformation defined as

$$\mathcal{T}:\ \sigma_z\mathbf{H}_0^*\sigma_z=\mathbf{H}_0 \tag{32}$$

since the complex conjugation reverses the sign of $\mathbf{p}=-i\nabla$, and conjugation by σ_z brings in a second compensating sign change (σ_x and σ_y are chosen real.) This implies a Kramers degeneracy in the spectrum: ψ and $\sigma_z\psi^*$ are degenerate eigenstates. This discrete symmetry is not to be confused with the time-reversal operation of the original lattice model [Eq. (2)] as defined in Eq. (11). However, it does correspond to an effective time-reversal symmetry for the massless sector of this theory.

(ii) It has a particle-hole symmetry

$$\mathcal{PH}:\ \sigma_z\mathbf{H}_0\sigma_z=-\mathbf{H}_0\ , \tag{33}$$

which means that every eigenstate ψ of energy E is accompanied by a partner $\sigma_z\psi$ with energy $-E$.

(iii) It has a parity invariance under y reflections,

$$\mathcal{P}_y:\ \Pi_y\sigma_x\mathbf{H}_0\sigma_x\Pi_y=\mathbf{H}_0\ , \tag{34}$$

and parity invariance under x reflections. (Recall that in two dimensions, the parity operation reflects only x or y since changing the sign of both is equivalent to a rotation by π.)

The Dirac mass term

$$\mathbf{H}_M=M\sigma_z \tag{35}$$

is odd under $\mathcal{P}_x$ and $\mathcal{P}_y$. It does not respect particle-hole symmetry $\mathcal{PH}$. It has no time-reversal symmetry since, having chosen the other two Pauli matrices real, we cannot also choose σ_z real.

Let us now compare the role of M in our problem to the role of the topological parameter θ in the nonlinear-sigma-model of Levine, Libby, and Pruisken.[3]

(i) The topological term, with its single derivative in x and y, is odd under parity. The same goes for the mass term, as shown above.

(ii) Under parity, the θ parameter that multiplies the topological term changes sign. Since adding 2π to it makes no difference, we might as well say $\theta\to 2\pi-\theta$. This amounts to a reflection about the point $\theta=\pi$. The fixed point of this transformation is $\theta=\pi$ which is where the Hall transition is located. In our problem, M behaves like $\theta-\pi$, and the transition is at $M=0$.

Returning to Figure 2, let us ask what happens if we vary M from negative to positive values at $E\neq 0$, say $E>0$. In this case the system undergoes two transitions, one at $E=-M$ as it goes from the gap with positive Hall conductance to the gapless region where the Hall conductance is zero, and then once more at $E=M$ from the gapless region to the gapped region with negative Hall conductance. Thus we have "halved" the Hall transition, since the jump is half as much in each stage.

III. TRANSPORT PROPERTIES OF THE PURE SYSTEM

A. Kubo conductivities

We have already computed the dc Hall conductivity to establish the phase diagram of the pure system in the $M-E$ plane. Now we limit ourselves to $E=0$, and compute the frequency dependent longitudinal and Hall conductivities.

The ac conductivities as obtained from the Kubo formula are

$$\sigma_{\mu\nu}(\omega,M)=\frac{e^2}{h}(2\pi/\omega)\int d^2\mathbf{r}\,d\tau\langle J_\mu(\mathbf{r},\tau)J_\nu(\mathbf{0},0)\rangle[e^{-i\omega\tau}-1]\ ,\qquad \mu,\nu=x,y\ , \tag{36}$$

where the subtraction arises from the diamagnetic contribution. Here ω is a Matsubara frequency. We evaluate this explicitly in the Appendix for the theory of a single noninteracting Dirac fermion. The result [with $Z=(\omega/M)^2$] is

$$\sigma_{xy}(\omega,M)=-\frac{e^2}{h}\frac{\mathrm{sgn}(M)}{2}f[Z]\ , \tag{37}$$

$$\sigma_{xx}(\omega,M)=\frac{e^2}{h}\frac{1}{2\sqrt{Z}}\left[\frac{Z}{4}f[Z]+(1-f[Z])\right]\ , \tag{38}$$

where

$$f[Z]=\frac{2}{\sqrt{Z}}\arcsin\left[\frac{Z}{\sqrt{Z(Z+4)}}\right]\ . \tag{39}$$

For small and large arguments we have

$$f[Z]=\begin{cases}1, & \text{if } Z\to 0\\ \sim(\pi/\sqrt{Z}\,) & \text{as } Z\to+\infty\ .\end{cases} \tag{40}$$

For the dc conductivities of the single Dirac fermion, this implies

$$\sigma_{xy}^{(\mathrm{dc})}=\begin{cases}\dfrac{e^2}{h}\dfrac{-\mathrm{sgn}(M)}{2} & \text{if } M\neq 0\\ 0 & \text{if } M=0\ ,\end{cases} \tag{41}$$

$$\sigma_{xx}^{(\mathrm{dc})}=\begin{cases}0 & \text{if } M\neq 0\\ \dfrac{e^2}{h}\pi/8 & \text{if } M=0\ .\end{cases} \tag{42}$$

In our problem we have two Dirac fermions, one with mass M which changes sign at the transition, the other always having mass $\tilde{M}$ of fixed sign (say $\tilde{M}<0$). Thus for the total dc conductivities we obtain

$$\sigma_{xy}^{(\mathrm{dc})}=\begin{cases}\dfrac{e^2}{h}(\tfrac{1}{2})[1-\mathrm{sgn}(M)] & \text{if } M\neq 0\\ \dfrac{e^2}{h}(\tfrac{1}{2}) & \text{if } M=0\ ,\end{cases} \tag{43}$$

$$\sigma_{xx}^{(\mathrm{dc})}=\begin{cases}0 & \text{if } M\neq 0\\ \dfrac{e^2}{h}\pi/8 & \text{if } M=0\ .\end{cases} \tag{44}$$

We remark that the Hall conductivity $\sigma_{xy}^{(\mathrm{dc})}$ should equal $(\frac{1}{2})e^2/h$ at the transition ($M=0$) due to particle-hole symmetry.[21] This is reproduced precisely by our calculations.

B. Alternate definition of longitudinal conductivity

So far we have used the Kubo formula to calculate the conductivities. This required us to integrate over frequency ω, which posed no problem in the case without disorder [since described by a (2+1)-dimensional Gaussian theory]. We now develop another definition of the longitudinal dc conductivity, which we call $\bar{\sigma}_{xx}$, involving only the fields at $\omega=0$. The latter is more tractable when randomness is present. In this section we evaluate it in the free theory, and find $\bar{\sigma}_{xx}=1/\pi$ [see Eq. (59)].

Excluding interactions among the particles the action is, as discussed in Sec. II, a sum of independent actions S_ω, one for each Matsubara frequency:

$$S=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}S_\omega\ , \tag{45}$$

where

$$S_\omega=\int\frac{d^2\mathbf{p}}{(2\pi)^2}\bar{\psi}_\omega(\mathbf{p})\mathbf{H}_\omega\psi_\omega(\mathbf{p}) \tag{46}$$

and

$$\mathbf{H}_\omega=\boldsymbol{\sigma}\cdot\mathbf{p}+M\sigma_z-i\omega\ . \tag{47}$$

Note that for each value of ω we get a (2+0)-dimensional theory, i.e., a two-dimensional classical statistical mechanics model, or, equivalently, a one-dimensional quantum field theory. The Matsubara frequency plays the role of a coupling constant, like a mass term.

Ultimately we are interested in the properties of the physical 2D Hamiltonian $\mathbf{H}_{\omega=0}$. We see from Eq. (46) that the single-particle Green's functions of field operators at the same frequency just give the single-particle Green's functions of this 2D Hamiltonian:

$$\langle\psi_\omega(\mathbf{p})\bar{\psi}_\omega(\mathbf{p})\rangle=\frac{1}{\mathbf{H}_\omega} \tag{48}$$

or, using a mixed representation in 2D position space $\mathbf{r}=(x,y)$, and 1D Matsubara-frequency space (ω),

$$G_{ab}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle\psi_a(\omega,\mathbf{r}_1)\bar{\psi}_b(\omega,\mathbf{r}_2)\rangle=\sum_n\frac{\Psi_{na}(\mathbf{r}_1)\Psi_{nb}^{\dagger}(\mathbf{r}_2)}{E_n-i\omega}\ , \tag{49}$$

where we have used an orthonormal set of (spinor-) wave functions

$$\Psi_n(\mathbf{r})=\begin{bmatrix}\Psi_{n1}(\mathbf{r})\\ \Psi_{n2}(\mathbf{r})\end{bmatrix}\ , \tag{50}$$

$$\mathbf{H}_{\omega=0}\Psi_n(\mathbf{r})=E_n\Psi_n(\mathbf{r})\ . \tag{51}$$

Thus the single-particle Green's function of the (2+0)-dimensional Lagrangian of Eq. (46) give us the Green's function of the Hamiltonian $\mathbf{H}_{\omega=0}$. Of course, this is not new (see, e.g., Refs. 13 and 22). The novelty of our approach consists (partly) in the fact, as we will describe in more detail later, that the (2+0) Lagrangian is a critical theory for $\omega=0$, and is therefore conformally invariant. This will remain true when specific forms of randomness are included. The frequency ω is then a particular (in general massive) perturbation of this conformal field theory. We will exploit this in Sec. VI C.

Let us now discuss the longitudinal conductivity in this language: introduce the function (see, e.g., Refs. 13 and 23)

$$K(\mathbf{q},\omega)=\tfrac{1}{2}\int d^2r\,e^{i\mathbf{q}\cdot\mathbf{r}}G_{ab}(0,\mathbf{r};\omega)G_{ba}(\mathbf{r},0;-\omega)\ . \tag{52}$$

From the spectral decomposition Eq.(49), we get

$$K(\mathbf{0},\omega)=\frac{\pi\rho(\omega)}{2|\omega|}\ , \tag{53}$$

where $\rho(\omega)=-(1/\pi)\mathrm{Tr}G(r,r;\omega)$. More generally one expects the expansion (see, e.g., Ref. 13)

$$K(\mathbf{q},\omega)=\frac{\pi\rho(\omega)}{2|\omega|+Dq^2+\cdots}=\frac{\pi\rho(\omega)}{2|\omega|}-\frac{\pi\rho(\omega)}{4\omega^2}Dq^2+\cdots\ ,\qquad \frac{q^2}{\omega}\ll 1\ , \tag{54}$$

with D an expansion coefficient. Consider now the quantity

$$\bar{\sigma}_{xx}=(-1)\frac{e^2}{h}\lim_{\omega\to 0}8\omega^2\left[\frac{\partial}{\partial q^2}\right]_{|q=0}K(\mathbf{q},\omega) \tag{55}$$

$$=\lim_{\omega\to 0}\frac{e^2}{h}\omega^2\int d^2r\,r^2G_{ab}(0,\mathbf{r};\omega)G_{ba}(\mathbf{r},0;-\omega)\ . \tag{56}$$

When the density of states is nonzero, $\rho(0)\neq 0$, the dc conductivity can be written as $\sigma_{xx}=\bar{\sigma}_{xx}=\rho(0)D(e^2/\hbar)$, which follows from an Einstein relation (see, e.g., Ref. 13), D representing a diffusion coefficient. However, when the density of states vanishes, $\rho(0)=0$, such a simple interpretation of $\bar{\sigma}_{xx}$ in Eq. (56) is not appropriate. This is the case for the massless Dirac equation and for other situations with disorder discussed below. In those cases the finite limit on the right-hand side of Eq. (56) defines a universal scaling amplitude, which characterizes the theory with $\omega\neq 0$ in the scaling region. This definition of conductivity is often used in the sigma-model literature of localization (see, e.g., Refs. 13 and 24). Since this amplitude is universal, it may be compared with results obtained using other techniques. Further-

more, when the density of states is finite, $\rho(0)\neq 0$ [as is in fact the case at one particular point $\Delta_A=\pi$ on the line of fixed points we find in the presence of a random vector potential (Sec. VI)], $\bar{\sigma}_{xx}$ should agree with the conductivity as computed from the Kubo formula.

We now compute $\bar{\sigma}_{xx}$ in Eq. (56) for the pure model at criticality ($E=M=0$). Using Eqs. (46) and (47) we can write

$$\left[\frac{\partial}{\partial q^2}\right]_{|q=0} K(\mathbf{q},\omega)=\frac{1}{2}\left[\frac{\partial}{\partial q^2}\right]_{|q=0}\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{\mathrm{Tr}\{[\sigma\cdot(\mathbf{p}+\mathbf{q}/2)+i\omega][\sigma\cdot(\mathbf{p}-\mathbf{q}/2)-i\omega]\}}{[(p+q/2)^2+\omega^2][(p-q/2)^2+\omega^2]}\ . \tag{57}$$

After performing the trace and expanding for small q^2, we find

$$\mathrm{RHS}=\int\frac{d^2\mathbf{p}}{(2\pi)^2}\left\{\frac{(-\frac{1}{4})}{(p^2+\omega^2)^2}-\frac{(\omega^2/2)}{(p^2+\omega^2)^3}\right\}=\frac{(-1)}{8\pi\omega^2}\ . \tag{58}$$

We thereby obtain for the critical theory ($E=M=0$)

$$\bar{\sigma}_{xx}=\frac{1}{2\pi^2}\frac{e^2}{\hbar}=\frac{1}{\pi}\frac{e^2}{h}\ . \tag{59}$$

It is noteworthy that this value of $\bar{\sigma}_{xx}$ is different from the critical value of σ_{xx} obtained from the Kubo formula, although the two numerical values are actually quite close.

IV. THE MODEL WITH VARIOUS FORMS OF DISORDER

We now consider incorporating quenched disorder into the above description. Of primary interest to us is how the disorder will effect the integer quantum Hall phase transition at $E=M=0$. In general, quenched disorder will generate spatially random perturbations to the pure Lagrangian (at $E=M=0$):

$$L_0=\bar{\psi}(i\omega-\boldsymbol{\sigma}\cdot\mathbf{p})\psi\ . \tag{60}$$

Specifically, in the Dirac language we consider three types of randomness:

$$\text{(I)}:\ L_A=\bar{\psi}\boldsymbol{\sigma}\cdot\mathbf{A}(x,y)\psi\ , \tag{61}$$

$$\text{(II)}:\ L_M=\bar{\psi}M(x,y)\sigma_z\psi\ , \tag{62}$$

$$\text{(III)}:\ L_V=\bar{\psi}\psi V(x,y)\ , \tag{63}$$

where the functions $\mathbf{A}$, M, and V are nonuniform and random in space, but constant in time. Thus they mix up the momenta but not the frequencies. To be concrete we will assume that all three random potentials are symmetrically distributed about zero mean.

Let us understand the significance of these random perturbations in terms of the original lattice spinless fermions, and in terms of the symmetries they break.[25]

The perturbation L_A is the most apparent: it corresponds to a random vector potential. We know this because the coupling of the gauge field is uniquely given by the principle of minimal coupling: $\mathbf{p}\rightarrow\mathbf{p}-\mathbf{A}$. This term preserves the $\mathcal{PH}$ symmetry of $\mathbf{H}_0$. Notice that this is true sample by sample, since for every realization of the random potential we have

$$-\sigma_z\mathbf{H}\sigma_z=\mathbf{H}\ . \tag{64}$$

However, L_A violates both time reversal and parity in a fixed sample, but not on the average.

Next consider L_M, the addition of a spatially inhomogeneous Dirac mass term. If we go back to the spinless fermion Hamiltonian, it is clear that with all other parameters fixed, varying the staggered chemical potential μ in the Hamiltonian amounts to varying M [see Eq. (15)]. Since the Dirac mass couples to the two components of the spinor with opposite sign (due to σ_z), and the staggered chemical potential couples to the two sublattices with opposite sign, it is clear that the two components stand for the two sublattices within each unit cell. Now one may ask about the fact that each unit cell had two sites from the same sublattice. The answer is that within the eigenspace $\tau_y=1$, in which we are working, the variables on the two sublattices are slaved to each other by the eigenvalue equation. (The other eigenvalue represents heavy modes which will not be excited.) The random Dirac mass term breaks $\mathcal{P}$, $\mathcal{PH}$, and $\mathcal{T}$.

Let us finally consider the random term L_V. Since it couples to the two spinor components equally, it corresponds to a smooth nonstaggered potential that varies very little over each unit cell of the original lattice. It is readily shown that this term preserves $\mathcal{T}$, but violates both $\mathcal{PH}$ and $\mathcal{P}$.

The symmetry properties of these various random potentials are summarized in Table I. They play a fundamental role in the following sections.

V. RANDOM DIRAC MASS: EDGE STATES AND QUANTUM PERCOLATION

In this section we specialize to a case where the only randomness present in the model is in the Dirac mass term. Of the three types of randomness considered in

TABLE I. Symmetries of the free massless Dirac Hamiltonian.

Symmetry	$\sigma\cdot\mathbf{p}$	$M\sigma_z$	$M(x,y)\sigma_z$	$V(x,y)$	$\sigma\cdot\mathbf{A}$
Parity	Yes	No	No	No	No
Time reversal	Yes	No	No	Yes	No
Particle hole	Yes	No	No	No	Yes

Sec. IV, a random Dirac mass is the only term which breaks time-reversal invariance, parity, and particle-hole symmetry in a fixed sample (see Table I, Sec. IX). One might therefore guess that a random Dirac mass term would be sufficient to place the system in the generic quantum Hall universality class. However, as we shall now show, at least for weak randomness this is in fact not the case, since this type of disorder is insufficient to generate a nonvanishing density of states (at $E=0$). Nevertheless, the model with random Dirac mass alone is an instructive case to explore, since it exhibits edge states and allows for contact with the quantum percolation pictures of the quantum Hall transition.[26]

As we saw in Sec. II, for the pure system in the absence of any disorder one could pass through a quantum Hall transition by changing the sign of the Dirac mass M. Thus in our lattice model the Dirac mass, which tunes through the transition, plays a role analogous to the Fermi energy in a conventional discussion of the quantum Hall transition. A spatially varying potential energy in a conventional treatment of electrons in a Landau band would thus correspond in our model to a spatially varying Dirac mass term.

Let us elaborate on this. Consider a *single* Dirac particle of mass M in a constant scalar potential V. The dispersion relation is

$$P^2+M^2=(E-V)^2 , \tag{65}$$

from which it follows that

$$P=\sqrt{(E-V)^2-M^2} . \tag{66}$$

First let $V=0$. We see that if $E<M$, P is imaginary, and the wave function is suppressed exponentially. Consider now a region in which M rises as we move away from the origin as shown in Fig. 3(a). It is clear that the particle will be confined to the region where $E>M$.

Compare this with a case where V rises from zero as we move away from the origin, but M is constant as in Fig. 3(b). We see from Eq. (66) that P is real when $V-E>M$. The reason is of course that in a region of large positive V we can have a negative- (kinetic-) energy state of sufficiently large momentum.

So far we have considered the nature of single-particle wave functions. Let us consider now the response of the filled and empty Dirac states below and above the filled Fermi sea, to a spatially varying V and M depicted in Figs. 4(a) and 4(b). To be concrete we are assuming the Fermi energy is at zero energy. Figure 4(a) shows the band structure of a massless Dirac field subject to a spatially varying M. Notice that with the Fermi energy at zero, low-energy excitations exist only where $M(x)$ vanishes. On the other hand, we see in Fig. 4(b) that when a spatially varying potential V is applied, both the filled and empty bands bend in the same direction, and gapless excitations exist everywhere at the Fermi energy as $M\to 0$.

Consider finally Fig. 5, which shows a mass which is negative within a finite region around the origin and positive beyond. The only gapless excitations exist on the $M=0$ contour. Except for the replacement $V\to M$, this

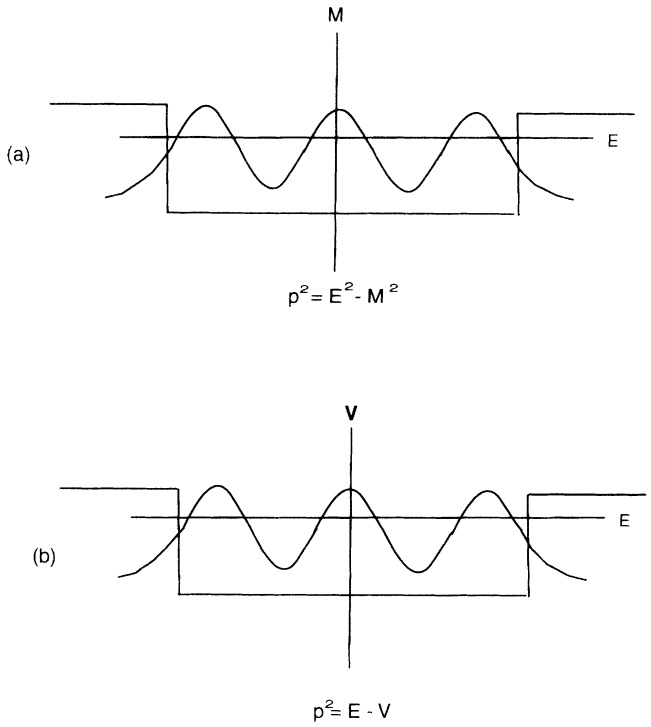

FIG. 3. Comparison of relativistic and nonrelativistic fermions. The former are confined by a mass term M, and the latter by a scalar potential V.

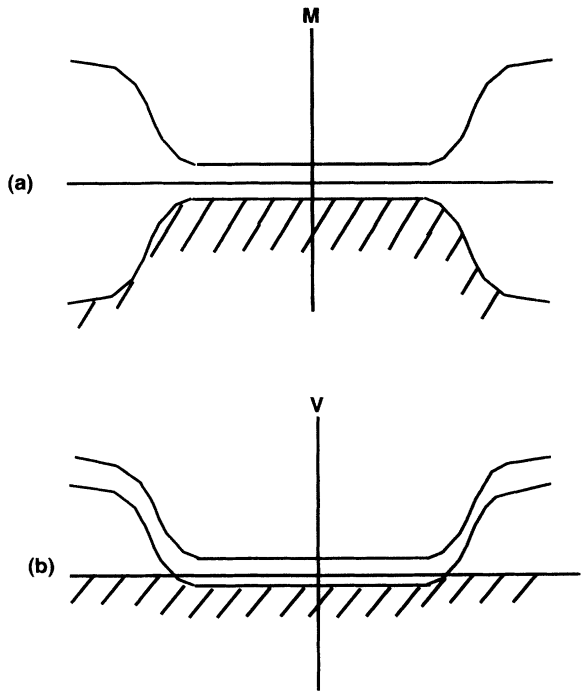

FIG. 4. Schematic representation of the effects of a confining potential on relativistic fermions. Notice that a scalar potential V is unable to confine the electrons since the negative (kinetic) energy electrons are not forbidden in the region of positive V.

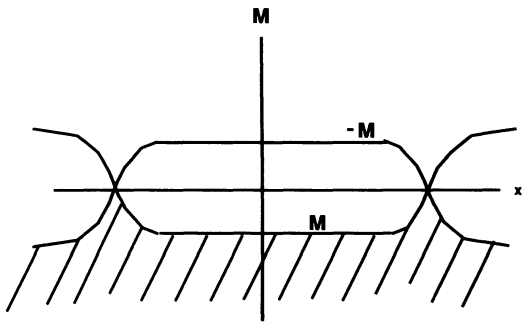

FIG. 5. With a spatially varying Dirac mass M, the states are filled as shown. Notice that the only gapless excitations are at the edges where $M=0$. The gap is sensitive only to the magnitude of M.

is exactly what is found with nonrelativistic fermions in the lowest Landau level and a confining potential V.

To reiterate, M in our problem plays the role of V in the conventional formulation of the Hall effect.

We now turn our attention to the edge states that live near the $M=0$ contours. Consider a Hamiltonian of the form

$$\mathbf{H}=\boldsymbol{\sigma}\cdot\mathbf{p}+M(x,y)\sigma_z \ , \tag{67}$$

where the Dirac mass term depends on the spatial coordinates. As discussed above, we anticipate that when the spatial variation of the Dirac mass is slow, the eigenstates of this Hamiltonian resemble edge states confined to the equipotential, $M(x,y)=0$. This can be seen most readily by considering a Dirac mass which depends only on one coordinate, say y, and is given by

$$M(x,y)=f(y) \ , \tag{68}$$

where f changes sign at $y=0$ as shown in Fig. 6. Let us

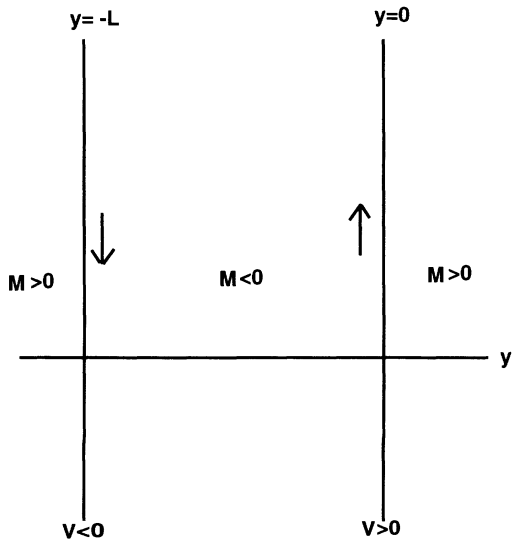

FIG. 6. A system with two edges caused by a spatially varying Dirac mass. A slight bias in the potential across the system causes a quantized edge current, as explained in the text.

assume, in addition, that there is also a scalar potential which depends only on x, $V(x)$. In this case an exact eigenstate, confined to the right edge (i.e., which falls off exponentially as one moves away from the edge at $y=0$), with arbitrary energy E, can be written

$$\Psi_E=e^{i\int_0^x[E-V(x')]dx'}\exp\left[-\int_0^y f(y')dy'\right]\begin{bmatrix}1\\1\end{bmatrix} . \tag{69}$$

If we consider a constant V, we have a propagating plane wave of momentum $k_x=E-V$ along the edge. These eigenstates have an amplitude which varies in the y direction. The eigenstate resembles an "edge state." Note that the spinor of this state is in an eigenstate of σ_x (eigenvalue $+1$). The spin is thus parallel to the edge.

Let us assume now that far to the left, at $y=-L$, we have another edge, where M changes back to positive. The wave function there is

$$\Psi_E=e^{-i\int_0^x[E-V(x')]dx'}\exp\left[\int_{-L}^y f(y')dy'\right]\begin{bmatrix}1\\-1\end{bmatrix} . \tag{70}$$

We could either obtain this solution from scratch or, better still, obtain it from the following argument. Suppose we slowly bend the right edge till it reverses direction to become the left edge. Assuming the solution found for the straight edge evolves adiabatically, we end up with the above result. Note in particular that the spin, which is always parallel to the edge, has undergone a rotation by 180°. It is interesting that upon fully encircling an equipotential that closes on itself, the spinor will rotate by 2π, and thus change the sign of the wave function. It is as if there is one-half of a magnetic flux quantum penetrating such closed loops. This causes a destructive interference which suppresses the zero-energy eigenstates on such isolated closed loops. Note also that there are no random phases for the $E=0$ closed loops in the absence of the potential V [from Eq. (69)]. This explains the presence of a universality class for the integer Hall transition with random M alone (see below), which is not in the generic universality class, such as that in the Chalker-Coddington model.[16,17]

The edge states allow us to gain a physical understanding of the behavior of σ_{xy}, computed earlier from the Kubo formula in Sec. II B. To this end consider a system with two edges as in Fig. 6. The edges are subject to a bias voltage which is $\pm V$ at the right/left. From Eqs. (69) and (70) we see that the additional current on each edge flows in the same direction, causing a net Hall current. If we now reverse M globally, the Hall current will change sign explaining the sgn$M$ factor in σ_{xy}. (To see this we must first find solutions that once again are confined to the edges. These are just the old solutions with right and left edges exchanged.)

For a random potential $M(x,y)$ with mean zero and a symmetric distribution, the equipotential $M(x,y)=0$ will percolate throughout the entire system. For a slowly varying (and possibly large) random Dirac mass, the zero-energy eigenstate will tend to be confined to this percolating backbone, as in the original quantum percolation

picture.[26] However, this semiclassical percolation picture has to break down at the saddle points in $M(x,y)$, where the equipotential lines intersect, due to quantum tunneling effects. In fact we now show that when the random Dirac mass is small, the tunneling dominates, driving the effective randomness to zero at long-length scales.

To analyze the effects of the random Dirac mass term at small randomness, we introduce a generating functional Z, which can be used to extract Green's functions of the Hamiltonian in Eq. (67):

$$Z = \int D\psi D\bar{\psi} e^{-S} , \tag{71}$$

with action

$$S = \int d^2x \, \bar{\psi}[\sigma \cdot \mathbf{p} + M(\mathbf{x})\sigma_z]\psi , \tag{72}$$

where ψ and $\bar{\psi}$ are independent two-component Grassmass fields. We shall assume that the random Dirac mass is Gaussian with mean zero and variance Δ_M. Before ensemble averaging it is convenient to let $\bar{\psi} \to i\bar{\psi}\sigma_z$ and to rotate by $\pi/2$ about the z axis to put the action in the form

$$S = \int d^2x \, \bar{\psi}[\sigma \cdot \mathbf{p} + iM(\mathbf{x})]\psi . \tag{73}$$

Notice that this transformation eliminated the σ_z multiplying the Dirac mass, and replaced it with an i.

We now use the replica trick, and ensemble average over the random Dirac mass. The effective action finally becomes

$$S = \int d^2x \, \bar{\psi}^{\alpha}(\sigma \cdot \mathbf{p})\psi_{\alpha} + \Delta_M \int d^2x \, (\bar{\psi}^{\alpha}\psi_{\alpha})(\bar{\psi}^{\beta}\psi_{\beta}) , \tag{74}$$

where a sum over the replica indices α and β from 1 to n is understood. This replicated action is formally equivalent to the classical two-dimensional random bond Ising model, where Δ_M is a measure of the strength of randomness in the Ising bonds.[6] The corresponding Ising model is at its critical point. The connection with percolation discussed above thus becomes apparent. With increasing random bond strength, the transition temperature in the $2d$ Ising model is driven down, and is eventually driven to zero at the classical percolation point. The zero-temperature percolation critical properties are known to be unstable to thermal fluctuations.[7] Moreover, following Ref. 6, one can easily show that a perturbative RG analysis on the above action gives

$$\frac{\partial \Delta_M}{\partial l} = -\frac{\Delta_M^2}{\pi} , \tag{75}$$

so that weak randomness is irrelevant, and the critical properties are determined by the pure $2d$ Ising model. If this transition is observed, it will be interesting to look for the calculable logarithmically decaying corrections to the pure Ising results.[6]

The work of LeDoussal and Georges[8] on diluted Ising models indicates that the irrelevant flow into the Ising fixed point originates at the classical percolation fixed point, as shown in Fig. 7. There are no intermediate fixed points. In terms of our quantum percolation picture of edge states, this means that quantum tunneling between the edges dominates at large length scales, driving the system into the pure Ising fixed point.

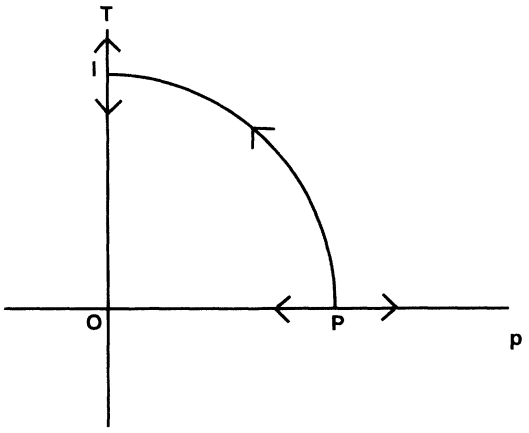

FIG. 7. A schematic flow diagram for the diluted Ising magnet as a function of temperature T and dilution probability p, with the ferromagnetic phase at small T and p. The fixed points denoted I and P correspond to the pure Ising fixed point and the percolation fixed point.

This analysis then strongly suggest that for the critical properties of the quantum Hall transition described by the Hamiltonian Eq. (67), the random Dirac mass term is always irrelevant. The transition is described by the free Dirac theory, with logarithmic corrections. Since the free Dirac theory has a vanishing density of states (at $E=0$), it is apparent that this is not in the same universality class as the generic disordered quantum Hall transition. (It is less clear whether or not there will be a finite, noncritical, background density of states for the random Dirac mass model.)

How then is this transition to be observed? A possible way is to design an array of quantum Hall droplets whose edge states enclose half a unit of flux. As the particle density is varied, the droplets will percolate and the transition should lie in the class described. There is also numerical evidence[27] that if a Chalker-Coddington[16,17] model is constructed with the random-phase factors removed, the transition has $\nu=1$. By the same token, if we introduce random phases into our closed loops, we presumably crossover to the generic Hall transition. From the preceding analysis it is clear that if a random V or nonzero E is introduced it will generate [see Eq. (69)] such random phases $\oint (E-V)dx$ depending on the details of each closed loop. Thus adding both random M and random V to our Dirac theory is presumably sufficient to put the model into the basin of attraction of the generic Hall fixed point.

We also remark that the percolation picture permits to produce approximate estimates for the *diagonal* conductivity.[28] Since we are focusing here on exact methods, we have not attempted to apply these approximate methods to our model. Nevertheless, it might be interesting to follow this route in future work.

VI. THE RANDOM VECTOR POTENTIAL FIXED LINE

In this section we specialize to the case where the only disorder present is in the vector potential. We drop from consideration completely the random Dirac mass term and the random scalar potential. As we shall see, with this simplification our random Hamiltonian becomes analytically tractable.

Let us then consider the Hamiltonian

$$\mathbf{H}=\boldsymbol{\sigma}\cdot\mathbf{p}+\boldsymbol{\sigma}\cdot\mathbf{A}\ , \tag{76}$$

where the vector potential is taken as a quenched random variable which satisfies a Gaussian white-noise distribution with mean zero and variance Δ_A. Physical properties can of course depend only on the magnetic field, $B=\epsilon_{ij}\partial_i A_j$, which is then also Gaussian with variance:

$$\overline{B(k)B(k')}=(2\pi)^2\delta^2(k+k')\Delta_A k^2\ . \tag{77}$$

Notice that the average field $B(k=0)$ has no fluctuations.

As in the pure model studied in Secs. II and III, the above Hamiltonian describes a critical theory (i.e., a quantum Hall transition), because the Dirac mass has been set to zero. Below we will focus on the associated random critical behavior, but it is instructive first to consider the massive theory in the presence of the random vector potential. Specifically, we examine the spectra of $H_M=H+M\sigma_z$, where H is the above Hamiltonian, and M is a uniform Dirac mass. It is straightforward to show that $H_M^2=H^2+M^2$, for every given realization of the random vector potential. Thus all eigenstates of the random Hamiltonian H_M have energies E_M which satisfy $E_M^2\geq M^2$. Thus the band gaps in the pure model (at $E^2\leq M^2$) are not filled in by a random vector potential term, no matter how large Δ_A is. As a result, at zero energy the Hall conductance for nonzero M will take on the same quantized values as in the pure case (see Fig. 2). As one tunes M through zero the quantized Hall conductance will jump by e^2/h. But now at $M=0$ we have a random Hall critical point.

As we shall show below, despite the presence of the random term this critical point is quite tractable. In Sec. VI A we show that (for $M=0$) it is possible to obtain exact zero-energy wave functions for the Hamiltonian, for arbitrary randomness Δ_A. This is a nice example of the so-called Atiyah-Singer index theorem. These wave functions are neither plane waves nor exponentially localized, but rather are a pedagogical example of multifractals.[9,11,12] We work out explicitly the multifractal $f(\alpha)$ spectrum, which is an exact parabola in this case, reflecting the log-normal distribution of the wave functions. We also compute all participation ratios. The presence of exact "extended" wave functions for this two-dimensional random Hamiltonian, in a given realization of disorder and at arbitrary disorder strength, establishes the presence of a nontrivial fixed line of quantum Hall phase transitions (parametrized by Δ_A). While we do not believe that the generic and physical integer quantum Hall effect (IQHE) phase transitions lie on this fixed line of critical points, it is certainly noteworthy that numerical calculations[10] for the generic IQHE transition have revealed extended wave functions with similar multifractal scaling properties.

In Secs. VI B and VI C we turn to properties of the random Hamiltonian (with $M=0$) which do not follow simply from the properties of the zero-energy wave function. We analyze single-particle properties, focusing on the ensemble-averaged single-particle Green's function and the resulting density of states (DOS) and mean free path. We show that the DOS is singular at zero energy, with a power-law dependence on energy with an exponent that varies continuously with Δ_A along the fixed line. There is one special point along the fixed line where the DOS is in fact constant (i.e., the power is zero).

Lastly we focus on two-particle properties with an emphasis on the electrical conductivity. We show that the longitudinal conductivity defined from the DOS and the diffusion constant, via an Einstein relation (as discussed in Sec. III B), is independent of disorder along the fixed line and exactly equal to $e^2/\pi h$. This result is established using methods from the conformal field theory, and exploits an underlying SU(2) symmetry between positive and negative frequency fields along the critical line.

A. Exact zero-energy wave functions

We first discuss the two-component eigenfunctions $\Psi(x)$ of the random Hamiltonian in a fixed realization of disorder, which satisfy $\mathbf{H}\Psi=E\Psi$ with zero energy, $E=0$. First we note that since σ_z anticommutes with the Hamiltonian, we have the particle-hole symmetry $\mathcal{PH}$ as established earlier in Sec. III: for every eigenfunction Ψ with energy E there is a corresponding eigenfunction $\tilde{\Psi}=\sigma^z\Psi$ which has energy $-E$. Thus the number of zero-energy eigenfunctions is even. This is also implied by the Atiyah-Singer index theorem (see, e.g., Ref. 29), which tells us that the number of right-handed and left-handed solutions ($\sigma_z=\pm 1$) at zero energy must differ by the total number of flux quanta, which is zero in our case.

In order to write an exact zero-energy eigenfunction it is convenient to specialize to the Coulomb gauge, $\partial_i A_i=0$. In this gauge one can express the vector potential in terms of a scalar field Φ:

$$A_i=\epsilon_{ij}\partial_j\Phi\ . \tag{78}$$

The magnetic field satisfies $B=\nabla^2\Phi$, so that we must take

$$\overline{\Phi(k)\Phi(k')}=(2\pi)^2\delta^2(k+k')\frac{\Delta_A}{k^2}\ , \tag{79}$$

It is then straightforward to verify that the random Hamiltonian annihilates the two wave functions

$$\Psi_{\pm}=(\text{const})(1\pm\sigma_z)\begin{bmatrix} e^{\Phi(x)} \\ e^{-\Phi(x)} \end{bmatrix}\ , \tag{80}$$

so that they are exact zero-energy eigenfunctions, in a given realization of disorder.

Notice that these wave functions are real and nodeless.

Although they are highly inhomogeneous, it is clear that they are not exponentially localized. In fact, in the thermodynamic limit they are non-normalizable, as is apparent from Eq. (80) since even the logarithm of the wave function has an infinite variance. In order to quantify the "extended" nature of these wave functions it is useful[9] to consider a *normalized* wave function in an $L \times L$ box with periodic boundary conditions:

$$\Psi(x)=\frac{e^{-\Phi(x)}}{\left[\int_{x'} e^{-2\Phi(x')}\right]^{1/2}} , \tag{81}$$

where we are now focusing exclusively on Ψ_+, and so Ψ in Eq. (81) is a scalar. The integration above is understood to be over the $L \times L$ square. In order to satisfy the periodic boundary conditions we require that $\Phi(x)$ in Eq. (80) is also periodic, and moreover take $\Phi(k=0)=0$ in every member of the ensemble.

In order to characterize the wave function it is useful[9] to consider the *participation ratios*, i.e., moments of the square of the wave function:

$$P_q(L)\equiv\overline{|\Psi(x)|^{2q}} \text{ (participation ratios)} , \tag{82}$$

where the power q need not be integer. Here it is important that $\Psi(x)$ is a normalized wave function in the $L \times L$ square. The large L behavior of $P_q(L)$ characterizes the degree to which the wave function is "extended." For example, for a plane wave one would have $P_q \sim 1/L^{2q}$, whereas $P_q \sim 1/L^2$ for an exponentially localized state. More generally one expects

$$P_q(L)\sim\frac{1}{L^{2+\tau(q)}}\sim\frac{1}{L^{2+(q-1)D(q)}} , \tag{83}$$

with some function $\tau(q)$ and $D(q)$. Were the wave function a simple fractal, $D(q)$ would be independent of q. But in a multifractal, D will depend on q, and equivalently $\tau(q)$ will be a nonlinear function of q. It is then customary to perform a Legendre transform on $\tau(q)$, by defining $\alpha=d\tau(q)/dq$ and a function

$$f(\alpha)=\alpha q-\tau(q) . \tag{84}$$

The general meaning of $f(\alpha)$ can be understood as follows.[9,12] Let $\Pi_L(Y)$ denote the probability distribution of $Y=|\Psi|^2$, the square of the wave function, so that

$$P_q(L)=\int dY\, Y^q \Pi_L(Y) . \tag{85}$$

Define $\alpha=-\ln(Y)/\ln(L)$. Using a simple steepest descent argument, one can show that in order to obtain the correct power-law dependence of $P_q(L)$ on L, the distribution function of α defined via

$$\tilde{\Pi}_L(\alpha)d\alpha=\Pi_L(Y)dY \tag{86}$$

must be

$$\tilde{\Pi}_L(\alpha)\sim e^{f(\alpha)\ln(L)} . \tag{87}$$

For large L this is dominated by the value of α, $\alpha_{\max}$, which maximizes the function $f(\alpha)$. It is thus clear (see, e.g., Ref. 12) that $\alpha_{\max}$ is the exponent for the typical value of $Y=|\Psi|^2$:

$$P_{\text{typical}}(L)\equiv\exp(\overline{\ln(|\Psi|^2)})\sim\frac{1}{L^{\alpha_{\max}}} . \tag{88}$$

We now proceed to evaluate the $f(\alpha)$ spectrum for our exact zero-energy wave function. Although Φ is assumed to have a Gaussian distribution, the average one needs to perform in Eq. (82) is not completely trivial since the wave function must be normalized first. The averaging can, though, be carried out straightforwardly, as shown in the Appendix, to yield

$$\tau(q)=2(q-1)+\frac{\Delta_A}{\pi}q(1-q) . \tag{89}$$

Notice that the term proportional to Δ_A is nonlinear in q, which indicates that the wave function is indeed a multifractal for nonzero disorder Δ_A. The resulting $f(\alpha)$ spectrum is (exactly) parabolic:

$$f(\alpha)=2-\frac{(\alpha-\alpha_{\max})^2}{4(\alpha_{\max}-2)} , \tag{90}$$

with

$$\alpha_{\max}=2+\frac{\Delta_A}{\pi} . \tag{91}$$

We thus see that the exact zero-energy wave functions are multifractals, and that the typical (local) value of the probability varies with an exponent $\alpha_{\max}$, for large system size L. Notice that $\alpha_{\max}$ increases as we increase the disorder strength moving along the fixed line. (For zero disorder, the probability distribution Π becomes trivial.) In several recent numerical calculations[10] on models of two-dimensional noninteracting electrons moving in magnetic field and random potential, it was found that the "extended" wave functions at the middle of the disorder-broadened Landau levels were indeed multifractals. Moreover, all simulations showed a roughly parabolic $f(\alpha)$ with $\alpha_{\max}\approx 2.3$. [Deviations from exact parabolic form (e.g., the presence of cubic terms) could not be confirmed with certainty because of numerical errors.[30]]

This behavior would be consistent with one particular point on our fixed line. However, we believe that the generic IQHE critical point probed in the numerical work does not lie on the fixed line of the Hamiltonian Eq. (76).

B. Single-particle properties

1. Perturbative RG

We now study the scaling properties of averaged single-particle quantities of the random-vector-potential model through perturbation theory in the disorder strength Δ_A. The starting point is the generating functional $Z=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_A}$, where S_A is the effective action written in terms of the independent Grassmann variables ψ and $\bar{\psi}$, for frequency ω:

$$S_A\equiv\int d^2\mathbf{p}\bar{\psi}(\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p}-i\omega)\psi(\mathbf{p})$$

$$+\int d^2x\,\bar{\psi}(\mathbf{x})\boldsymbol{\sigma}\cdot\mathbf{A}(\mathbf{x})\psi(\mathbf{x}) . \tag{92}$$

One can then average over the randomness with the replica method to arrive at the replicated effective action

$$S_A^{n=0}=\int_{\mathbf{p}}\sum_{\alpha}\bar{\psi}_\alpha(\sigma\cdot\mathbf{p}-i\omega)\psi_\alpha$$
$$-\Delta_A\int_{\mathbf{x}}\sum_{\alpha,\beta}\sum_{\mu=x,y}\bar{\psi}_\alpha\sigma_\mu\psi_\alpha\bar{\psi}_\beta\sigma_\mu\psi_\beta\ . \tag{93}$$

Here the replica indices α and β are summed from 1 to n, and the $n\to 0$ limit is to be taken, as usual, at the end of the calculation. In deriving this last equation we have integrated over all gauges of the vector potential.

Standard methods[31] can now be applied to generate, perturbatively in Δ_A, RG recursion relations for ω, and the disorder strength Δ_A, and RG equations for single-particle correlation functions of physical interest, such as the impurity-averaged density of states at energy E:

$$\rho(E)\equiv\lim_{\omega\to 0}\mathrm{Im}\,\mathrm{Tr}\bar{G}(i\omega-E,\mathbf{x}=0)/\pi\ ; \tag{94}$$

here $\bar{G}(i\omega-E,\mathbf{x})$ is the impurity-averaged, single-particle Green's function.

The diagrams of Figs. 8(a) and 8(b), respectively, control the first-order renormalization of the terms of the effective action $S_A^{n=0}$, that are quadratic and quartic in the Grassmann variables ψ and $\bar{\psi}$. Here the solid lines represent the bare propagator, which in momentum space takes the form

$$G_0(i\omega-E,\mathbf{p})=(\sigma\cdot\mathbf{p}+i\omega-E)/[p^2-(i\omega-E)^2]\ , \tag{95}$$

while the dashed line represents the coupling constant Δ_A and the σ matrices in Eq. (93). We use the RG rescalings $p=b^{-1}p'$ and $\psi(\mathbf{p})=b^{\zeta}\psi'(\mathbf{p}')$, choosing ζ to keep the coefficient of $\sigma\cdot p$ in Eq. (93) fixed. One thereby straightforwardly obtains from the diagrams in Fig. 8 that $\zeta=\frac{3}{2}$, and the following recursion relations:

$$\omega'=b^z\omega\ , \tag{96}$$

$$\Delta_A'=\Delta_A\ . \tag{97}$$

Here the dynamical exponent z is given by

$$z=1+\Delta_A/\pi\ . \tag{98}$$

Since ω and E always occur in the combination $i\omega-E$ in all Green's functions, the recursion relation

$$E'=b^zE \tag{99}$$

follows from Eq. (96).

While the result for ζ depends on the choice of gauge for the random vector potential, the recursion relations for E, ω, and Δ_A, and in particular the value of the exponent z are gauge invariant. Note that the impurity strength Δ_A does not renormalize to lowest order. It is easy to check directly from the form of S_A that Δ_A is dimensionless in 2D, which accounts for the lack of renormalization to zeroth order; to first order, however, the invariance is less trivial, requiring the cancellation of the contributions from the three diagrams of Fig. 8(b). We show in Sec. VI B 2 through bosonization that the recursion relations above and the expression for z actually hold to all orders in Δ_A, so that $E=\omega=0$ for arbitrary Δ_A is a true fixed line of the RG transformation.

From the above scaling follows immediately a result for the asymptotic behavior of $\rho(E)$ for small E: The recursion relation for E (or ω) implies the scaling equation $\int d^2x\bar{\psi}\psi\sim b^z\int d^2x'\bar{\psi}'\psi'$. Since $\rho(E)\sim\mathrm{Im}\langle\bar{\psi}\psi\rangle$, this yields the exact result $\rho(E)\sim b^{z-2}\rho(E')$, whereupon, from Eq. (99),

$$\rho(E)\sim E^{(2-z)/z}\ . \tag{100}$$

Equation (98) then shows that the density of states develops a singularity, i.e., an infinite derivative, at $E=0$ for any arbitrarily weak disorder Δ_A. [For the pure system, $\Delta_A=0$, and $\rho(E)$ is simply linear in E, consistent with the result derived in Sec. II.] This singularity becomes stronger with increasing Δ_A [see Eq. (98)], corresponding to the progressive filling in of the hole in the density of states at zero energy. Finally, at $\Delta_A=\pi$, where $z=2$, $\rho(E)$ approaches a constant as $E\to 0$, thereby developing at $E=0$ the finite density of states expected in the quantum Hall transition. (Recall from the Atiyah-Singer index theorem of Sec. VI A that there exist extended states at $E=0$ for all Δ_A.) For $\Delta_A>\pi$, the density of states diverges at $E=0$.

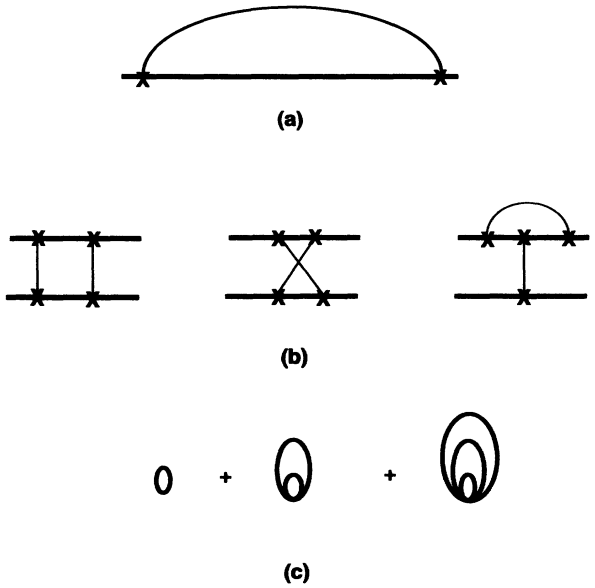

FIG. 8. Feynman diagrams contributing to the lowest-order RG calculation of (a) the self-energy for the propagator in the model of Eq. (93). Solid lines represent the bare propagator, and dashed lines the strength Δ_A of the random vector potential. (b) The renormalization of Δ_A in Eq. (93). (c) The renormalization of the frequency ω in model of Eq. (118).

2. *Diverging lengths near $E=0$*

Since the states at nonzero energy for any Δ_A are presumably localized, it is natural to assume that the singularity in $\rho(E)$ at $E=0$ reflects the localization transition at $E=0$. Except at the special value $\Delta_A=\pi$, where there is no singularity, this behavior represents a

departure from the conventional localization transition described by the nonlinear sigma model,[22,13] which is characterized by a finite, nonzero density of states through the mobility edge. There is another important difference: The correlation lengths characterizing the decay of the single-particle Green's function are finite through the mobility edge in the standard localization picture,[22,13] whereas they diverge in our model, even at $\Delta_A=\pi$. One can infer the occurrence of single-particle lengths that diverge at $E=0$ in the random-vector-potential model from the RG calculation just described. In particular, consider the recursion relation Eq. (99) which shows that E grows like b^z under length rescaling. The value of b at which, starting from some infinitesimal initial value E_0, the renormalized E becomes of order unity, then defines a characteristic length $\xi_1 \sim E_0^{-\nu_1}$, with

$$\nu_1 = 1/z \ . \tag{101}$$

To make these notions more explicit, it is instructive to calculate perturbatively in Δ_A the single-particle, impurity-averaged Green's function $\bar{G}(i\omega - E, \mathbf{x})$ in the limit $\omega \to 0$. The main physics of the spatial decay being independent of the gauge, we simplify the computation by averaging over all gauges. To lowest nontrivial order in Δ_A, the only diagram contributing to the one-particle irreducible self-energy matrix Σ, in Dyson's equation, $\bar{G}^{-1}(i\omega - E, \mathbf{p}) = G_0^{-1}(i\omega - E, \mathbf{p}) - \Sigma(i\omega - E, \mathbf{p})$, is shown in Fig. 8(a). Since this diagram does not have any dependence upon external momentum, it is easy to see that to this order

$$\bar{G}^{-1} \sim \boldsymbol{\sigma} \cdot \mathbf{p} - (\tilde{E} + i\tilde{\omega}) \ , \tag{102}$$

where, in the limit $\omega \to 0$, $\tilde{E} = E[1 + \Delta_A \ln(\Lambda/E)/\pi]$, and $\tilde{\omega} = \mathrm{sgn}(\omega)\Delta_A E/2$; here Λ is the high-momentum cutoff. Straightforward Fourier transformation then shows that the exponential piece of the averaged Green's function behaves like

$$\bar{G}(\mathbf{x}) \sim e^{ix/\lambda} e^{-x/\xi_1} \ , \tag{103}$$

where, for small E,

$$\lambda \sim E^{-(1-\Delta_A/\pi)} \ , \tag{104}$$

$$\xi_1 \sim E^{-1}/\Delta_A \ . \tag{105}$$

The physical significance of the two lengths λ and ξ_1 is clear: λ is just the wavelength with which a typical wave function oscillates in space, and so is finite even in the pure system ($\Delta_A = 0$), where all the states are extended, diverging only at $E=0$; ξ_1, on the other hand, is the elastic phase-coherence length, and so diverges in the pure system limit, as well as at $E=0$. Given the RG arguments above, we expect both of these lengths to diverge as $E \to 0$ like $E^{-\nu_1}$, where, from Eqs. (98) and (99), $\nu_1 = (1 + \Delta_A/\pi)^{-1}$. The prefactor of ξ_1 should itself diverge as $1/\Delta_A$ as $\Delta_A \to 0$, as seen from Eq. (105). To the order in which we are working, the results in Eqs. (103) and (105) are both consistent with this expectation.

3. *Exact scaling from bosonization*

We now verify to all orders in Δ_A the existence of the fixed line at $E=\omega=0$, and the result $z = 1 + \Delta_A/\pi$. To do so requires recasting the generating functional Z for fermions in 2D in terms of a scalar field θ. Recall the rules for (abelian) bosonization:[32]

$$\int [D\psi D\bar{\psi}] \to \int [d\theta] \ , \tag{106}$$

$$\bar{\psi}\boldsymbol{\sigma} \cdot (-i\boldsymbol{\nabla})\psi \to \tfrac{1}{2}(\nabla_v \theta)^2 \ , \tag{107}$$

$$\bar{\psi}\sigma_i \psi \to i \frac{1}{\sqrt{\pi}} \varepsilon^{ij} \partial_j \theta \ . \tag{108}$$

We will find it convenient to work with left- and right-moving fermions in 1D, and express them in terms of left- and right-moving boson fields $\theta_{L/R}$. To write Z in the conventional form in terms of the Grassmann variables, ψ_R and ψ_L, for right- and left-moving electrons, respectively, in 1D, make the change of variables

$$(-i)(\bar{\psi}_1, \bar{\psi}_2)\sigma_y \equiv (\bar{\psi}_R, \bar{\psi}_L) \ , \tag{109}$$

while

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \equiv \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix} \ . \tag{110}$$

This has the effect of eliminating the Pauli matrices (which we take in the standard form for this purpose) from the action, which then takes the form

$$\begin{aligned} S \sim & \int dx\, dy 2 \left[\overline{\psi_L} \frac{\partial}{\partial z^*} \psi_L + \overline{\psi_R} \frac{\partial}{\partial z} \psi_R \right] \\ & + \int dx\, dy [\, A_x(\overline{\psi_R}\psi_R - \overline{\psi_L}\psi_L) + iA_y(\overline{\psi_R}\psi_R + \overline{\psi_L}\psi_L) \\ & - i\omega(\overline{\psi_L}\psi_R - \overline{\psi_R}\psi_L)] \ , \end{aligned} \tag{111}$$

where

$$z \equiv y + ix, \quad z^* \equiv y - ix \ . \tag{112}$$

Treating the y direction as a (imaginary) time axis (i.e., letting $ip_y \to \nabla_t$) gives this equation an obvious interpretation in terms of two sets of 1D fermions moving to the right (R) (i.e., with momentum $+p_x$) and left (L) ($-p_x$). Thus the problem has been cast in a form to which bosonization in terms of right and left movers can be applied. We sketch the procedure only briefly; further details can be found in, e.g., Refs. 33–35 and 32.

Writing ψ_R and ψ_L in terms of the boson fields ϕ_R and ϕ_L defined by

$$\psi_R \sim e^{i\pi^{1/2}\phi_R} \ , \tag{113}$$

$$\psi_L \sim e^{i\pi^{1/2}\phi_L} \ , \tag{114}$$

one can express the kinetic-energy term (first line) of Eq. (111) in the quadratic form

$$S_{\rm KE}\sim\tfrac{1}{4}\int dx\,d\tau[i\nabla_x\phi_R\nabla_\tau\phi_R-i\nabla_x\phi_L\nabla_\tau\phi_L+(\nabla_x\phi_R)^2+(\nabla_x\phi_L)^2]\ . \tag{115}$$

The densities $\bar\psi_R\psi_R$ and $\bar\psi_L\psi_L$, respectively, are given by $\nabla_x\phi_R$ and $\nabla_x\phi_L$, while the "backscattering" operator $(\bar\psi_L\psi_R-\bar\psi_R\psi_L)$ is simply proportional to $\sin[\pi^{1/2}(\phi_R-\phi_L)]$. With a modest amount of algebra, one can then express the effective action in terms of the fields $\phi\equiv(\phi_R+\phi_L)/2$ and $\theta\equiv(\phi_R-\phi_L)/2$:

$$S=\int dx\,d\tau\{i\nabla_x\phi\nabla_\tau\theta+[(\nabla_x\phi)^2+(\nabla_x\theta)^2]/2+i(A_x\nabla_x\phi+A_y\nabla_x\theta)/\pi^{1/2}+\omega\sin(2\pi^{1/2}\theta)\}\ . \tag{116}$$

Since no terms higher than quadratic in the field ϕ appear in this Hamiltonian, one can eliminate ϕ via direct integration, producing the following rather simple effective action

$$S_\theta=\int dx\,d\tau[(\nabla_\mu\theta)^2/2+iB(x,\tau)\theta/\pi^{1/2}+\omega\sin(2\pi^{1/2}\theta)]\ , \tag{117}$$

where B is the magnetic field $\nabla_x A_\tau-\nabla_\tau A_x$. This result agrees with that quoted in the bosonization dictionary [Eq. (108)] at the outset of this subsection.

One is now in a position to use replicas to average over the randomness. The manifest gauge invariance of the action Eq. (117) allows one to treat A_x and A_y as independent random variables, and so to average over all gauges, producing the replicated action

$$S_\theta^{n\to0}=\int dx\,d\tau\left[\sum_{\alpha\beta}(\nabla_\mu\theta_\alpha\nabla_\mu\theta_\beta)(\delta_{\alpha\beta}+\Delta_A/\pi)/2+\omega\sum_\alpha\sin(2\pi^{1/2}\theta_\alpha)\right]\ . \tag{118}$$

In the limit of vanishing ω in which we are ultimately interested, this action is purely quadratic. In consequence, the only possible renormalization of Δ_A comes from its "engineering dimension." Since, as noted earlier, Δ_A is dimensionless, one concludes that it does not renormalize to any order in perturbation theory, i.e., that Eq. (97) is indeed exact.

The fact that Δ_A enters Eq. (118) only in a quadratic term likewise enables one to compute the exponent z governing the renormalization of ω near $\omega=0$ (or that of E for small E) to all orders in Δ_A. Figure 8(c) shows the diagrams governing the renormalization of ω to $O(\omega)$. Using the result for the quadratic part of the action, viz.

$$\langle\theta_\alpha(q)\theta_\beta(-q)\rangle_0=(\delta_{\alpha\beta}+\Delta_A/\pi)\ , \tag{119}$$

valid in the $n\to0$ limit, one readily arrives at the exact expression

$$\omega'=b^{1+\Delta_A/\pi}\omega\ . \tag{120}$$

Comparison with Eq. (96) then confirms the validity of (98) to all orders in Δ_A.

4. *Correlation length exponent of constant Dirac mass*

In the presence of a constant Dirac mass term $M\bar\psi\sigma_z\psi$ (see Sec. II), in the action, calculations very similar to those described above can be performed to calculate the flow of M away from the fixed line. While this can be done perturbatively in the fermion representation, it can be accomplished to all orders in Δ_A through the bosonization methods just discussed. Using (113) and (114) and the fact that the Dirac mass term can be expressed (using the standard basis of Pauli matrices) as $M(\bar\psi_R\psi_L+\bar\psi_L\psi_R)$, one readily writes this term in the form $M\cos(2\pi^{1/2}\theta)$. At this point the computation of the leading flow of M away from the fixed line at $M=E=\omega=0$ for arbitrary Δ_A is seen to be virtually identical to that of ω, just described, and has the same result, viz.

$$M'=b^{1+\Delta_A/\pi}M\ . \tag{121}$$

Thus the exponent ν, governing the divergence of the length associated with the $M\to0$ limit, is

$$\nu=\frac{1}{1+\Delta_A/\pi}\ . \tag{122}$$

C. Exact two-particle properties

In this section we solve the two-particle properties exactly along the random-vector-potential fixed line. In particular, we compute the exact value of $\bar\sigma_{xx}$ (see Sec. III B for its definition), using a new nonperturbative conformal sum rule, mentioned above. We find $\bar\sigma_{xx}=(1/\pi)(e^2/h)$, independent of the random-vector-potential disorder.

The main technical tool for deriving this result is non-Abelian bosonization,[36,37] and the non-Abelian SU(2) symmetry rotating positive into negative frequency fields. We choose, for convenience, the replica method to average over disorder, but the same result would be obtained by just perturbing in the disorder strength. The SU(2) symmetry is present in every member of the ensemble, and is independent of the replica method.

1. *Exact solution by quadrature*

For the evaluation of the quantity $\bar\sigma_{xx}$ of Eq. (56) with random vector potentials (discussed in Sec. III B), it will be very convenient to use non-Abelian bosonization. Here we first introduce the basic ideas in the case without disorder.

The main technical complication in computing $\bar\sigma_{xx}$ is that it involves fields at *two* frequencies $\pm\omega$:

$$\psi_j(\mathbf{r}) \equiv \psi_{j\omega}(\mathbf{r}) , \tag{123}$$

where $j=\pm 1$, denotes positive and negative frequency fields $\pm\omega$.

The action that governs the dynamics of the two fields, $\psi_j(\mathbf{r})$, has an extra non-Abelian SU(2) symmetry, which rotates positive into negative frequency fields. In order to exhibit this symmetry, we use the notation of Eq. (111), Sec. VI B 3. We first make a trivial change of variables multiplying all right-moving, negative frequency fields by (-1):

$$\begin{aligned} &\psi_{R,-1}(\mathbf{r}) \to (-1)\psi_{R,-1}(\mathbf{r}) , \\ &\bar{\psi}_{R,-1}(\mathbf{r}) \to (-1)\bar{\psi}_{R,-1}(\mathbf{r}) \end{aligned} \tag{124}$$

while all remaining fields are unchanged.

It is convenient to introduce the following compact notation:

$$\Phi(\mathbf{r}) \equiv \bar{\psi}^j_R \psi_{L,j} - \bar{\psi}^j_L \psi_{R,j} , \tag{125}$$

$$\Phi^x(\mathbf{r}) \equiv \bar{\psi}^j_R \left[\frac{\sigma^x}{2}\right]^j_{\bar{j}} \psi_{L,j} + \bar{\psi}^{\bar{j}}_L \left[\frac{\sigma^x}{2}\right]^j_{\bar{j}} \psi_{R,j} , \tag{126}$$

$$\Phi^y(\mathbf{r}) \equiv \bar{\psi}^{\bar{j}}_R \left[\frac{\sigma^y}{2}\right]^j_{\bar{j}} \psi_{L,j} + \bar{\psi}^{\bar{j}}_L \left[\frac{\sigma^y}{2}\right]^j_{\bar{j}} \psi_{R,j} , \tag{127}$$

$$\Phi^z(\mathbf{r}) \equiv \bar{\psi}^{\bar{j}}_R \left[\frac{\sigma^z}{2}\right]^j_{\bar{j}} \psi_{L,j} + \bar{\psi}^{\bar{j}}_L \left[\frac{\sigma^z}{2}\right]^j_{\bar{j}} \psi_{R,j} . \tag{128}$$

From Eq. (111) we thus obtain the following form of the action, governing the dynamics of the two $(\pm\omega)$ fields (summation over repeated indices):

$$S^{(2)}_\omega = \frac{1}{2\pi} \int d^2r \{ 2\bar{\psi}^j_R \partial/\partial z \psi_{R,j} + 2\bar{\psi}^j_L \partial/\partial z^* \psi_{L,j} - i\omega\Phi(\mathbf{r}) \} . \tag{129}$$

[Note that we have rescaled the fields ψ by $\sqrt{2\pi}$, such that $\langle \bar{\psi}^j_R(z^*)\psi_{R,i}(0)\rangle = \delta^j_i/z^*$, $\langle \bar{\psi}^j_L(z)\psi_{L,i}(0)\rangle = \delta^j_i/z$, where $z=y+ix$.] In this notation, we obtain from Eq. (56), taking into account the change of variables in Eqs. (109) and (110) as well as Eq. (124):

$$\begin{aligned} &G_{ab}(0,\mathbf{r};\omega)G_{ba}(\mathbf{r},0;-\omega) \\ &= \frac{1}{(2\pi)^2} \langle [\Phi^x(\mathbf{0}) - i\Phi^y(\mathbf{0})][\Phi^x(\mathbf{r}) + i\Phi^y(\mathbf{r})] \rangle_\omega , \end{aligned} \tag{130}$$

where the Green's function is to be evaluated with the action of Eq. (129).

We see from Eq. (123) that the action in Eq. (129) is invariant under the following SU(2) transformations:

$$\begin{aligned} &\begin{pmatrix} \psi_{L,+1} \\ \psi_{L,-1} \end{pmatrix} \to \mathbf{U} \begin{pmatrix} \psi_{L,+1} \\ \psi_{L,-1} \end{pmatrix} , \\ &(\bar{\psi}_{L,+1}\bar{\psi}_{L,-1}) \to (\bar{\psi}_{L,+1}\bar{\psi}_{L,-1})\mathbf{U}^\dagger , \end{aligned} \tag{131}$$

where U is an SU(2) matrix. The right-moving fields are rotated simultaneously with the *same* matrix U.

Due to this SU(2) invariance of the action in Eq. (129), we have in fact

$$G_{ab}(0,\mathbf{r};\omega)G_{ba}(\mathbf{r},0;-\omega) = \frac{2}{(2\pi)^2} \langle \Phi^x(\mathbf{0})\Phi^x(\mathbf{r}) \rangle_\omega . \tag{132}$$

In Sec. VI C 3 we show that the SU(2) invariance of the action of Eq. (129) permits us to evaluate exactly the integral in Eq. (56), in the presence of random vector potentials, with the *same* numerical value as for the nonrandom system, independent of randomness. This calculation rests on a nonperturbative result in conformal field theory, similar in spirit to Zamolodchikov's c theorem.[14]

The action for the two-particle properties, in the presence of a random vector potential, $\mathbf{A}=(A_x, A_y)$, is

$$S = S_0 + \delta S_\omega + \delta S_A , \tag{133}$$

where S_0 and δS_ω are defined in Eq. (129), and

$$\delta S_A = \frac{i}{2\pi} \sum_{j=\pm 1} \int d^2r [A_L \bar{\psi}^j_R \psi_{R,j} + A_R \bar{\psi}^j_L \psi_{L,j}] . \tag{134}$$

Here we have introduced a complex notation for the components of the vector potential:

$$A_R \equiv A_y + iA_x , \quad A_L \equiv A_y - iA_x . \tag{135}$$

Notice that the change of variables made in Eq. (124), does not affect δS_A: we were able to redefine the fields as in Eq. (124) and introduce the above-mentioned SU(2) symmetry because the gauge interaction did not mix left and right movers. Thus the techniques that follow, for the computation of $\bar{\sigma}_{xx}$, cannot be readily applied to the problem where the interaction mixes right and left movers.

Next we use Abelian bosonization[38] for the Fermion bilinears in Eq. (134), $\bar{\psi}^j_R \psi_{R,j} = i\partial_{z^*}\theta^{(j)}$ and $\bar{\psi}^j_L \psi_{L,j} = -i\partial_z \theta^{(j)}$, to obtain

$$\delta S_A = \frac{(-1)}{2\pi} \int d^2r \sum_{j=\pm 1} [A_L \partial_{z^*} \theta^{(j)} - A_R \partial_z \theta^{(j)}] . \tag{136}$$

Here $\theta^{(j)} = \theta^{(j)}_L + \theta^{(j)}_R$, where $j=\pm 1$, denote positive and negative frequency parts, and we have used the fact that left (right) fields depend only on z (z^*). Integrating by parts and using $B = \partial_x A_y - \partial_y A_x = (-i)[(\partial/\partial z^*)A_L - (\partial/\partial z)A_R]$, after introducing replicas $\alpha = 1, \ldots, n$ and averaging over the Gaussian disorder of the magnetic field in the Coulomb gauge,[39] we obtain

$$\overline{B(\mathbf{r}_1)B(\mathbf{r}_2)} = \frac{2\Delta_A}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial z^*} \delta^{(2)}(\mathbf{r}_1 - \mathbf{r}_2) \tag{137}$$

(the overbar denotes the disorder average), and from Eq. (136),

$$\exp\{\delta S_{r,A}\}=\overline{\exp\left\{\frac{i}{2\pi}\int d^2r\, B(\mathbf{r})\left[\sum_{\alpha=1}^{n}\sum_{j=\pm1}\theta^{(\alpha,j)}(\mathbf{r})\right]\right\}} \tag{138}$$

$$=\exp\left\{\frac{\Delta_A}{2\pi^2}\int d^2r\left[\frac{\partial}{\partial z}\sum_{\alpha,i}\theta_L^{(\alpha,i)}(z)\right]\left[\frac{\partial}{\partial z^*}\sum_{\beta,j}\theta_R^{(\beta,j)}(z)\right]\right\}. \tag{139}$$

The full action describing the random system is obtained by replicating the S_0 and δS_ω parts in Eq. (133) as well. We denote the full replicated action by

$$S_r=S_{r,0}+\delta S_{r,\omega}+\delta S_{r,A}\ . \tag{140}$$

The effect of the random-vector-potential term can be taken into account exactly by a quadratic transformation (see, e.g., Ref. 37). To this end it is easiest to consider the (1+1) Hamiltonian, corresponding to the free replicated action $S_{r,0}$. This amounts to considering the x coordinate as space and the y coordinate as the imaginary time of a one-dimensional quantum system. This Hamiltonian is (see, e.g., Ref. 36)

$$\mathbf{H}_0=(1/2\pi)\int_{-\infty}^{\infty}\left[\bar\psi_L^{\alpha j}i\frac{d}{dx}\psi_{L,\alpha j}-\bar\psi_R^{j}i\frac{d}{dx}\psi_{R,j}\right]$$

$$=\mathbf{H}_{0L}+\mathbf{H}_{0R}\ . \tag{141}$$

Clearly this Hamiltonian is invariant under $U(1)\times SU(2n)$ transformations of L and R fermions, where U(1) describes multiplication by a phase. It is well known[40–42] that it is possible to write it in terms of the Noether currents that generate these symmetries[43] (Sugawara construction[42]).

Explicitly, these Noether currents have a simple form in terms of the fermionic variables:

$$J_L(z)\equiv\bar\psi_L^{\alpha j}\psi_{L,\alpha j},\quad \mathrm{U(1)},\ \text{charge}\ , \tag{142}$$

$$\mathbf{I}_L(z)\equiv\bar\psi_L^{\alpha i}(\mathbf{T})_{\alpha i}^{\beta j}\psi_{L,\beta j},\quad \mathrm{SU}(2n),\ \text{spin} \tag{143}$$

[$\mathbf{T}$ are $(2n)^2-1$ generator matrices of SU(2n)], and similar expressions for the right-moving components. The Fourier modes of these currents satisfy the well-known Kac-Moody commutation relations.[41] SU(2n) has a subgroup SU(2), which transforms only the $j=\pm1$ frequency indices, but does not act on the replica indices α,β. Therefore, the Noether current $\mathbf{J}$ associated with this SU(2) subgroup is a particular linear combination of the Noether currents $\mathbf{I}_L(z)$ of Eq. (143). In terms of fermions, it takes on the form

$$\mathbf{J}_L(z)\equiv\tfrac{1}{2}\bar\psi_L^{\alpha j}(\boldsymbol{\sigma})_j^i\psi_{L,\alpha i},\quad \mathrm{SU(2)},\ \text{spin}\ , \tag{144}$$

which can be seen by applying Noether's theorem to the (Lagrangian) action corresponding to Eq. (141).

Straightforward algebra, using Wick's theorem, shows that (see, e.g., Ref. 44)

$$\mathbf{H}_0=(1/2\pi)\int_{-\infty}^{\infty}\left[\frac{1}{4n}J_LJ_L+\frac{1}{2n+1}\mathbf{I}_L\cdot\mathbf{I}_L+(L\leftrightarrow R)\right]. \tag{145}$$

The first term describes a Luttinger liquid,[34] the second a critical SU(2n) spin chain,[37] and the two terms commute. It is important to realize that this Hamiltonian describes the *same* theory as Eq. (141), and therefore the replica limit $n\to0$ does not introduce ill-defined expressions.[45]

Since the Hamiltonian corresponding to $S_{r,A}$ takes the form

$$\mathbf{H}_A=\frac{\Delta_A}{2\pi^2}\int dx\, J_LJ_R\ , \tag{146}$$

it can be incorporated into the U(1) charge part of $\mathbf{H}_0$ by a canonical transformation:[46,37] Defining

$$\begin{bmatrix}\tilde J_L\\ \tilde J_R\end{bmatrix}\equiv\begin{bmatrix}\cosh(\vartheta) & \sinh(\vartheta)\\ \sinh(\vartheta) & \cosh(\vartheta)\end{bmatrix}\begin{bmatrix}J_L\\ J_R\end{bmatrix}, \tag{147}$$

we obtain

$$\frac{1}{4n}\left[J_LJ_L+J_RJ_R+\frac{4n\Delta_A}{\pi}J_LJ_R\right]$$

$$=\frac{1}{4n}\frac{1}{\cosh(2\vartheta)}[\tilde J_L\tilde J_L+\tilde J_R\tilde J_R]\ , \tag{148}$$

where

$$2\tanh(2\vartheta)=\frac{4n\Delta_A}{\pi}\ . \tag{149}$$

In the limit $n\to0$ this becomes

$$\vartheta=\frac{n\Delta_A}{\pi}[1+O(n^2)]\ . \tag{150}$$

The effect of the canonical transformation on the fermion fields is as follows: The fermion can be thought to be made out of a charge part θ and a spinon field part $\mathbf{g}_{j\alpha}$. Explicitly we have[40,41,44]

$$\psi_{L,\alpha j}\propto e^{-i\theta_L/\sqrt{2n}}\mathbf{g}_{L,j\alpha},\quad \bar\psi_L^{\alpha j}\propto e^{i\theta_L/\sqrt{2n}}\mathbf{g}_L^{\dagger j\alpha}\ , \tag{151}$$

$$\psi_{R,\alpha j}\to e^{i\theta_R/\sqrt{2n}}\mathbf{g}_{R,\alpha j},\quad \bar\psi_R^{\alpha j}\to e^{-i\theta_R/\sqrt{2n}}\mathbf{g}_R^{\dagger\alpha j}\ , \tag{152}$$

where $\dagger$ denotes that the corresponding field transforms in the complex-conjugate representation of the unitary group. These equations reflect the separation of charge and spin degrees of freedom in Eq. (141). (The correlation functions of the spinon fields are discussed in Ref. 41.)

The charge boson is related to the charge current by[41]

$$J_L(z)\equiv(-i)\sqrt{2n}\frac{\partial}{\partial z}\theta_L(z)\ , \tag{153}$$

$$J_R(z) \equiv i\sqrt{2n}\,\frac{\partial}{\partial z^*}\theta_R(z^*)\ , \tag{154}$$

and a similar relation for tilde variables.[47]

The transformation of the fermion fields under the canonical transformation follows from Eq. (147):

$$\begin{bmatrix}\theta_L\\ \theta_R\end{bmatrix} = \begin{bmatrix}\cosh(\vartheta) & -\sinh(\vartheta)\\ -\sinh(\vartheta) & \cosh(\vartheta)\end{bmatrix}\begin{bmatrix}\tilde{\theta}_L\\ \tilde{\theta}_R\end{bmatrix}\ . \tag{155}$$

Inserting these expressions into Eq. (152), and using

$$\langle e^{ip\tilde{\theta}_L(z)} e^{-ip\tilde{\theta}_L(0)}\rangle \propto 1/z^{p^2} \tag{156}$$

(and a similar expression for right movers) permits us to compute all Green's functions of the fermion fields.

We remark that the prefactor $\cosh^{-2}(2\vartheta)=1+O(n^2)$ in Eq. (148) corresponds to a change of the Fermi velocity of the charge field θ, while there is no change of the Fermi velocity of the spinon fields. However, from now on we will replace $\cosh^{-2}(2\vartheta)$ by $=1$: This does not modify our conclusions[48] in the replica limit $n\to 0$, since the difference corresponds to a term in the action of order $O(n^2)$.

Let us pause for a moment to determine the scaling dimension of the operator coupling to the frequency, ω, in the replicated theory. In fact, we may consider the more general operator, which reads, using Eqs. (151), (152), and (155),

$$\bar{\psi}_R^{i,\beta}\psi_{L,j\alpha} = \exp\{-ie^{-\vartheta}[\tilde{\theta}_L+\tilde{\theta}_R]\}(\mathbf{g}_R^{\dagger i\beta}\mathbf{g}_{L,j\alpha})\ . \tag{157}$$

This couples to the Matsubara frequency, when traced over all indices. The $\mathbf{g}$ fields are not affected by the canonical transformation, and therefore their scaling dimensions remains unchanged. From Eqs. (152) and (156) we find that the charge part has suffered a change in scaling dimension of $(1/2n)[e^{-2\vartheta}-1]$, as can be seen by computing the two-point function of Eq. (157), using Eqs. (156) and (151) and (152). Its total scaling dimension (being equal to 1 without disorder) is therefore

$$x = 1-\frac{1}{2n}[1-e^{-2\vartheta}]\ , \tag{158}$$

which is always smaller than 1, since by Eq. (149) $\vartheta>0$ for nonzero disorder. Thus the operator becomes more relevant for nonzero disorder Δ_A (for arbitrary number n of replicas). In particular, from Eq. (149) we obtain

$$x \to 1-\Delta_A/\pi \tag{159}$$

in the replica limit $n\to 0$.

Notice that the discussion just outlined describes one-particle properties, when SU($2n$) is replaced by SU(n). All results obtained in Sec. VI B can be recovered, using this non-Abelian formalism.

2. *Correlation length exponent of the Dirac mass*

A constant Dirac mass M enters the two-particle action in the form $-M[\bar{\psi}_R^j\psi_{L,j}+\bar{\psi}_L^j\psi_{R,j}]$, as can be seen from Eqs. (47) and (109), using the standard basis of Pauli matrices.

It thus acquires, after the change of variables in Eq. (124), the form $-2M\Phi^z$ [see Eq. (128)]. This operator has the same scaling dimension as the operator in Eq. (157), along the fixed line. Therefore, the scaling dimension of M is the same as the scaling dimension of the Matsubara frequency ω, Eq. (159). The correlation length introduced by a constant Dirac mass into one- and two-particle Green's functions therefore diverges with the same exponent ν given in Eq. (122). This is quite different from conventional localization, where the one-particle properties remain noncritical.

We remark that this exponent ν seems to violate, at first sight, the inequality of Chayes *et al.*[49] However, the exponent ν considered by Chayes *et al.* refers to a situation where one tunes through the transition of the random system by varying the probability distribution of disorder. This is different from the situation discussed here, where our exponent ν arises from tuning through the transition by varying M, while the distribution for the random vector potential is kept fixed. The inequality therefore does not apply to the exponent ν we are considering.[50]

3. *Exact computation of $\bar{\sigma}_{xx}$ along the random vector potential fixed line*

So far we have solved the random-vector-potential model along its critical line. Next we will consider a nonzero frequency ω, which is a massive perturbation off this fixed line: We consider the replicated theory with random vector potential [Eqs. (148) and (145)] for an integer number n of replicas. We add the frequency perturbation

$$\mathbf{H}_\omega = -i\left[\frac{\omega}{2\pi}\right]\int dx\ \Phi \tag{160}$$

to the Hamiltonian of this replicated theory, where

$$\Phi = \bar{\psi}_R^{j,\alpha}\psi_{L,j\alpha} - \bar{\psi}_L^{j,\alpha}\psi_{R,j\alpha}\ , \tag{161}$$

as discussed in Eq. (125). In particular, we wish to compute [see Eqs. (132) and (56)]

$$\bar{\sigma}_{xx} = \frac{e^2}{h}\lim_{n\to 0}\frac{2}{n}\left[\frac{\omega}{2\pi}\right]^2\int d^2r\, r^2\langle\Phi^x(\mathbf{0})\Phi^x(\mathbf{r})\rangle_\omega \tag{162}$$

for nonzero ω, where

$$\Phi^x = \bar{\psi}_R\frac{\sigma^x}{2}\psi_L + \bar{\psi}_L\frac{\sigma^x}{2}\psi_R \tag{163}$$

(all $j=\pm$ indices are contracted against the Pauli matrix, and all replica indices are contracted against each other). For this calculation we will use the modified action, where the charge velocity $\cosh^{-2}(2\vartheta)$ is replaced by $=1$. As discussed above, this does not modify our conclusions in the replica limit $n\to 0$.

It is remarkable that an exact expression can be obtained for this integral. That this is possible rests on a general theorem of conformal field theory, the Kac-Moody sum rule (for a more extensive discussion, see Ref. 51). It exploits the fact that the massive perturbation ω does not break the SU(2) symmetry, transforming posi-

tive into negative frequency fields, present for nonzero disorder and nonzero frequency ω [as can be seen from Eqs. (161), (145), and (146)]. The conservation equation for the associated SU(2) Noether current allows us to compute the integral exactly.

We start by outlining the proof of this sum rule. The Noether current in question is

$$\mathbf{J}_\mu(\mathbf{r}), \quad \mu=x,y \ , \tag{164}$$

which is related to the L and R components of the SU(2) current in Eq. (144) via

$$\mathbf{J}_R \equiv \mathbf{J}_y + i\mathbf{J}_x, \quad \mathbf{J}_L \equiv \mathbf{J}_y - i\mathbf{J}_x \ . \tag{165}$$

It satisfies the conservation equation

$$\partial_x \mathbf{J}_x + \partial_y \mathbf{J}_y = 0 \ , \tag{166}$$

which reads, in terms of L and R components,

$$\frac{\partial}{\partial z}\mathbf{J}_R(\mathbf{r}) + \frac{\partial}{\partial z^*}\mathbf{J}_L(\mathbf{r}) = 0 \ . \tag{167}$$

This equation is true, for ω zero or not, since the frequency perturbation does not break SU(2) invariance. When $\omega=0$, Eq. (165) has the form of Eq. (144), and depends only on z^* and z, respectively. Using Eq. (144) and Wick's theorem, one can verify that

$$\begin{aligned} \langle J_R^a(z^*) J_R^b(0)\rangle_{\omega=0} &= \frac{n\delta^{ab}/2}{(z^*)^2} \ , \\ \langle J_L^a(z) J_L^b(0)\rangle_{\omega=0} &= \frac{n\delta^{ab}/2}{z^2}, \quad a=x,y,z \end{aligned} \tag{168}$$

in the absence of the random vector potential. Since, as we have seen above, the random vector potential affects only the charge fields, Eq. (168) is also valid at any point on the fixed line.[52]

We will consider the z derivative of this current, which also forms a conserved current, for any ω:

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\mathbf{J}_R(\mathbf{r})\right] + \frac{\partial}{\partial z^*}\left[\frac{\partial}{\partial z}\mathbf{J}_L(\mathbf{r})\right] = 0 \ , \tag{169}$$

as follows immediately from Eq. (167). The basis of the following analysis is that a current, being a generator of a symmetry, cannot become renormalized.[53] This means that even in the perturbed theory $\omega \neq 0$, the current still maintains its scaling dimension $=1$ (in units of inverse spatial distance), and still transforms like $e^{-i\gamma}$ under rotations (angle γ) of 2D position space. Furthermore it always transforms as a vector under SU(2). Using these ingredients, the Green's functions of the currents must have the following form:

$$\langle \partial_z J_L^x(\mathbf{r}) \partial_z J_L^x(\mathbf{0})\rangle_\omega = \frac{F_1[\ln(zz^*)]}{z^4} \ , \tag{170}$$

$$\langle \partial_z J_R^x(\mathbf{r}) \partial_z J_L^x(\mathbf{0})\rangle_\omega = \frac{F_2[\ln(zz^*)]}{z^3 z^*} \ , \tag{171}$$

$$\langle \partial_z J_R^x(\mathbf{r}) \partial_z J_R^x(\mathbf{0})\rangle_\omega = \frac{F_3[\ln(zz^*)]}{(zz^*)^2} \ , \tag{172}$$

where $F_i[\ln(zz^*)] = \mathcal{F}_i[zz^*\omega^{2/y}]$ and $i=1$, 2, and 3 are scaling functions. Here $y=2-x=1+(1/2n)[1-e^{-2\vartheta}]>1$ for $\Delta_A>0$ [from Eq. (158)], where $\vartheta \geq 0$ is given in Eq. (149), is the scaling dimension (in units of inverse length) of the perturbation ω.

Based on a power counting argument,[54] possible since the current operator maintains its scaling dimension in the perturbed theory, one can relate the current to the operator Φ^x.

Consider the perturbation expansion in terms of ω:

$$\partial_z J_R^x(\mathbf{r}) = \frac{-i\omega}{2\pi} A_{(1)}^x + \left[\frac{-i\omega}{2\pi}\right]^2 A_{(2)}^x + \cdots \ . \tag{173}$$

Using the scaling dimension $(=2)$ of the left-hand side, and the scaling dimension y of ω, as well as identical behavior under spatial and spin rotation of both sides, all operators multiplying nonlinear powers of ω would have scaling dimensions that do not exist in the spectrum of the theory.[56] Therefore, only the linear term is present. To find the corresponding operator $A_{(1)}^x$, consider the perturbative expansion of the two-point function:

$$\langle J_R^x(\mathbf{r}) J_R^x(\mathbf{0})\rangle_\omega \ . \tag{174}$$

The nth-order term is proportional to

$$\int d^2r_1 \cdots \int d^2r_n \langle J_R^x(z^*)\Phi(\mathbf{r}_1)\cdots\Phi(\mathbf{r}_n) J_R^x(0)\rangle_{\omega=0} \ . \tag{175}$$

Since this is evaluated in the $\omega=0$ theory, this has the simple form[58]

$$\int d^2r_1 \cdots \int d^2r_n \left\{ \sum_{m=1}^{n} \frac{1}{(z^*-z_m^*)} \langle \Phi(\mathbf{r}_1)\cdots\Phi^x(\mathbf{r}_m)\cdots\Phi(\mathbf{r}_n) J_R^x(0)\rangle_{\omega=0} \right\} , \tag{176}$$

where the Kac-Moody Ward identity[41] was used. Note that the right-moving current operator acts in Eq. (175) as a generator of SU(2) on the right-moving spinon field contained in Φ [defined in Eqs. (161) and (152)], which transforms as a doublet under SU(2), giving the Pauli matrix in Φ^x, while it leaves the left-moving spinon field untouched.

Taking the derivative ∂_z of Eq. (176), we obtain

$$\partial_z \int d^2r_m \Theta[(z-z_m)(z^*-z_m^*)-a^2] \times \frac{1}{(z^*-z_m^*)} \Phi^x(z_m, z_m^*) \ , \tag{177}$$

where Θ is the step function, providing a rotationally invariant short-distance cutoff. This gives a δ function upon differentiation such that Eq. (177) becomes

$$\pi\Phi^x(z,z^*) . \tag{178}$$

Since the cutoff a was just a tool to do the integral, it may now be set to zero. Putting this result into Eqs. (174), (175), and (176), we find

$$\langle \partial_z J_R^x(\mathbf{r}) J_R^x(\mathbf{0})\rangle_\omega = \pi \left[\frac{-i\omega}{2\pi}\right] \langle \Phi^x(\mathbf{r}) J_L^x(\mathbf{0})\rangle_\omega . \tag{179}$$

This identifies the operator $A_{(1)}^x$ in Eq. (173), i.e.,

$$\partial_z J_R^x = \pi \left[\frac{-i\omega}{2\pi}\right] \Phi^x . \tag{180}$$

Note that Φ^x is the only operator in the theory with this scaling dimension and the required behavior under spatial and spin rotations.

Equation (180) relates the scaling function F_3 to the Green's function occurring in $\bar{\sigma}_{xx}$. Next we use the conservation law Eq. (169), providing a relation between Eqs. (170) and (171):

$$F_1' = 3F_2 - F_2' \tag{181}$$

[the prime denotes the derivative with respect to $\ln(zz^*)$], and a relation between Eqs. (171) and (172):

$$(-1)F_2 + F_2' = 2F_3 - F_3' . \tag{182}$$

Defining

$$K[\ln(zz^*)] \equiv F_1 - 2F_2 - 3F_3 , \tag{183}$$

Eqs. (181) and (182) give

$$K' = -6F_3 . \tag{184}$$

Using Eq. (172),

$$K[l_2] - K[l_1] = (-6)\int_{l_1}^{l_2} d[\ln(zz^*)] F_3 \tag{185}$$

$$= (-6)\int_{\exp(l_1)}^{\exp(l_2)} d(r^2) r^2 \langle \partial_z J_R^x(\mathbf{r}) \partial_z J_R^x(\mathbf{0})\rangle_\omega$$

$$= 6\left[\frac{\omega}{2\pi}\right]^2 \pi^2 \int_{\exp(l_1)}^{\exp(l_2)} d(r^2) r^2 \langle \Phi^x(\mathbf{r})\Phi^x(\mathbf{0})\rangle_\omega \tag{186}$$

where Eq. (180) was used. Now let $l_1 \to -\infty$: This describes the short-distance fixed point, which is the random-vector-potential fixed line which we know in detail. From Eq. (168) and the fact that the R-L Green's functions of a current vanish at a fixed point (see, e.g., Ref. 37), we obtain

$$F_3[-\infty] = F_2[-\infty] = 0, \quad F_1[-\infty] = -3n . \tag{187}$$

Then let $l_2 \to +\infty$: This describes the long-distance strong-coupling fixed point, which has no gapless critical excitations, implying

$$F_3[+\infty] = F_2[+\infty] = F_1[+\infty] = 0 . \tag{188}$$

Using this in Eq. (186) finally gives

$$\bar{\sigma}_{xx} = \lim_{n\to 0} \frac{2}{n} \left[\frac{\omega}{2\pi}\right]^2 \int d^2r\, r^2 \langle \Phi^x(\mathbf{r})\Phi^x(\mathbf{0})\rangle_\omega = 1/\pi$$

$$(\text{units of } e^2/h) . \tag{189}$$

This shows that $\bar{\sigma}_{xx}$ is a constant all along the random-vector-potential critical line, having the same value as for the nonrandom model [Eq. (59)].

Let us recall that it was possible to calculate $\bar{\sigma}_{xx}$ because it was expressed as an integral of an exact differential in Eqs. (185) and (186) due to the SU(2) symmetry of the theory, even for nonzero frequency ω. At one limit of integration, the integrand was given by the short-distance limit, the fixed line (which we know in detail), while the other end was given by the strong-coupling fixed point at large ω which has no low-energy excitations that could contribute to F_i and K. Thus the value of the integral was determined by the SU(2) symmetry and the two fixed points.

VII. RANDOM SCALAR POTENTIAL V

Here we consider briefly the case where the only randomness present is a random scalar potential V. Both random Dirac mass and random vector potentials are set to zero.

For small random scalar potential V we can use directly the perturbative results obtained for the random Dirac mass case in Sec. V, as follows. First recall that these two terms enter the Lagrangian as

$$L = \bar{\psi}(\boldsymbol{\sigma}\cdot\mathbf{p} + M\sigma_z + V)\psi . \tag{190}$$

Consider now putting $V = 0$. Upon making the change

$$\bar{\psi} \to -\bar{\psi} i\sigma_z \tag{191}$$

(with ψ unchanged) we find that

$$L \to \bar{\psi}(\boldsymbol{\sigma}\times\mathbf{p} + iM)\psi . \tag{192}$$

A spatial rotation by 90° then maps the problem with nonzero M to one with a purely imaginary V. Upon replicating and ensemble averaging, this corresponds to a negative disorder strength, Δ_V. Equivalently, a positive Δ_V problem maps on to the negative Δ_M problem. Since the latter was a marginal coupling, the flow will now be away from the origin, rather than toward it. In other words, a random V drives us to some strong-coupling regime. We do not expect a fixed point in the single-particle theory. Instead we expect the generation of a nonzero density of states. If the two-particle properties are calculated using $\pm\omega$ frequencies, a sigma model in the symplectic ensemble should emerge due to the time-reversal invariance of the random V model for every member of the ensemble.

VIII. GLOBAL PHASE DIAGRAM

In the previous three sections we have considered the model separately in the presence of the three types of disorder introduced in Sec. IV. Generically, for a real phys-

ical system of electrons, one would expect all three types of disorder to be simultaneously present. Here we discuss briefly the expected nature of the quantum Hall transition in this general case.

To this end consider first the case where both a random scalar potential V *and* a random Dirac mass M are *simultaneously* present. As discussed in Sec. V, within the edge state picture the motion along the edges (at zero energy) will now be accompanied by random phase factors due to the random scalar potential V. Similarly, if we add in the random-vector-potential term, edge states which close upon themselves will pick up a random-phase factor from the enclosed magnetic flux. In either case, the model then becomes equivalent to the Chalker-Coddington model,[16,17] and one expects a generic Hall transition with $\nu \sim \frac{7}{3}$. Perturbative RG calculations for small Δ_M, Δ_V, and Δ_A support this conclusion, since when all three disorder strengths are nonzero the flows are away from the fixed line (Δ_A line) and the zero-disorder fixed point. Unfortunately, since the RG flows carry one into an uncontrolled strong-coupling regime, we have so far not been able to access the generic critical point analytically.

In view of the above results we propose the global phase diagram in Fig. 9. The main features are (1) A fixed line at $\Delta_M = \Delta_V = 0$, which is unstable to both Δ_M and Δ_V; (2) a $2d$ Ising fixed point at the origin, stable when only Δ_M is nonzero; (3) a fixed point on the Δ_V axis which is probably in the universality class of the symplectic nonlinear sigma model; and (4) the generic, and so far unaccessible, quantum Hall fixed point which presumably attracts all the other flows.

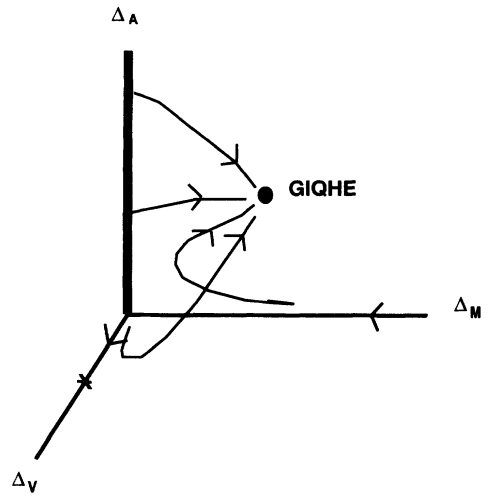

FIG. 9. Conjectured global flow diagram. The fixed line on the Δ_A axis is unstable to both Δ_M and Δ_V. The fixed point at the origin is stable along the Δ_M axis, but unstable along the Δ_V axis. The perturbatively inaccessible fixed point on the Δ_V axis is argued to be unstable to either Δ_M or Δ_A. In the text we argue that all other flows terminate at a strong coupling GIQHE (generic integer quantum Hall effect) fixed point.

IX. SUMMARY AND OUTLOOK

A class of models was introduced and studied which exhibits an IQHE transition as a control parameter is varied. The model consists of nonrelativistic (spinless) fermions hopping on a square lattice, with nearest-neighbor hopping $t=1$, diagonal hopping t', a staggered potential $(-1)^{x+y}\mu$, and half a unit of magnetic flux per square plaquette. An effective low-energy theory appropriate near the quantum Hall transition was derived, and expressed in terms of two free Dirac fields (whose masses differed by the staggering potential). At zero energy ($E=0$), when the mass M of one of the Dirac fields changes, its contribution to the Hall conductivity σ_{xy} jumps from $-\frac{1}{2}$ to $\frac{1}{2}$ in units of e^2/h. The heavier Dirac particle, making a constant contribution of $\frac{1}{2}$ to σ_{xy}, leads to a jump in the total σ_{xy} from 0 to 1, so that the model exhibits an integer Hall transition. In the absence of disorder, the phase diagram in the $M-E$ plane was determined explicitly. The correlation length exponent $\nu=1$ and the longitudinal conductivity $\sigma_{xx}=\pi/8$ at the transition were computed. In addition, frequency- and energy-dependent responses were computed.

The free massless Dirac fixed point describing the transition in the pure system is particle-hole, parity, and time-reversal invariant. The Dirac mass term, which serves as the control parameter to tune through the transition, breaks both parity and time-reversal invariances, very much like $\theta-\pi$ in the nonlinear-sigma-model treatment of Levine, Libby, and Pruisken.[3] In this respect, the Dirac mass term plays the role of energy in the usual treatment of the IQHE.

Since our lattice model already exhibits an IQHE transition even with no disorder present, it serves as a very natural starting point for considering the IQHE transition in disordered systems. We considered disorder in the Dirac mass M, in a random scalar potential V, and in a random vector potential $\mathbf{A}$. Table I summarizes the symmetries which the free massless Dirac Hamiltonian as well as these three random potentials preserve in each given member of the ensemble.

We found that with a spatially varying and random Dirac mass $M(x,y)$, the model exhibits edge states with wave functions confined to the contours $M(x,y)=0$. In this case, with only random M present, the IQHE transition was naturally interpreted in terms of quantum percolation of these isolated edge states. The corresponding Chalker-Coddington model,[16,17] however, has random tunneling amplitudes but no random phases. For this reason, the model does not exhibit the generic quantum Hall transition with $\nu \sim \frac{7}{3}$, but rather was found to give $\nu=1$. This was established by exploiting the equivalence between a Dirac field with a random mass and an Ising model with bond dilution. In the latter case, disorder is known to be marginally irrelevant, implying the same for the IQHE transition with random M. Thus at this transition $\nu=1$, with logarithmic corrections coming from the marginally irrelevant disorder. More generally all exponents and transport properties at the transition are those of the pure system, up to logarithmically vanishing corrections. Recently Lee has analyzed numerically a

Chalker-Coddington model in which the random phases are dropped,[27] retaining only random amplitudes. In this case he finds an exponent $\nu \sim 1$. Our analysis strongly suggests that this transition is in the universality class of the $2d$ Ising model.

The case with only a random scalar potential V present was mapped onto a random Dirac mass problem with purely imaginary mass. The marginal irrelevance of the variance of a random, real M, then implies immediately the marginal relevance of a weak random scalar potential. It was argued that in this case the transition at strong coupling would be described by a symplectic nonlinear sigma model.

The case in which the only randomness present was in the vector potential had many remarkable properties.

(i) Every realization of random **A** led to a particle-hole symmetric spectrum.

(ii) The exact zero-energy wave function was found for every realization.

(iii) The wave function was shown to be extended, and its multifractal scaling behavior was characterized fully by the explicit calculation of its $f(\alpha)$ spectrum.

(iv) Ensemble averaged properties were found to be described by a fixed line, parametrized by the strength of the random vector potential.

(v) The density of states was found to exhibit a power-law dependence on energy near $E=0$, with an exponent which varies continuously upon moving along the fixed line.

(vi) The diagonal conductivity was calculated using a hidden SU(2) symmetry and a conformal sum rule, and found to be constant along the fixed line.

Recently, a model exhibiting an IQHE transition in the presence of random potentials which respect particle-hole symmetry in every realization of disorder was studied by Gade.[59] She found a vanishing RG β function, signalling a fixed point. The density of states was found to be divergent. On universality grounds we may identify her fixed point with a point on our random-vector-potential fixed line. Indeed, in a $1/N$ expansion on a related model believed to be in the same universality class, Hikami, Shirai, and Wegner[24] found a β function vanishing in $1/N$ and a longitudinal conductivity $\bar{\sigma}_{xx}$ which was identical to our result of $(1/\pi)e^2/h$ in Eq. (189). Furthermore their $1/N$ expansion indicated a diverging density of states. This would seem to indicate that their model corresponds to one point on our fixed line with $\Delta_A/\pi > 1$ [see Eqs. (98) and (100)].

It is instructive also to consider the related numerical work of Refs. 60–62, where nonrelativistic fermions on a lattice were studied in the presence of a random magnetic flux per plaquette, uniformly distributed between 0 and 2π. This problem also has a particle-hole symmetry at zero energy in each realization of the disorder. Their numerical analysis for this model revealed evidence for extended states in a range of energies centered around zero. In our model with a random vector potential, we also found an extended state at $E=M=0$.[63]

It would be interesting to extract the value of $\bar{\sigma}_{xx}$ in their model to compare it to the universal value of $\bar{\sigma}_{xx}=1/\pi$ that we have obtained here. Furthermore, since the density of states is finite at one particular point $\Delta_A=\pi$ on the fixed line in our model, the conductivity as obtained from the Kubo formula should also equal $\sigma_{xx}=1/\pi$ at that point. We also remark that at this point on our fixed line the lowest participation ratio $|\Psi|^4$ [Eq. (82)] scales like $1/N^{\mu}$ (N is the system size), where $\mu=1$, in reasonable agreement with the behavior observed numerically in Ref. 64. However, it should be borne in mind that in our model the distribution of magnetic flux per plaquette is peaked around π and not uniformly distributed in the interval $[0,2\pi]$.

Finally, our analysis revealed that in the generic disordered case, in which all three types of disorder are simultaneously present, the RG flows take us to a strong-coupling regime. Using an edge-state picture we argued that this strong-coupling fixed point was correctly described in terms of the Chalker-Coddington model and should thus have $\nu \sim \frac{7}{3}$. Unfortunately, we have been unable so far to access this generic quantum Hall fixed point analytically.

In conclusion, we have found a family of IQHE transitions with interesting critical behavior. It would be most interesting to see if they could be realized experimentally. In Sec. V we proposed one such scheme for doing this. We trust that our experimental colleagues can readily improve upon this.

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APPENDIX

1. Transport properties of pure system

In this appendix we derive the expressions for σ_{xy} and σ_{xx} quoted in Sec. III. However, we follow a different and more powerful approach than the one used in Sec. II B. Specifically, we will use a vector potential, rather than the scalar potential, to apply the electric field. This means that we have to subtract the diamagnetic contribution.

Let us first make some notational changes to introduce a three-dimensional formalism, which will facilitate the evaluation of integrals. Going back to the action density of Sec. II B,

$$\bar{\psi}(i\omega-\boldsymbol{\sigma}\cdot\mathbf{p}-M\sigma_z)\psi\ , \tag{A1}$$

and making the change

$$\bar{\psi}\rightarrow\bar{\psi}(-i\sigma_z)\ , \tag{A2}$$

we obtain a propagator

$$G=\frac{1}{\omega\sigma_z-p_x\sigma_y+p_y\sigma_x+iM}\ , \tag{A3}$$

which becomes, upon making the change $p_x\to -p_y$, $p_y\to p_x$ (a 90° rotation),

$$G=\frac{1}{\not{p}-m}\ , \tag{A4}$$

where

$$m=-iM\ , \tag{A5}$$

$$\not{p}=\gamma_\mu p_\mu,\quad \mu=x,y,z\ , \tag{A6}$$

$$p_\mu=(p_x,p_y,\omega)\ , \tag{A7}$$

$$\gamma_\mu=(\sigma_x,\sigma_y,\sigma_z)\ . \tag{A8}$$

Corresponding to the propagator in Eq. (A4) is an action density

$$\bar{\psi}(\not{p}-m)\psi\ . \tag{A9}$$

Using these notations we obtain, from the Kubo formula,

$$\omega\sigma_{\mu\nu}(\omega,M)=\int d^2r\,d\tau\langle(\bar{\psi}\gamma_\mu\psi)(\mathbf{r},\tau)(\bar{\psi}\gamma_\nu\psi)(\mathbf{0},0)\rangle[e^{-i\omega\tau}-1]$$

$$=\int\frac{d^3p}{(2\pi)^3}\left[\frac{\mathrm{Tr}\{\gamma_\mu[\not{p}+\not{q}+m]\gamma_\nu[\not{p}+m]\}}{[(p+q)^2-m^2][p^2-m^2]}-\{\text{same with } q=0\}\right], \tag{A10}$$

where $q=(0,0,\omega)$. Here we have used $(\not{p}+m)(\not{p}-m)=p^2-m^2$. Next, we use the identities[65]

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu]=2\delta_{\mu\nu},\quad \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho]=2i\epsilon_{\mu\nu\rho}\ , \tag{A11}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma]=2\delta_{\mu\rho}\delta_{\nu\sigma}-2\delta_{\mu\nu}\delta_{\rho\sigma}+2\delta_{\mu\sigma}\delta_{\rho\nu}\ . \tag{A12}$$

This leads to

$$\omega\sigma_{\mu\nu}(\omega,M)=2\int\frac{d^3p}{(2\pi)^3}\left[\frac{(2p_\mu p_\nu-\delta_{\mu\nu}p^2)+(p_\mu q_\nu+q_\mu p_\nu)-\delta_{\mu\nu}qp-M\epsilon_{\mu\nu\rho}q^\rho-M^2\delta_{\mu\nu}}{[(p+q)^2+M^2][p^2+M^2]}-\{\text{same with } q=0\}\right]. \tag{A13}$$

For the conductivities σ_{xx} and σ_{xy}, we have

$$\mu,\nu\in\{1,2\},\quad q=(0,0,\omega)\ . \tag{A14}$$

Consequently, the third term in the numerator above is always absent for these components of $\sigma_{\mu\nu}$. We obtain the following expressions.

(i): σ_{xy}

$$\sigma_{xy}=(2/\omega)\int_0^1 ds\int\frac{d^3p}{(2\pi)^3}\frac{-M\omega}{[p^2+2p_3s\omega+(s\omega^2+M^2)]^2} \tag{A15}$$

(the subtraction from the diamagnetic term vanishes in this case, and we have introduced a Feynman parameter[67] s to rewrite the denominator). After a shift of the integration variable,[68] $p_3\to p_3-s\omega$, the integral over $|p|$ (absolute value of the 3D vector p) is elementary:

$$\sigma_{xy}=(-M/4\pi)\int_0^1 ds\frac{1}{[M^2+\omega^2s(1-s)]^{1/2}}=(1/4\pi)\frac{-M}{|M|}f[(\omega/M)^2]\ , \tag{A16}$$

where we have introduced the scaling function

$$Z=(\omega/M)^2\ , \tag{A17}$$

$$f[Z]\equiv\int_0^1 ds\frac{1}{[1+Zs(1-s)]^{1/2}}=(2/\sqrt{Z}\,)\arcsin\left[\frac{Z}{\sqrt{Z(Z+4)}}\right]. \tag{A18}$$

The last equality follows, e.g., from Eq. (2.261) in Ref. 69. We observe that

$$f[Z]=1\ \text{ if } Z\to 0\qquad\sim(\pi/\sqrt{Z}\,)\ \text{ as } Z\to+\infty\ . \tag{A19}$$

In conclusion, we have obtained

$$\sigma_{xy}(\omega,M)=\frac{-\mathrm{sgn}(M)}{4\pi}f[(\omega/M)^2]\ . \tag{A20}$$

(ii): σ_{xx}

$$\sigma_{xx}=(2/\omega)\int\frac{d^3p}{(2\pi)^3}\left[\frac{-p_3^2-\omega p_3-M^2}{[(p+q)^2+M^2][p^2+M^2]}+\frac{\frac{1}{3}p^2+M^2}{[p^2+M^2]^2}\right], \tag{A21}$$

which may be rewritten as

$$I_1+I_2\equiv(2/\omega)\int\frac{d^3p}{(2\pi)^3}\left[\frac{-p_3^2-\omega p_3-M^2+\frac{1}{3}\{(p+q)^2+M^2\}}{[(p+q)^2+M^2][p^2+M^2]}+\frac{2}{3}\frac{M^2}{[p^2+M^2]^2}\right]. \tag{A22}$$

The first term I_1 now converges[70] for large values of p, since it can be expressed as

$$I_1=(2/3\omega)\int\frac{d^3p}{(2\pi)^3}\frac{(p^2-3p_3^2)-\omega p_3+(\omega^2-2M^2)}{[(p+q)^2+M^2][p^2+M^2]}. \tag{A23}$$

Upon introducing a Feynman parameter s, as above, and a subsequent shift $p_3\rightarrow(p_3-s\omega)$, we obtain

$$I_1=(2/3\omega)\int_0^1 ds\int\frac{d^3p}{(2\pi)^3}\frac{(p^2-3p_3^2)-p_3\omega(4s-1)+\omega^2[(1-s)+2s(1-s)]-2M^2}{[p^2+\omega^2s(1-s)+M^2]^2}. \tag{A24}$$

All p-dependent terms in the numerator vanish due to rotational invariance. Scaling out M^2, introducing Z as for σ_{xy}, and doing the (elementary) integral over $|p|$, we obtain

$$\sigma_{xx}=J_1+J_2+J_3 \tag{A25}$$

$$=(1/12\pi)\frac{1}{\sqrt{Z}}\int_0^1 ds\left[Z\frac{(1-s)}{\sqrt{1+Zs(1-s)}}+2Z\sqrt{1+Zs(1-s)}-4\frac{1}{\sqrt{1+Zs(1-s)}}+2\right]. \tag{A26}$$

The integrals J_1 and J_2 are elementary [using, e.g., Eqs. (2.264/2) and (2.262/1), respectively, of Ref. 69], and J_3 is discussed in σ_{xy}. This gives the final form

$$\sigma_{xx}(Z)=(1/4\pi)\frac{1}{\sqrt{Z}}\left[\frac{Z}{4}f[Z]+(1-f[Z])\right], \tag{A27}$$

$$Z=\left[\frac{\omega}{M}\right]^2,$$

where $f[Z]$ is the scaling function introduced for σ_{xy}.

We note that

$$\sigma_{xx}(0)=0, \tag{A28}$$

$$\sigma_{xx}(\infty)=\tfrac{1}{16}. \tag{A29}$$

2. Multifractal properties of the zero-energy wave function in a random vector potential

Consider

$$P_q=\left\langle\frac{\frac{1}{L^2}\int e^{2q\Phi(x)}d^2x}{\left[\int e^{2\Phi(x)}d^2x\right]^q}\right\rangle, \tag{A30}$$

where $\langle\cdots\rangle$ stands for the average over the distribution of Φ given by Eq. (79). We now use the replica trick, multiplying the numerator and denominator by $[\int e^{2\Phi(x)}d^2x]^{n-q}$ with $n=0$. This leads to

$$P_q=\frac{1}{L^2}\left\langle e^{2q\Phi(x)}\prod_{j=1}^{n-q}e^{2\Phi(x_j)}d^2x\,d^2x_1\cdots d^2x_{n-q}\right\rangle. \tag{A31}$$

The average over the scalar field distribution gives

$$\left\langle e^{2q\Phi(x)}\prod_{j=1}^{n-q}e^{2\Phi(x_j)}\right\rangle=(L/a)^{q^2\Delta/\pi}(L/a)^{\Delta/\pi(n-q)}f(x_i/L), \tag{A32}$$

where f is a function of the x's divided by the system size L, which serves as the cutoff. The L dependence of P_q now follows readily:

$$P_q\simeq\frac{1}{L^2}\int^1\frac{d^2x}{L^2}\cdots\frac{d^2x_{n-q}}{L^2}(L)^{2+2(n-q)}\times(L/a)^{\Delta/\pi(q^2+n-q)}f(x/L)|_{n=0} \tag{A33}$$

$$\simeq L^{-2q+(\Delta/\pi)q(q-1)}, \tag{A34}$$

which concludes the proof.

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