

Renormalization Group Analysis of Scalar Field
Theories

Second
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by
Kenneth Halpern

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

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Abstract

We employ Wilson's renormalization group procedure in an attempt to classify and understand the physics of the continuum limits of scalar field theories. Analysis of the flows near the Gaussian fixed point reveals the existence of an infinite set of asymptotically free continuum limits. We study the associated physics by calculating scattering cross sections and the 1-loop effective potential. Examination of the latter provides evidence for the existence of a phase boundary in parameter space between those theories with broken symmetry and those with unbroken symmetry. We extend the flow analysis near the Gaussian fixed point to Bose/Fermi theories possessing arbitrary internal symmetries. Fermionic interactions are found to decouple in parameter space. The behavior of renormalization group trajectories near the Gaussian fixed point is solely determined by the Bosonic structure of the theory.

Thesis Supervisor: Kerson Huang
Title: Professor

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Chapter 1

Introduction

1.1 Renormalization Group and Its Uses

The Renormalization Group procedure (RG) is a method for relating a field theory defined at one energy scale to a physically equivalent “effective” theory at a different energy scale. By examining the high energy limits of sequences (“flows”) of such theories, we obtain “continuum” theories that describe interactions at arbitrarily high energies. All continuum limits are thought to be “fixed point” theories, upon which the renormalization group operation has no effect. The quantum field theory that best describes the universe is believed to lie at or near a fixed point.

Both a search for fixed points and a study of the directions of approach to them in the space of theories are important. In addition to helping us understand why renormalization works, such analyses may provide us with physical predictions. Present day quantum field theory assumes that the Gaussian fixed point (free theory) is the fixed point of interest. In such a scheme, certain unknown parameters, such as masses and couplings, are inputs and cannot be predicted *ab initio*. A non-Gaussian fixed point could provide such information. In an ideal scenario, the existence of a non-Gaussian fixed point in the Standard Model would tell us all of the eighteen “fundamental” parameters in terms of one unknown parameter. The beauty of such a solution is that it would come from within the Standard Model. Most other theories that have the potential to explain the fundamental masses and couplings treat the Standard Model as an approximation to a more general theory. The renormalization group approach has the potential to explain the Standard Model from only a few basic assumptions.

1.2 Goals and Methods

When we first embarked on the present course of research, the primary intention was to locate new fixed points, and, ideally, use these to compute physical quantities such as the Higgs mass. It soon became apparent that such an approach was fraught with difficulties. In particular, the space of theories that we were required to consider was large and intractable. We could not restrict ourselves to a simpler subspace without jeopardizing the validity of our calculations. This precluded a search for non-Gaussian

fixed points. However, we were able to show that near the Gaussian fixed point a simplified calculation was legitimate. In that case, it was valid to restrict ourselves to a simple subspace of theories. We used the Wegner-Houghton infinitesimal RG equations to study the flows projected into this subspace. From this, all of our results followed by straightforward analysis.

1.3 Preview

For a detailed summary of results, we refer the reader to chapter eight. Here we only provide a general preview of the most salient points.

While examining the renormalization group flow structure of $O(N)$ symmetric scalar field theories near the Gaussian fixed point, we discover an infinite set of asymptotically free eigentheories, many of which exhibit symmetry breaking. By proving that the space of theories under consideration is closed at linear order near the Gaussian fixed point, we demonstrate that the theories discovered are true eigendirections rather than artifacts. We calculate, to lowest order in a perturbation expansion, the scattering amplitudes and cross sections for the new theories, and we find the high energy scaling of the cross sections to differ significantly from that of ordinary ϕ^4 theory. This could have important ramifications for the behavior of the Higgs sector of the Standard Model.

Examination of the one-loop effective potential reveals the existence of a phase boundary in parameter space between symmetry broken and unbroken theories.

We extend our renormalization group flow calculations to theories consisting of Bose and Fermi fields with arbitrary internal symmetries and find that the eigenstructure near the Gaussian fixed point depends only on the internal symmetry structure of the Bose interactions.

1.4 Prospectus

This thesis is divided into two parts. The first part provides a general discussion of the renormalization group (chapter 2) and its application to renormalization theory (chapter 3). No prior knowledge of the subject is assumed, although some knowledge of quantum field theory is necessary. The second part of the thesis is an exposition of the author's research on the renormalization group flow structure of field theories near the Gaussian fixed point. A derivation of the infinitesimal flow equations for scalar field theories leads to the discovery and classification of asymptotically free nonpolynomial theories (chapter 4). The scattering amplitudes and cross sections for such theories are computed at lowest order perturbatively (chapter 5), and the one-loop effective potential is examined—revealing some interesting symmetry breaking behavior (chapter 6). We begin to extend the subject of this thesis to theories possessing Fermi and Bose fields and arbitrary internal symmetries (chapter 7). A summary of the results obtained, along with a discussion of possible future directions of research, is provided in chapter 8.

The author’s intention is that each chapter be as self-contained as possible. To this end, results and concepts are often replicated between chapters. Certain calculations and explanations that are not central to the thesis have been relegated to the appendices (appendices A, B, and C). A summary of useful mathematical formulae is provided (appendix D). For easy reference, a list of notations (appendix E) and a glossary of terms (appendix F) are also included. In an effort to increase readability, chapters have been subdivided to the greatest extent possible.

1.5 Basic Notation and Conventions

In this section, we describe most of the notation and specific terminology that is employed throughout the thesis. We defer some definitions to later sections, as it makes little sense to discuss objects prior to development of the concepts upon which they are predicated. A comprehensive glossary of symbols is provided in appendix E.

- We work in d -dimensional Euclidean space-time. All objects are analytically continued from (integer) n -dimensions and are Wick rotated. The metric is positive definite.
- Most calculations are performed in momentum space rather than position space. Momentum space objects are obtained by fourier transforming their position space counterparts.
- For part of the thesis, our system will be bounded by a space-time box of side L . This allows us to perform calculations that would otherwise be ambiguous. A space-time boundary imposes a lattice structure in momentum space, with lattice spacing $\frac{2\pi}{L}$.
- Scalar fields are denoted $\phi_i(x)$, with i an index over field components. The Fourier transform is denoted $\phi_i(k)$ in the infinite volume case and $\phi_{i,k}$ in the finite volume case. See appendix D.7 for our fourier transform conventions in both cases. Except in chapter 7, we work with theories possessing an $O(N)$ internal symmetry, often with $N = 1$. All scalar fields are real.
- Fermion fields, which appear only in chapter 7, are denoted $\psi_\alpha^*(x)$ and $\psi_\alpha(x)$. They are complex grassman variables, with α denoting both spinor and internal indices. The fourier transforms are $\psi_\alpha^*(k)$ and $\psi_\alpha(k)$.
- A theory is defined by both an action $S[\phi]$ —a general functional of the fields—and a cutoff Λ . The cutoff is a sphere in Euclidean momentum space such that if $k^2 > \Lambda^2$, $\phi_i(k) = 0$. This spherical domain is denoted Ω . The most general scalar theory we consider has action

$$S[\phi] = \sum_{n=0}^{\infty} \int d^d x_1 \cdots d^d x_n \phi_{i_1}(x_1) \cdots \phi_{i_n}(x_n) u_{i_1 \dots i_n}^{(n)}(x_1 \cdots x_n), \quad (1.1)$$

where the $u^{(n)}$ may be distributions rather than functions.

- We assume the action to be parametrizable by a set of parameters that span a linear vector space. If we add the non-kinetic part of two actions, we expect the associated parameters to add as well. In general, the set of parameters will be nondenumerably infinite. Note that each $u^{(n)}$ in the general action 1.1 encompasses an infinite number of parameters. These may be extracted by expansion in a complete set of basis functions (or distributions). However, such a basis may be difficult to specify.
- Most of our research involves a restricted subspace of theories. These theories, herein referred to as “local, non-derivative theories,” have actions of the form

$$S = \int d^d x [a_{ij} \partial_\mu \phi_i(x) \partial^\mu \phi_j(x) + U(\phi(x))] \quad (1.2)$$

for bosons, and

$$S = \int d^d x [b_{\alpha\beta}^\mu \psi_\alpha^*(x) \partial_\mu \psi_\beta(x) + U(\psi(x), \psi^*(x))] \quad (1.3)$$

for fermions. These general forms are employed in chapter seven. In all other sections, we consider $O(N)$ symmetric bose theories. For these,

$$S = \int d^d x [\frac{1}{2} \partial_\mu \phi_i(x) \partial^\mu \phi_i(x) + U(\phi_i(x) \phi_i(x))]. \quad (1.4)$$

- The fields, parameters in the action, positions, and momenta in our theory are dimensionless. The associated dimensional parameters may be obtained through scaling by appropriate powers of the cutoff Λ . Because the action is dimensionless to begin with, Λ will not appear in our calculations. Our use of dimensionless parameters is equivalent to a choice of $\Lambda = 1$ as the unit of momentum. For the purposes of exposition, we will sometimes employ Λ explicitly. It will be clear from the context that this is for explanatory reasons only.

Chapter 2

Renormalization Group

Contained herein is a brief introduction to the renormalization group and its application to particle physics. No prior knowledge of the renormalization group is assumed, although some acquaintance with field theory is necessary.

2.1 Historical Introduction

The renormalization group is the culmination of a series of attempts to use scaling as a tool for studying critical phenomena. Researchers were aware for many years that there is an important connection between the scale of density fluctuations in a system and critical phenomena. As a phase transition is approached, density fluctuations in the order parameter (the density, magnetism, etc.) grow very long. Points that are widely separated become correlated and influence one another's behavior. Sometimes this behavior manifests itself in spectacular ways, such as when a transparent fluid spontaneously becomes opaque. In that case, the density fluctuations are on the order of optical wavelengths.

In the early 1970's, Wilson[23] consolidated the ideas of Kadanoff and others, and formulated in concrete mathematical terms the renormalization group procedure for studying scaling behavior. In its original formulation, the RG involved successive coarse grainings of spins on a lattice. This is now referred to as the position space RG. One examines a system of discrete points at lower and lower resolutions. For example, starting with a lattice of magnetic spins, one defines a block lattice on which each spin is an average over several of the spins on the original lattice. This is similar to the method employed in defining such classical fields as polarization, magnetization, and fluid density. Since most classical systems are discrete at small enough scales, one typically defines a field as an average over discrete points. The RG does this repetitively and asks the question: How do short distance correlations (eg. between spins) influence long distance correlations? Answering this and other similar questions provides much information about the nature of critical phenomena. In particular, one can identify theories that are unchanged by coarse graining. In such theories the length scales decouple, and a phase transition may be present. Wilson[23] used the RG procedure to find, among other things, such a "fixed point" in magnetic

systems that was not obvious from conventional analyses.

One may formulate the renormalization group in terms of energy scales rather than position scales by noting that short distances correspond to high frequencies. The momentum space RG is the counterpart of the position space RG but with everything fourier transformed. Wilson[23], Wegner and Houghton[22], and others developed techniques for examining the momentum space RG. Lattices are discrete, but the momentum space RG can be made continuous by studying infinitesimal changes in energy scale. These and other practical considerations make the momentum space RG easier to study analytically than the position space RG. Much research, driven primarily by Lattice QCD, is performed using computers to study the position space RG. However, since we will primarily be concerned with analytic results, this thesis will focus on the momentum space RG.

Once the renormalization group procedure was developed, condensed matter theory saw enormous growth. It had been empowered with the ability to perform a large variety of new calculations. Unfortunately, the RG techniques— so exciting to condensed matter theorists— were largely ignored by particle theorists. Renormalization had been used successfully for over two decades and few researchers actively tried to fathom the reasons for its success. Several people developed scaling equations connecting renormalized parameters at different energy scales. Among others, the Callan-Symanzik equation and Gellman-Low equation have found great application in particle physics. Much of the terminology (including the name “renormalization group”) was adapted from condensed matter physics. However, the basic idea and technique of the Wilson RG went unused. In 1984, Polchinski[18], in keeping with some comments Wilson had made regarding applications to renormalization[23], used the RG to prove, simply and intuitively, certain theorems about renormalizability that had previously required elaborate diagrammatic arguments. After this, an ever increasing number of workers have turned their energies toward application of the RG to particle physics. As will be discussed in the next chapter, renormalization cast into the language of the RG no longer seems mysterious. Although it is difficult to prove general theorems about the validity of renormalization, the intuitive picture thus provided makes plausible a technique of an otherwise dubious nature.

In the past few years a number of efforts have focused on finding new fixed points which, as will be described, may represent new physical theories. The primary obstacle is the difficulty of calculation. Even scalar field theories present substantial hurdles, and a successful RG analysis of gauge theories remains elusive. Ironically, the position space RG, though intractable analytically, provides a natural vehicle for studying gauge theories.

2.2 Basic Idea

A “theory” consists of an action, characterized by a point p in parameter space, and a cutoff Λ . The renormalization group procedure is a prescription for constructing a theory p' at a cutoff $\Lambda' < \Lambda$ that is physically equivalent to p at energy scales $E < \Lambda'$. The high energy degrees of freedom are absorbed into a modification of

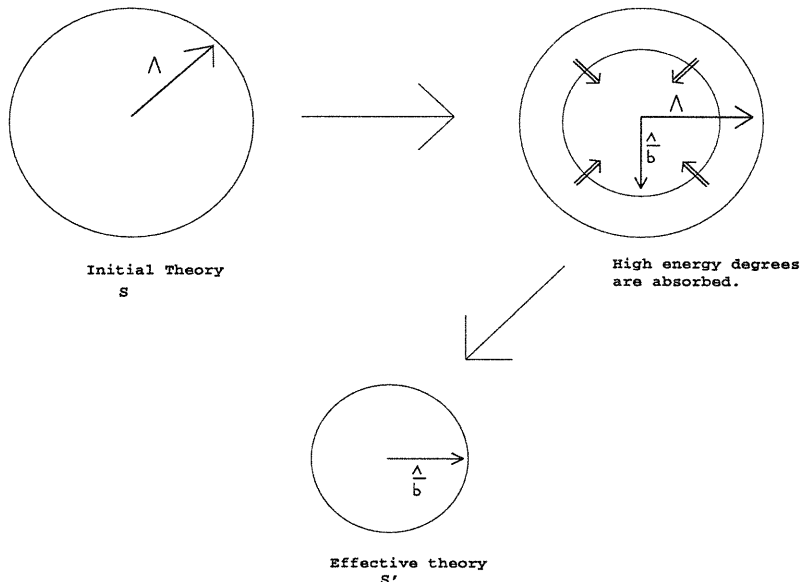


Figure 2-1: Illustration of the general idea behind the RG procedure.

the parameters in the action, and we are left with an “effective” low energy theory. Greens functions with external momenta less than the lower cutoff are identical, up to a field scale factor, in the two theories.

2.3 Notation

We parameterize the RG procedure with a real number $b \geq 1$ which denotes the scaling of the effective cutoff $\Lambda' = \frac{\Lambda}{b}$. For infinitesimal scaling, we define $b \equiv e^t \approx 1 + t$ with $t \geq 0$. The fields are separated into “fast” components, with momenta greater than Λ' , and “slow” components, with momenta less than Λ' :

$$\phi(k) = \phi_s(k) + f(k) \quad (2.1a)$$

$$\phi_s(k) = \phi(k)\theta(\Lambda' - |k|) \quad (2.1b)$$

$$f(k) = \phi(k)\theta(|k| - \Lambda'), \quad (2.1c)$$

with $\theta(x)$ defined to be one if $x \geq 0$ and zero otherwise. We denote the shell $\frac{\Lambda}{b} < |k| < \Lambda$ by $\partial\Omega$.

2.4 Procedure

The basic principle of the RG procedure is depicted in figure 2-1. Consider an object of physical interest represented as a functional integral

$$G(k_1 \cdots k_n) = \frac{\int [D\phi] e^{-S[\phi]} F(\phi, k_1 \cdots k_n)}{\int [D\phi] e^{-S[\phi]}}, \quad (2.2)$$

where F is some function of the slow¹ fields and momenta.

The renormalization group procedure consists of three steps. The first involves integration over the fast degrees of freedom—the fields $f(k)$. This results in a modified action $S'[\phi_s]$.

$$\begin{aligned} G(k_1 \cdots k_n) &= \left[\frac{\int [D\phi_s][Df] e^{-S[\phi_s+f]} F(\phi_s, k_1 \cdots k_n)}{\int [D\phi_s][Df] e^{-S[\phi_s+f]}} \right] \\ &= \left[\frac{\int [D\phi_s] e^{-S'[\phi_s]} F(\phi_s, k_1 \cdots k_n)}{\int [D\phi_s] e^{-S'[\phi_s]}} \right] \end{aligned} \quad (2.3)$$

This must hold for any F , so

$$S'[\phi_s] = -\ln \int [Df] e^{-S[\phi_s+f]}. \quad (2.4)$$

The field interactions are of three types: slow-slow, fast-fast, and slow-fast². Because our physical quantities only depend on ϕ_s , the fast-fast interactions do not contribute to G . Through integration, we have converted the fast-slow interactions amongst the fields into a modified set of slow-slow interactions.

The second step involves a rescaling of the cutoff. As defined in our initial theory, Λ (chosen to be 1) serves a dual role. It is both the momentum space cutoff and the momentum scale (unit of mass). The ability to perform successive RG operations depends on our restoring the former to its initial value. Integration lowers the momentum space cutoff to $\frac{\Lambda}{b}$. Therefore, we must scale the momentum space cutoff to Λ . No reference is made in the RG procedure to the momentum scale. We work with dimensionless objects so it should never arise. Therefore we are free to redefine our unit of momentum so that the cutoff has the value $\Lambda = 1$ once again. This affects how we recover physical quantities from the dimensionless ones, but has no impact on the RG procedure itself.

The final step consists of a rescaling of the field. The first two steps may have modified the kinetic term in the action. In order to maintain canonical fields, we must restore proper normalization. We scale the fields, and this leads to a redefinition of the parameters in the action. It is this scaling that gives rise to the anomalous dimensions of couplings. Denoting by Z the requisite scaling factor, the combined effect of the second and third steps may be expressed as

$$\phi'(k) \equiv Z\phi\left(\frac{k}{b}\right) \quad (2.5a)$$

$$S''[\phi'] \equiv S'[\phi_s]. \quad (2.5b)$$

Of course, any fields that appear in Greens functions must be modified as well. If $G^{(n)}$ is a Greens function that involves a product of n fields (with $k_i < 1$),

¹It is not useful to apply the RG procedure to objects involving fast fields because its effect will then be specific to the object in question.

²Here, “slow” and “fast” refer respectively to any product of ϕ_s fields and any product of f fields.

$$G_{new}^{(n)}(k_1 \cdots k_n) = Z^n G^{(n)}\left(\frac{k_1}{b} \cdots \frac{k_n}{b}\right). \quad (2.6)$$

2.5 Physical scales and Energies

The renormalization group procedure involves a redefinition of the unit of mass. Consider an energy E . Before the RG operation, E is measured in units of Λ , which serves as both the cutoff and the unit of mass. After integration, the cutoff is $\frac{\Lambda}{b}$ and the unit of mass remains Λ . E is defined relative to the unit of mass, so it is unchanged. When we scale the cutoff back to Λ , the unit of mass scales to $\frac{\Lambda}{b}$. In the new units, the same physical energy has the value bE .

A simple way to visualize the rescaling is by focusing on the surface of the cutoff sphere. Initially, the surface corresponds to energy Λ . After the RG procedure, it corresponds to physical cutoff $\frac{\Lambda}{b}$ but has the same value Λ . Therefore, the values of the physical energies have scaled as $E \rightarrow bE$.

2.6 Infinitesimal Flows and Trajectories

Given an initial point p_0 in parameter space at cutoff Λ , the RG procedure associates with each $b > 1$ a theory $p(b)$. This defines a trajectory in parameter space.

Performing functional integrals over a macroscopic shell is generally not tenable. If it were, we could solve the field theory directly. Infinitesimal transformations are calculable and provide some information about the scaling behavior of theories. An infinitesimal RG transformation is the tangent to the RG trajectory at a given point in parameter space. This tangent is unique. Mathematically, we are calculating

$$\left. \frac{dp}{db} \right|_{b=1} = \left. \frac{dp}{dt} \right|_{t=0} \quad (2.7)$$

The latter parameterization is more convenient for infinitesimal analyses. The infinitesimal “flow equations” are of the form

$$\left. \frac{dp}{dt} \right|_{t=0} = f(p), \quad (2.8)$$

where both p and f are vectors in parameter space. The flow comprises a (infinite) set of autonomous nonlinear first-order differential equations.

2.7 Mathematical Issues

Before discussing the nature of flows in parameter space, there are certain mathematical issues that should be addressed. We do not attempt a rigorous exposition, as such does not exist.

2.7.1 Closure

Consider the effect of the RG procedure on a pure ϕ^4 theory. New terms (see chapter 4) arise that weren't present in the original action. These, in turn, modify the ϕ^4 interactions during successive RG transformations. A space of theories p is "closed" under the RG procedure if it comprehends all interactions that can arise. Otherwise, the flows move out of the space into a larger space. Because the flow equations are coupled, interactions that have not been accounted for can influence the projection of the flow onto a subspace. We cannot study a subspace of theories in isolation unless that subspace is closed. If we are employing an approximation, we must ensure that the space is closed to whatever order we are calculating.

2.7.2 Invertibility

Although the RG procedure defines a unique trajectory for $b > 1$, it is not obvious what happens when we attempt to invert it. We may ask the question "What theories at cutoff Λ give rise to a given theory at cutoff $\frac{\Lambda}{b}$ under the RG operation?" The infinitesimal RG gives us a tangent vector at each point in parameter space. In any finite dimensional space, such information would divide space into non-intersecting orbits and we could trace our trajectory backward as well as forward. Therefore, we might be led to believe that the RG procedure is invertible. It is easy to see that this is not the case. Consider a macroscopic transformation. We integrate out an infinite number of degrees of freedom over a finite shell. There are many ways to restore these without altering the low energy behavior. The RG procedure loses information, and is not invertible. So why were we deceived in the infinitesimal case? We are not dealing with a finite dimensional space. The smallest closed space of interacting theories is infinite dimensional. At every point in this space, there is a confluence of an infinite number of flows. We attempt to depict this in figure 2-2.

2.8 Fixed Points and Eigendirections

2.8.1 Fixed Points

There exist points in parameter space at which the tangent to the flow vanishes. At these "fixed points," the renormalization group procedure has no effect. It is commonly believed that all trajectories begin and end on such points, although no rigorous proof to this effect is extant. The defining equation for a fixed point p^* is

$$\left. \frac{dp^*}{dt} \right|_{t=0} = f(p^*) = 0. \quad (2.9)$$

2.8.2 Eigendirections

As with any system of first order autonomous differential equations, we study the behavior of flows near a fixed point by examining the eigenstructure of the linearized

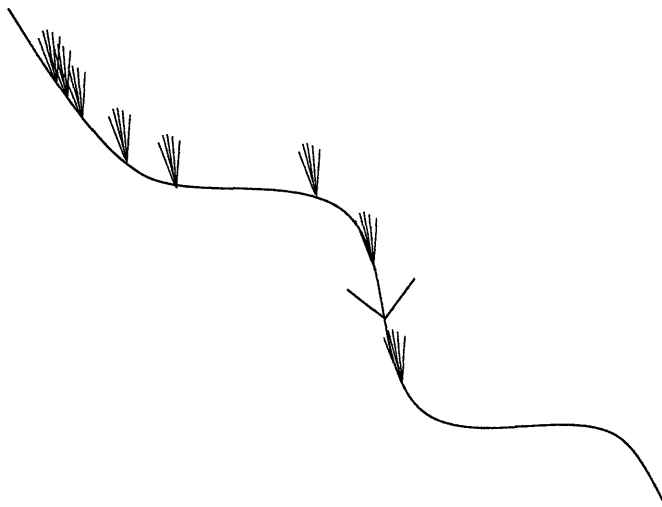


Figure 2-2: An attempt to depict the infinite number of trajectories converging at every point in parameter space. The arrow points in the direction of increasing b (decreasing cutoff).

equations. Let p^* be our fixed point and let $p = p^* + p'$ be a point nearby³:

$$\left. \frac{dp}{dt} \right|_{t=0} = f(p)$$

$$\left. \frac{d(p^* + p')}{dt} \right|_{t=0} = f(p^* + p').$$

At the fixed point

$$\left. \frac{dp^*}{dt} \right|_{t=0} = f(p^*) = 0,$$

so

$$\left. \frac{dp'}{dt} \right|_{t=0} = f(p^* + p').$$

Linearizing, we obtain (the index is over parameters)

$$\left. \frac{dp'_i}{dt} \right|_{t=0} = M_{ij} p'_j + O(p'^2) \tag{2.10a}$$

$$M_{ij} = \left. \frac{\partial f_i}{\partial p_j} \right|_{p^*} \tag{2.10b}$$

The eigenvalues of the matrix M determine the flow structure near the fixed point. Suppose \mathbf{m} is an eigenvector of M , with eigenvalue λ . If we start with a theory a distance r along \mathbf{m} , the linearized RG flow is (with $\mathbf{p}(0) \equiv r\mathbf{m}$)

³Though there is no metric on parameter space, the concept of “linear order” in the parameters is well defined.

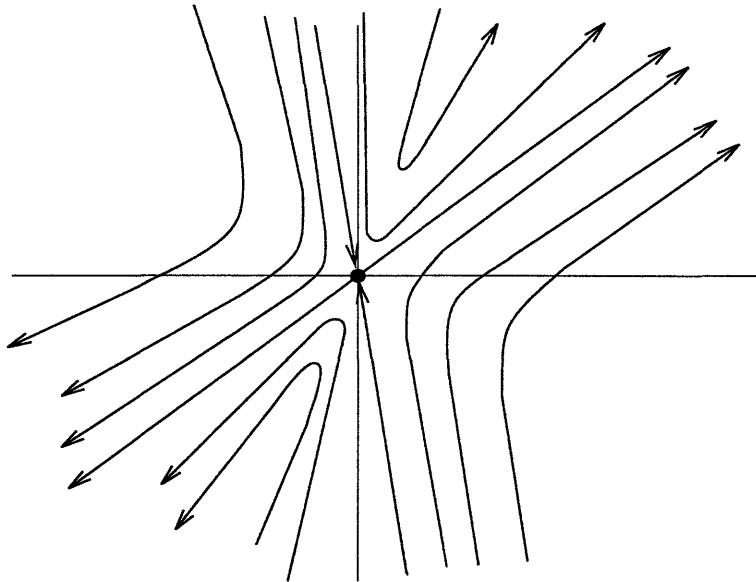


Figure 2-3: An example of flows near a fixed point. The arrows point in the direction of decreasing effective cutoff. Flows in are “irrelevant” and flows out are “relevant.”

$$\mathbf{p}(t) = (re^{\lambda t})\mathbf{m} \quad (2.11)$$

Theories along eigendirections with positive eigenvalues flow away from the fixed point under the RG procedure, while theories with negative eigenvalues flow toward the fixed point. The former are termed “relevant” directions by particle theorists and the latter are termed “irrelevant” directions⁴. An eigendirection with eigenvalue $\lambda = 0$ is termed “marginal” and corresponds to flow within a dense structure of fixed points. A simple example of flows near a fixed point is provided in figure 2-3. Complex eigenvalues are unphysical because the action is real.

2.8.3 Gaussian Fixed Point

The origin in parameter space, corresponding to a free massless theory, is a fixed point of any physical theory, and is called the “Gaussian fixed point.” Standard perturbation theory is an asymptotic expansion to the Gaussian fixed point. In four space-time dimensions, no other fixed points are known to exist in any physical theory.

2.8.4 Condensed Matter vs. Particle Physics limit

Condensed matter theorists are interested in the long wavelength correlations that characterize critical phenomena. Therefore they study low energy approaches to fixed points. To a condensed matter theorist, an eigendirection with negative eigenvalue is relevant because theories along that direction approach the fixed point in the low

⁴The opposite terminology is employed by condensed matter theorists.

energy limit.

A particle theorist, on the other hand, is interested in finding continuum limits. The object is to lift the cutoff and study high energy behavior. Continuum limits correspond to high energy approaches to a fixed point. To a particle physicist eigendirections with positive eigenvalue are relevant.

2.8.5 Searching for Fixed Points: Closure

A general search for fixed points is difficult. While restriction to a closed parameter subspace may simplify the problem, rarely is such a subspace easily identifiable. Therefore, it is important to understand what information may be obtained by restricting analysis to an unclosed subspace. Examples of such subspaces include the space of local, non-derivative theories and the two dimensional space of pure ϕ^4 theories.

Let us refer to the subspace of interest as G spanned by parameters g , and to the other parameters as h . For the purposes of discussion, we choose $h = 0$ in the subspace G . A point in parameter space is denoted (g, h) and a point in G is denoted $(g, 0)$. The equations that define a fixed point of the full theory are

$$\left. \frac{dg}{dt} \right|_{(g^*, h^*)} = \left. \frac{dh}{dt} \right|_{(g^*, h^*)} = 0, \quad (2.12)$$

while the analogous equations for the restricted theory are

$$\left. \frac{dg}{dt} \right|_{(g^*, 0)} = 0. \quad (2.13)$$

The equations defining a fixed point of the full theory include those defining a fixed point of the restricted theory. Not every solution of the latter is a solution of the former. In fact, it is possible that there are no common solutions. However, every solution for the full theory that obeys $h^* = 0$ must be a solution for the restricted theory. Therefore, if there are no fixed points of the restricted theory, there are no fixed points of the full theory that lie in the subspace G . In summary,

- A fixed point of the restricted theory is not necessarily a fixed point of the full theory.
- All fixed points of the full theory that lie within the restricted subspace must be fixed points of the restricted theory.

Note that if G is a closed subspace, the equation

$$\left. \frac{dh}{dt} \right|_{(g, 0)} = 0 \quad (2.14)$$

is obeyed, and any fixed point of the restricted theory is also a fixed point of the full theory.

2.9 Wegner-Houghton Approach

Shortly after Wilson's seminal papers[23] on the renormalization group, Wegner and Houghton developed a functional equation for the infinitesimal RG flow[22]. In this section we provide a qualitative discussion of the basic method, reserving detailed derivations for later.

2.9.1 Basic Idea

Consider the exponent that appears in the integration step of the RG procedure, equation 2.4. We Taylor expand⁵ $S[\phi_s + f]$ in f :

$$S[\phi_s + f] = S[\phi_s] + \sum_{n,k} \frac{\partial S}{\partial f_{n,k}} \Big|_{f=0} f_{n,k} + \frac{1}{2} \sum_{n,m,k,k'} \frac{\partial^2 S}{\partial f_{n,k'} \partial f_{m,k}} \Big|_{f=0} f_{n,k'} f_{m,k} + \dots \quad (2.15)$$

Wegner and Houghton [22] proved that no terms in this series with more than two derivatives contribute to the infinitesimal flow equations (i.e. at $O(t)$) and that, of the terms with two derivatives, only those with matching $(k, -k)$ momenta contribute. A heuristic justification and discussion of the Wegner-Houghton claims are provided in appendix A. The part of the Taylor series that contributes to S' to $O(t)$ is

$$S[\phi_s + f] = S[\phi_s] + \sum_{n,k} \frac{\partial S}{\partial f_{n,k}} \Big|_{f=0} f_{n,k} + \frac{1}{2} \sum_{n,m,k} \frac{\partial^2 S}{\partial f_{n,-k} \partial f_{m,k}} \Big|_{f=0} f_{n,-k} f_{m,k}. \quad (2.16)$$

The functional integral over fast components is now Gaussian. We may perform the integral explicitly, take the logarithm, and obtain an expression for $S'[\phi_s]$ in terms of the functional derivatives of S . The result is the lowest order term in an expansion of $S'[\phi_s]$ in powers of t . Equivalently, it is the tangent to the RG flow, as described earlier.

Computing $S'[\phi_s]$ from equation 2.4 is equivalent to a diagrammatic expansion. The vertices are the terms in S . Each external line corresponds to ϕ_s and each internal line corresponds to f . In this language, the Wegner-Houghton claim is that, to $O(t)$, the only diagrams that contribute to S' are tree (without any branches) and 1-loop, and that the external momenta at each vertex must sum to zero. The latter requirement is equivalent to demanding that the loop momentum remain uniform throughout the loop.

⁵For ease, we put our system in a box so that k is discrete. We treat $f(k)$ and $f(-k) = f^*(k)$ as independent complex variables and require $k_0 \geq 0$ in all sums.

2.9.2 Limitations

The Wegner-Houghton flow equations suffer a severe analytic limitation. They cannot be integrated. Because we demand that the partial sums of momenta be zero, we lose information that is essential for extension of the calculation beyond linear order in t . Consequently, we cannot integrate our tangent vectors to construct the flows. This is a problem with the sharp cutoff that we have employed. It leads to ambiguities involving nonlocal terms that arise, and it prevents us from extending our calculation to higher orders in t . Other cutoff schemes have been studied[16] but invariably render calculation intractable. Unfortunately, it is the very aspect of the sharp cutoff scheme which makes calculation feasible that is also responsible for the higher order ambiguities. Only by separating the degrees of freedom into disjoint sets, such as fast and slow components, can we reduce the information necessary to define an effective theory. Such a sharp division always leads to analytic ambiguities at the boundary.

Chapter 3

Renormalization

3.1 Introduction

Historically¹, renormalization was developed to extract physically useful information from seemingly divergent theories. Early in the development of quantum field theory it was realized that a Lagrangian with finite mass and coupling parameters leads to divergent scattering amplitudes. The origin of this problem² is the incorporation of interactions with arbitrarily high energies. Suppose we regulate the theory by imposing a Euclidean momentum-space cutoff on the fields. A propagator in a diagram corresponds, in the language of yesteryear, to a virtual particle. If we allow this virtual particle to possess arbitrarily high energies, the diagram may diverge when we lift the cutoff.

If we note that the fields and Lagrangian parameters are not directly observable, renormalization presents itself as a natural solution to the problem. We can absorb the divergences into a scaling of the fields and parameters order by order in perturbation theory. Generally, this is done by regulating the theory with a cutoff, imposing a set of normalization conditions (choices of experimentally determined scattering amplitudes and masses), and choosing the “bare” Lagrangian parameters and fields so that when we lift the cutoff, the theory obeys the normalization conditions.

An intuitive picture is often presented to help in understanding this process. An isolated particle in a free theory is “bare.” If we add couplings, but still consider a single particle state, interactions of the particle with itself via virtual particles (self energy diagrams) polarize the vacuum and “dress” the particle. A measurement of the mass or charge of the particle by an observer would be modified by the shell of polarization, and is “renormalized.” The higher the energy of our probe, the closer we get to the particle and the less the polarization shields it. Thus, as we raise the energy our measurements approach the bare values.

We have no reason to believe that field theory accurately describes physics at arbitrarily high energies. However, the success of renormalization indicates that at low energies an effective field theory describes the world surprisingly well. This will

¹This chapter is a supplement, rather than prerequisite, to the remainder of the thesis.

²We do not consider infrared divergences here.

be expanded on shortly, when we examine the same issue in the context of the renormalization group.

Once renormalization was adopted, its success was astounding. An entire machinery of diagrammatic analysis and dimension counting was developed to classify theories as renormalizable or non-renormalizable. Renormalizable theories are those in which one can absorb all of the divergent behavior into a finite number of bare parameters. Renormalizability was raised to the status of an axiom. As an assumption, its basis remains pragmatic. Nobody has successfully performed physically interesting calculations in a nonrenormalizable theory.

Part of the reason perturbative renormalization works is that the energy scale at which renormalization corrections become significant is extraordinarily high. People often claim that renormalized parameters are small and that bare parameters are divergent. They then proceed to manipulate the bare parameters as if these too were small. Surprisingly this works, and one reason for this is that at a cutoff equal to the energy scale of the entire universe the bare and renormalized parameters differ only slightly[20]. The bare parameters do diverge, but only at physically unattainable energies.

So far, in keeping with historical development, we have linked renormalization to perturbation theory. However, renormalization is far more fundamental than this. Perturbative divergences are only one manifestation of the need for it. There is no interacting theory in which the “masses” and “couplings” in the Lagrangian are those measured by experiment. Even in a field theory with no divergences (a “superrenormalizable” theory), we need to express our bare parameters as functions of physically measurable quantities. Otherwise, when we calculate scattering cross sections, the results will be functions of the unmeasurable bare parameters, and of little use. This is a reason why renormalization is necessary even outside of perturbation theory.

There is a yet deeper meaning to renormalization. This concerns the decoupling of theories at different energy scales or different length scales. The basic idea is that physics below an energy scale E can be described by an effective theory with a cutoff at E [17]. Virtual particles with energies higher than E have little effect on physics below E .

Whichever view we take, renormalization is a fundamental part of field theory. Through the renormalization group, this seemingly mysterious process finds a beautiful and intuitive expression.

3.2 Renormalizability

In its most basic form, renormalizability is the requirement that it be possible to extract finite physical quantities from a theory. All physical quantities are derivable from Greens functions, so these provide a natural vehicle for the mathematical expression of this requirement. The Greens functions are defined as

$$G^{(n)}(x_1 \cdots x_n) = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle. \quad (3.1)$$

Our intention is to define a renormalized theory with finite Greens functions. Let

us denote the Greens functions obtained from a theory with parameters g and cutoff Λ by $G^{(n)}(g, \Lambda, \{x_1 \cdots x_n\})$. We use g_0 and ϕ_0 to denote bare parameters and fields³. A theory is renormalizable if it is possible to define scale factors Z and renormalized parameters g_r at energy scale E

$$g_r = g_r(\Lambda, E, g_0) \tag{3.2a}$$

$$Z^{(n)} = Z^{(n)}(\Lambda, E, g_0), \tag{3.2b}$$

such that the renormalized Greens functions G_r obtained from

$$G^{(n)}(g_0, \Lambda, \{x_1 \cdots x_n\}) = Z^{(n)} G_r^{(n)}(g_r, E, \{x_1 \cdots x_n\}) \tag{3.3}$$

are finite as $\Lambda \rightarrow \infty$.

The idea is that all the divergent behavior of the Greens functions may be incorporated in scaling factors $Z^{(n)}$ and a redefinition of the parameters. The g_r are finite and the Z are divergent as $\Lambda \rightarrow \infty$. Note that the statement of renormalizability is not trivial; it is possible that the divergences involve x dependencies and cannot be absorbed into Z and g_r .

A more restricted, and pragmatic, definition of renormalizability is often used. This requires that the aforementioned condition hold for a theory with a finite number of bare and renormalized parameters. In practice, we usually fix the renormalized parameters g_r at energy scale E via a set of “normalization condition.” Then, we choose the bare parameters $g_0(\Lambda)$ to obey the normalization conditions at energy E .

For the purposes of our renormalization group discussion we adopt the following associated definition of renormalization: Given a set of normalization conditions at energy scale E , we define a set of bare parameters $g_0(\Lambda)$ such that the theory obtained from $g_0(\Lambda)$ with cutoff Λ obeys the normalization conditions to vanishing order in E/Λ . We will use $O(E/\Lambda)$ to denote a function that vanishes as $E/\Lambda \rightarrow 0$. Such a “weak” E/Λ dependence need not be linear.

3.3 Renormalization in the Language of the RG

3.3.1 Basics

The renormalization group procedure relates theories with different cutoffs, and provides a natural and elegant framework for the discussion of renormalization. For the purpose of exposition, we consider theories near the Gaussian fixed point; analogous analyses can be performed near other fixed points. The parameter space P may be written as a direct product of two spaces, G and H . We will later specify these as the spaces spanned respectively by the relevant⁴ and irrelevant eigendirections to the Gaussian fixed point. We denote by either p or (g, h) a point in P . By a “bare” theory we refer to a point $(g, 0) \in G$ and employ the notation $g \equiv (g, 0)$. To make

³i.e. the parameters and fields appearing in the action.

⁴and marginal

contact with traditional renormalization theory, we take G to be finite-dimensional.

To renormalize we must specify a sequence of bare theories g at progressively higher cutoffs Λ such that certain normalization conditions at energy E are satisfied to $O(E/\Lambda)$. At any cutoff Λ there are numerous theories that exactly obey the normalization conditions at energy E . Let us denote the associated set of points in parameter space p_Λ .

There are two approaches to constructing the requisite sequence of bare theories. They differ in the manner by which the $O(E/\Lambda)$ weak cutoff dependence is incorporated. One approach involves scaling $p_{\Lambda'}$ backwards to $\Lambda > \Lambda'$ under the RG and requiring that the resultant set of theories come within $O(E/\Lambda)$ of the space G , while the other approach⁵ involves constructing a sequence $g(\Lambda)$ that scales under the RG to within $O(\Lambda'/\Lambda)$ of $p_{\Lambda'}$. The latter approach is most closely linked to conventional renormalization.

The remainder of this section consists of a more detailed description of both viewpoints followed by a brief discussion of how renormalizability is related to the RG eigenstructure near the Gaussian fixed point. The explanations provided are intended to aid in forming an intuitive understanding of renormalization. They are plausibility arguments only. It is also important to keep in mind that the space P is infinite dimensional; for it is from this that most of our arguments derive.

3.3.2 Viewpoint 1: Backward

The set of theories at cutoff E that satisfy the normalization conditions exactly at energy E is p_E . For n normalization conditions, this set is a strip of codimension n in the space P . As mentioned in chapter 2, there exist multiple theories at cutoff Λ that give rise to a particular theory at cutoff $\Lambda' < \Lambda$ under RG scaling. This means that if we trace p_E backwards, we generate an ever expanding “cone” of theories⁶.

Renormalization is possible only if there exists a sequence of theories $p'(\Lambda) \in p_\Lambda$ that lies within $O(E/\Lambda)$ of G . If we denote $p'(\Lambda) = (g'(\Lambda), h'(\Lambda))$, then we require that $h'(\Lambda)$ be $O(E/\Lambda)$.

This notion of renormalization is illustrated in figure 3-1. Although the “cone” is depicted as being of fixed dimension, it is really a complicated object involving bifurcations and expansions into an ever-increasing number of dimensions. Rather than approaching G asymptotically (as is implied in the figure), it does so by increasing in dimension. It is difficult to develop intuition for an infinite-dimensional object, and this illustration is not intended to be an accurate depiction.

3.3.3 Viewpoint 2: Forward

Given a point $g \in G$ at cutoff Λ , we may trace forward along an RG trajectory until we reach energy E . Consider a sequence of theories $g(\Lambda) \in G$. Under the RG

⁵This view of renormalization was developed by Polchinski[18].

⁶We use the word “cone” for lack of a better term; the set generated is actually a complicated union of trajectories in the infinite-dimensional space of theories.

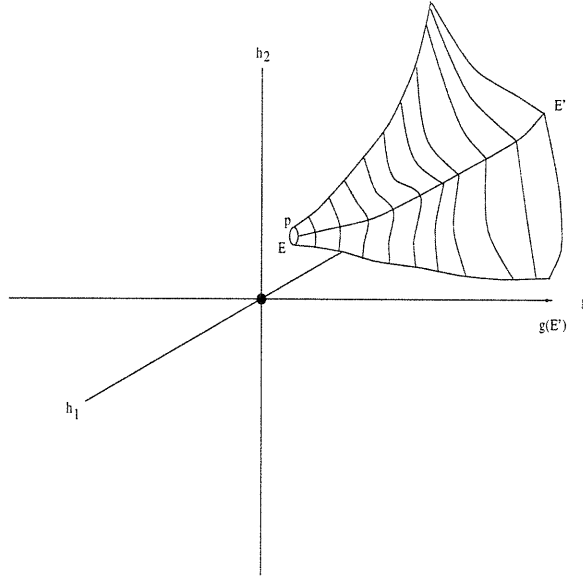


Figure 3-1: “Cone” of theories, with progressively higher cutoffs, that are physically equivalent to the original theories p_E at energy scale E . At $E' > E$, the cone comes within $O(E/E')$ of the g axis (the space of bare theories).

operation, this flows to a sequence of theories $p'(\Lambda)$ at energy E . Renormalization is possible only if there exists a sequence $g(\Lambda)$ whose associated sequence $p'(\Lambda)$ lies within $O(E/\Lambda)$ of p_E .

An illustration of this approach to renormalization is provided in figure 3-2. It is adapted from the paper by Polchinski[18].

3.3.4 Renormalizability

So far, we have recast the idea of renormalization into the language of the renormalization group via two different approaches. The question of renormalizability has not been addressed. We are now in a position to determine the characteristics that distinguish a renormalizable theory from a non-renormalizable one. As mentioned, we provide a heuristic discussion only⁷. For details we suggest the pioneering paper by Polchinski[18]. We denote by n the dimension of the space spanned by the relevant eigendirections to the Gaussian fixed point.

Flows converge toward the n -dimensional space of relevant directions as they approach the Gaussian fixed point. To within $O(E/\Lambda)$ all information is absorbed into the relevant parameters. A large number of theories at high cutoff coalesce at low cutoff⁸. Consequently, the low energy behavior of a theory is governed by only n parameters⁹.

⁷Also see Peskin’s book[17].

⁸This is a property of the region near the fixed point, and complements the information loss described in chapter 2.

⁹“Relevance” is a purely local concept in the space of theories, but renormalizability is not.

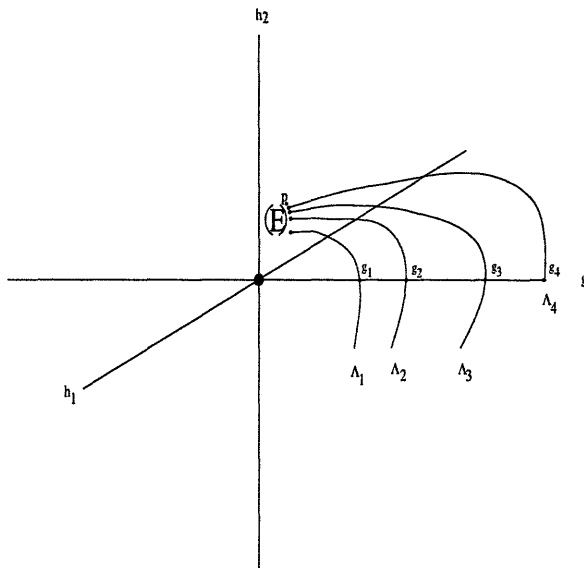


Figure 3-2: Bare Parameters for $\Lambda_4 > \Lambda_3 > \Lambda_2 > \Lambda_1 > E$. The trajectory passing through $g(E_i)$ scales down to within $o(E/E_i)$ of p at energy scale E .

Suppose that we impose n normalization conditions on the Greens functions at energy E . These fix the low energy behavior of our theory because we need only n pieces of information to do so. A bare theory at cutoff Λ obeys these normalization conditions to within $O(E/\Lambda)$. The $O(E/\Lambda)$ correction is the penalty for ignoring irrelevant parameters in our bare theory. The Greens functions at low cutoff are finite. Any corrections that arise from raising the cutoff (employing a bare theory) are $O(E/\Lambda)$. Therefore the Greens functions remain finite and the theory is renormalizable.

Now suppose that we wish to include irrelevant parameters in our space of bare theories. Let m be the total number of parameters in our bare theory space. The imposition of $m > n$ normalization conditions requires that we fix $m - n$ irrelevant parameters at low energy. The strip p_E , of codimension n , still specifies the low-energy behavior of the theory. Any irrelevant parameters present in a theory at cutoff Λ will contribute at $O(E/\Lambda)$ to the low-energy behavior of the theory. In order for them to significantly contribute to the low-energy behavior, they must be initially large enough to compensate for the $O(E/\Lambda)$ scaling. However, far away from the Gaussian fixed point the eigenstructure is lost. It is not possible to have a few irrelevant parameters that are huge and expect the other irrelevant parameters to remain small under scaling toward E . But this is what would be required in order to construct a sequence of bare theories. As a result, the specification of $m > n$ normalization conditions on the Greens functions leads to an infinite number of bare parameters. If we artificially ignore all but a finite number of bare parameters, the Greens functions contain a strong Λ dependence and diverge.

To summarize: In addition to the information loss associated with RG scaling,

Rather, it is the local manifestation of a global property.

there is a compression of irrelevant parameter information due to the eigenstructure near the Gaussian fixed point. As a result, we only have enough information to specify the relevant parameters of the low-energy theory. Specification of irrelevant parameters requires an infinite number of bare parameters.

3.4 Triviality

Renormalization involves the solution of a set of simultaneous equations relating the bare parameters to the renormalized parameters. It is not always possible to solve these equations. An example is massive ϕ^4 theory. There does not exist a solution $g_0(\Lambda)$ for any non-zero choice of renormalized coupling. Perturbatively, this manifests itself as the “Landau singularity.” The bare coupling diverges for a finite value of Λ unless the renormalized coupling is chosen to be zero. In terms of renormalization group flows, triviality arises because there are no relevant eigendirections to the Gaussian fixed point except for the mass-axis.

Until recently, it was widely believed that there are no non-trivial scalar field theories in $d = 4$ dimensions. As will be discussed, our research leads to a different conclusion.

3.5 β Functions

In particle theory, a set of “renormalization group” equations have been developed¹⁰ relating renormalized parameters at different energy scales. Renormalized parameters (or normalization conditions) are chosen at some energy scale E . One then asks which choice of renormalized parameters at a different energy scale E' would lead to the same physics (i.e. same bare parameters). In analogy to the Wilson RG procedure, one can trace flows in the space of renormalized parameters—the only difference being that this space is usually finite-dimensional. The tangents to the flows are referred to as β -functions.

A natural question to ask is what, if any, relationship exists between the Wilson RG flows and the flows in renormalized-parameter space? In particular, do the Wilson RG flows provide us with enough information to determine the β -functions? Both approaches provide us with scaling information about Greens functions. Therefore, we expect that there is some relationship. However, it turns out that the Wilson RG flows alone are not enough to determine the renormalized-parameter β -functions. In addition, we need to know the relationship between the bare and renormalized parameters at some energy scale E .

¹⁰For a comprehensive listing, see the books by Zinn-Justin[24] and Amit[2].

3.6 Asymptotic Freedom

A relevant eigendirection to the Gaussian fixed point corresponds to an asymptotically free theory. In the high energy limit, the trajectory moves into the fixed point and the bare parameters vanish. Since the renormalized parameters scale the same way as the bare parameters along an eigendirection, these vanish in the high energy limit and the theory is asymptotically free. The reasons for such asymptotic freedom are very different from that of non-abelian gauge theories. In our case, the continuum limit happens to be the Gaussian fixed point. In the gauge case, the gauge fields induce a negative vacuum polarization.

Chapter 4

Scalar Theories Near the Gaussian Fixed Point

4.1 Overview

In this chapter we present our research[6, 7] on the flow structure of $O(N)$ symmetric scalar field theory near the Gaussian fixed point. We begin by providing a detailed derivation of the infinitesimal flow equations for scalar field theories. We note that the smallest closed space of interacting theories involves non-local couplings that cannot be unambiguously identified within our sharp cutoff scheme. This renders our methods unsuitable for the study of non-Gaussian fixed points. Near the Gaussian fixed point, we linearize the equations and examine the resultant flow eigenstructure. After demonstrating that to linear order near the Gaussian fixed point the space of local, non-derivative interactions is closed, we identify those eigendirections that lie within this space. We find that, in addition to a set of irrelevant directions corresponding to polynomial potentials, the eigenvectors include an infinite set of relevant directions corresponding to asymptotically free nonpolynomial theories. Some of the nonpolynomial theories exhibit symmetry breaking. For these we compute the broken potentials. In two space-time dimensions, the eigenpotentials are shown to be Sine-Gordon potentials.

Although we provide an unexpurgated derivation of the flow equations and eigenstructure, we relegate the calculation of symmetry-broken potentials to appendix B. Many of the mathematical tools necessary to reproduce our computations are provided in appendix D.

4.2 Infinitesimal Flow Equations

4.2.1 Basics

To derive infinitesimal flow equations, we employ the Wegner-Houghton approach described in section 2.9. Certain ambiguities arise if we do not impose a spatial boundary. Where necessary we will assume that the system lies within a space-

time box of volume $V = L^d$ with periodic boundary conditions. As mentioned, we work with dimensionless parameters, fields, and momenta. For convenience, we write $\phi^2 \equiv \sum_i \phi_i \phi_i$, as we are presently only examining $O(N)$ symmetric theories. Although much of our calculation is applicable to more general classes of theories, we restrict our attention to the space of local, non-derivative interactions, considering issues of closure afterward and deferring a discussion of general scalar theories until chapter 7.

4.2.2 The Action

Our object is to determine the functions $f(p)$ in the flow equations 2.8. The general $O(N)$ symmetric action is

$$S[\phi] = \sum_{n=1}^{\infty} S_{2n} \quad (4.1a)$$

$$S_{2n} = \int d^d x_1 \cdots d^d x_{2n} \phi_{i_1}(x_1) \phi_{i_1}(x_2) \cdots \phi_{i_n}(x_{2n-1}) \phi_{i_n}(x_{2n}) u_{2n}(x_1 \cdots x_{2n}). \quad (4.1b)$$

The parameters p in equation 2.8 are actually the coefficients of an expansion of the $u_{2n}(x_1 \cdots x_{2n})$ in a complete set of functions¹. However, to avoid the difficult task of determining a complete set of basis functions, we work directly with the $u_{2n}(x_1 \cdots x_{2n})$. We expect the theory to be translationally invariant², and require u_{2n} to be a function of $(x_2 - x_1) \cdots (x_{2n} - x_{2n-1})$ only.

The infinite-volume momentum-space form of equation 4.1 is

$$S_{2n} = (2\pi)^d \int_{\Omega} \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_{2n}}{(2\pi)^d} \phi_{i_1}(k_1) \phi_{i_1}(k_2) \cdots \phi_{i_n}(k_{2n-1}) \phi_{i_n}(k_{2n}) u_{2n}(k_1 \cdots k_{2n}) \delta^{(d)}(\sum_{j=1}^{2n} k_j), \quad (4.2)$$

where $u_{2n}(-k_1 \cdots -k_{2n})(2\pi)^d \delta^{(d)}(\sum k_i)$ is the fourier transform of $u_{2n}(x_1 \cdots x_{2n})$. The δ function arises from the invariance of $u_{2n}(x_1 \cdots x_{2n})$ under space-time translations.

4.2.3 Bounding Box

We begin by constructing a finite-volume action that attains the form 4.1 in the infinite volume limit. For convenience, we list our finite-volume fourier conventions and the transition maps between finite and infinite volume objects. The symbol \sim denotes equivalence in the infinite volume limit. k_{disc} and k_{cont} are respectively the finite-volume and infinite-volume momenta.

¹The u_{2n} are really distributions, but we will refer to them as functions for simplicity.

²We impose this and other space-time symmetries despite their explicit violation by our cutoff and bounding box. The justification comes when we verify that the symmetry restricted space of actions is closed under the RG.

$$\phi(x) = \frac{1}{V} \sum_k \phi_k e^{ik \cdot x} \quad (4.3a)$$

$$\phi_k = \int d^d x \phi(x) e^{-ik \cdot x} \quad (4.3b)$$

$$V \delta_{k,p} \sim (2\pi)^d \delta^{(d)}(k - p) \quad (4.4a)$$

$$\frac{1}{V} \sum_k \sim \int \frac{d^d k}{(2\pi)^d} \quad (4.4b)$$

$$k_{disc} \sim k_{cont} \quad (4.4c)$$

$$\frac{\partial}{\partial k_{disc}} \sim \frac{\partial}{\partial k_{cont}} \quad (4.4d)$$

$$\phi_k \sim \phi(k) \quad (4.4e)$$

$$V \frac{\partial}{\partial \phi_k} \sim \frac{\delta}{\delta \phi(k)}. \quad (4.4f)$$

Because we explicitly separated the momentum conserving δ function from the fourier transform of u_{2n} , the transition properties of $u_{2n}(k_1 \cdots k_{2n})$ will involve volume factors:

$$V u_{2n, k_1 \cdots k_{2n}} \delta_{\Sigma k_i} \sim u_{2n}(k_1 \cdots k_{2n}) (2\pi)^d \delta^d(k_1 \cdots k_{2n}). \quad (4.4g)$$

From these limit rules, we obtain the finite-volume momentum-space action³:

$$S[\phi] = \sum_{n=1}^{\infty} S_{2n} \quad (4.5a)$$

$$S_{2n} = V^{1-2n} \sum_{k_1 \cdots k_{2n} < 1} \delta_{\Sigma k_i} \phi_{k_1} \cdots \phi_{k_{2n}} u_{2n, k_1 \cdots k_{2n}}. \quad (4.5b)$$

The u_{2n} associated with a local, non-derivative action have the form

$$u_{2, k_1, k_2} = -\frac{1}{2} k_1 \cdot k_2 + u_2 \quad (4.6a)$$

$$u_{2n, k_1 \cdots k_{2n}} = u_{2n}. \quad (4.6b)$$

Because of the $\delta_{k_1+k_2}$ in the action's quadratic term, u_{2, k_1, k_2} may be written $u_{2, k} \equiv \frac{1}{2} k^2 + u_2$. Where convenient we shall use $r \equiv 2u_2$, in keeping with conventional notation for the mass term.

4.2.4 Modus Operandi

In deriving the infinitesimal flow equations, we consider a “fast shell” that involves momenta $1 > |k| > 1 - t$, and we perform calculations to $O(t)$. Our goal is to

³For simplicity we use δ_k to denote $\delta_{k,0}$.

determine⁴

$$\left. \frac{d}{dt} \right|_{t=0} u_{2n, k_1 \dots k_{2n}}$$

as a function(al) of the u 's.

As discussed in section 2.4, the renormalization group procedure involves three steps: integration, dimensional scaling, and field normalization. Consider an infinitesimal RG transform ($t \ll 1$). This generates a modification of u_{2n} :

$$u'_{2n, k_1 \dots k_{2n}} = u_{2n, k_1 \dots k_{2n}} + t\Delta u_{2n, k_1 \dots k_{2n}} + O(t^2). \quad (4.7)$$

Each step may be thought of as an operator acting on the action. Let us, for temporary illustrative purposes only, call A the integration operator, B the dimensional rescaling operator and C the field normalization operator. The RG procedure may be summarized as

$$S' = CBAS. \quad (4.8)$$

For infinitesimal RG transforms, all three operators produce $O(t)$ corrections to the action. We may write them as the identity, I , plus an operator that is $O(t)$: $A = I + ta$, $B = I + tb$, and $C = I + tc$. Then,

$$\begin{aligned} S' &= (I + ta)(I + tb)(I + tc)S \\ &\approx S + t(a + b + c)S + O(t^2). \end{aligned} \quad (4.9)$$

The point of this exposition is that to $O(t)$ we may consider the three steps of the RG procedure as acting independently on the original action. The effect of the RG procedure may be summarized, with Δ 's denoting operators, as

$$\left. \frac{dS}{dt} \right|_{t=0} = (\Delta_{int} + \Delta_{dim} + \Delta_{norm})S. \quad (4.10)$$

Although the Δ operators can mix parameters, we will use

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} u_{2n, k_1 \dots k_{2n}} &= \Delta_{int} u_{2n, k_1 \dots k_{2n}} + \Delta_{dim} u_{2n, k_1 \dots k_{2n}} \\ &\quad + \Delta_{norm} u_{2n, k_1 \dots k_{2n}} \end{aligned} \quad (4.11)$$

as a notation for the modifications to the parameters.

⁴As mentioned, we are implicitly considering the space of theories that are translationally invariant. We have not shown this space to be closed but, as will be evident, in none of our steps do nonhomogeneous interactions arise.

4.2.5 Step 1: Integration

The first step of the RG procedure involves integration over the fast components f_k with $1 > |k| > 1 - t$. $\Delta_{int}S$ is the $O(t)$ part of equation 2.4, the discretized form of which is

$$S'[\phi_s] = -\ln \int \prod_{k \in \partial\Omega} df_k e^{-S[\phi_s + f]}. \quad (4.12)$$

Volume factors that arise from the discretization of the functional integral contribute a constant to the action, and we ignore them. Taylor-expanding the action in the fast fields f , we only keep the following terms⁵:

$$\begin{aligned} S[\phi_s + f] &\approx S[\phi_s] + \sum_{n,k} \left. \frac{\partial S}{\partial f_{n,k}} \right|_{f=0} f_{n,k} \\ &+ \frac{1}{2} \sum_{n,m,k} \left. \frac{\partial^2 S}{\partial f_{n,-k} \partial f_{m,k}} \right|_{f=0} f_{n,-k} f_{m,k}. \end{aligned} \quad (4.13)$$

Let us rewrite this as

$$S[\phi_s + f] \approx S[\phi_s] + \sum_{k \in \partial\Omega} [B_k \cdot f_k + \frac{1}{2} f_{-k} A_k f_k], \quad (4.14a)$$

with

$$(A_k)_{nm} \equiv \left. \frac{\partial^2 S}{\partial f_{n,-k} \partial f_{m,k}} \right|_{f=0} \quad (4.14b)$$

$$(B_k)_n \equiv \left. \frac{\partial S}{\partial f_{n,k}} \right|_{f=0}. \quad (4.14c)$$

Our space-fields $\phi(x)$ are real, so their fourier transforms obey

$$\phi_{-k} = \phi_k^*. \quad (4.15)$$

S is real, so

$$A_k = A_k^\dagger \quad (4.16a)$$

$$B_{-k} = B_k^*. \quad (4.16b)$$

We may treat f_k and f_k^* as independent variables of integration in equation 4.12 if we restrict $k_0 \geq 0$. We denote this half-shell domain $\partial\Omega_+$. Equation 4.14 may be rewritten

$$S[\phi_s + f] \approx S[\phi_s] + \sum_{k \in \partial\Omega_+} [B_k \cdot f_k + B_k^* \cdot f_k^* + f_k^* A_k f_k]. \quad (4.17)$$

⁵As discussed in section 2.9, all other terms contribute to S' at $O(t^2)$ or higher.

The functional integral is a product of decoupled Gaussian integrals over the variables f_k and f_k^* :

$$\begin{aligned} S'[\phi_s] &= -\ln \int \prod_{k \in \partial\Omega_+} df_k df_k^* e^{-S[\phi_s] - \sum_{k \in \partial\Omega_+} [B_k \cdot f_k + B_k^* \cdot f_k^* + f_k^* A_k f_k]} \\ &= -\ln \left[e^{-S[\phi_s]} \prod_{k \in \partial\Omega_+} \int df_k df_k^* e^{-[B_k \cdot f_k + B_k^* \cdot f_k^* + f_k^* A_k f_k]} \right]. \end{aligned} \quad (4.18)$$

Using the formulae for Gaussian integrals provided in appendix D.2, we obtain

$$\int df_k df_k^* e^{-[B_k \cdot f_k + B_k^* \cdot f_k^* + f_k^* A_k f_k]} = \frac{(2i\pi)^n}{\det A_k} e^{B_k A_k^{-1} B_k^*}. \quad (4.19)$$

The modified action, to within an insignificant constant, is (using $\det A = e^{\text{tr} \ln A}$)

$$S'[\phi_s] = S[\phi_s] + \sum_{k \in \partial\Omega_+} [\text{tr} \ln A_k - B_k A_k^{-1} B_k^*]. \quad (4.20)$$

The sum $\sum_{k \in \partial\Omega_+}$ is proportional to the volume of the shell, and is $O(t)$. This is readily seen in the infinite volume limit, where the \sum_k becomes an integral over a shell of width t .

At this point, we could derive a general expression for $\Delta_{int} u_{2n}$ by substituting the action 4.5 into equation 4.20 and extracting contributions with $2n$ fields. We defer such an analysis to chapter 7 where it is conducted in much greater generality. At present we only consider the space of local, non-derivative interactions; so the $u_{2n, k_1 \dots k_{2n}}$ have the form given by equation 4.6. Denoting by A_{2n} and B_{2n} those terms in A and B that derive from S_{2n} , we find

$$\begin{aligned} (A_{2n, k})_{ij} &= 2n V^{1-2n} \sum_{k_1 \dots k_{2n-2} < 1-t} \delta_{\Sigma k_i} \left(\phi_{m_1, k_1} \phi_{m_1, k_2} \dots \phi_{m_{n-2}, k_{2n-5}} \phi_{m_{n-2}, k_{2n-4}} \right) \\ &\quad \cdot [(2n-2) \phi_{i, k_{2n-3}} \phi_{j, k_{2n-2}} + \delta_{ij} \phi_{l, k_{2n-3}} \phi_{l, k_{2n-2}}] u_{2n} \\ (A_{2, k})_{ij} &= V^{-1} \delta_{ij} (k^2 + 2u_2) \\ (B_{2n, k})_i &= 2n V^{1-2n} \sum_{k_1 \dots k_{2n-1} < 1-t} \delta_{k + \Sigma k_i} \left(\phi_{m_1, k_1} \phi_{m_1, k_2} \dots \right. \\ &\quad \left. \dots \phi_{m_{n-1}, k_{2n-3}} \phi_{m_{n-1}, k_{2n-2}} \right) \phi_{i, k_{2n-1}} u_{2n} \\ (B_{2, k})_i &= 0. \end{aligned} \quad (4.21)$$

The last result arises from differentiating $f(-k)f(k)$ with respect to $f(k)$ and setting $f = 0$. Note that, at the order to which we are calculating, $k^2 = 1$ on the shell. So

$$(A_{2,k})_{ij} = V^{-1} \delta_{ij} (1 + 2u_2). \quad (4.22)$$

We could proceed to expand A_k and B_k around their quadratic parts, insert the expansions into equation 4.20, and obtain a rather unwieldy expression. Since we will eventually work in linear approximation, we defer expansion until then. For now, we note that the result we would have obtained is a sum over products of $A_{k,2n}$ and $B_{k,n}$ factors.

4.2.6 Step 2: Dimensional Rescaling

To $O(t)$ we need only consider the effects of dimensional rescaling on $S[\phi_s]$. From equation 4.5,

$$S[\phi_s] = \sum_{n=1}^{\infty} S_{2n}[\phi_s] \quad (4.23a)$$

$$S_{2n}[\phi_s] = V^{1-2n} \sum_{k_1 \dots k_{2n} < 1-t} \delta_{\Sigma k_i} \phi_{k_1} \dots \phi_{k_{2n}} u_{2n, k_1 \dots k_{2n}}. \quad (4.23b)$$

We define⁶

$$k'_i \equiv k_i(1+t) \quad (4.24a)$$

$$\phi'(k') \equiv \phi(k'(1-t)) = \phi(k). \quad (4.24b)$$

We must treat the δ function carefully. If we scale the density of momentum space, the discrete δ will contribute for more choices of $k_1 \dots k_{2n}$. The proper manner in which to deal with the discrete δ is deducible from the analogous behavior of the Dirac δ . From the transition relations 4.4 we see that the two types of δ functions scale the same way, so

$$\delta_{k'(1-t)} = (1+t) \delta_{k'}. \quad (4.25)$$

Likewise, we expect the summations to be changed by a scaling of the momentum space density (they will now over-count by the scale factor). Again, from the transition relations we see that the sum scales like an integral in the limit. So for any function F ,

$$\sum_{k < 1-t} F_k = (1-t)^d \sum_{k' < 1} F_{k'(1-t)}. \quad (4.26)$$

Substituting these into equation 4.23, we find

⁶This definition is only precise in the infinite volume limit. For a finite volume, k is discrete and we assume the momentum lattice spacing to be sufficiently small to render our scaling equivalent. We later take the infinite volume limit anyway, so this is reasonable.

$$S_{2n}[\phi_s] = (1+t)^d (1-t)^{2nd} V^{1-2n} \sum_{k'_1 \dots k'_{2n} < 1} \delta_{\Sigma k'_i} \phi'_{k'_1} \dots \phi'_{k'_{2n}} u_{2n, k'_1(1-t) \dots k'_{2n}(1-t)}. \quad (4.27)$$

We Taylor expand u_{2n} as if the variables were continuous⁷:

$$u_{2n, k'_1(1-t) \dots k'_{2n}(1-t)} = \left[1 - t \sum_{i=1}^{2n} k'_i \frac{\partial}{\partial k'_i} \right] u_{2n, k'_1 \dots k'_{2n}} + O(t^2). \quad (4.28)$$

Combining these and extracting the $O(t)$ part, we obtain

$$\begin{aligned} S_{2n}^{dim}[\phi'] &= S_{2n}[\phi_s] \\ &= V^{1-2n} \sum_{k'_1 \dots k'_{2n} < 1} \delta_{\Sigma k'_i} \phi'_{k'_1} \dots \phi'_{k'_{2n}} \left[1 - 2ndt + dt - t \sum_{i=1}^{2n} k'_i \frac{\partial}{\partial k'_i} \right] u_{2n, k'_1 \dots k'_{2n}}. \end{aligned} \quad (4.29)$$

We read off Δ_{dim} from this as the coefficient of the $O(t)$ term:

$$\Delta_{dim} = -2nd + d - \sum_{i=1}^{2n} k_i \frac{\partial}{\partial k_i}. \quad (4.30)$$

This analysis may seem somewhat contrived; that is because we worked in a finite volume. This step may be performed without ambiguity in the infinite volume limit and then appears more natural.

4.2.7 Step 3: Field Normalization

We must now determine the effects of the first two steps on the kinetic term. As is evident from equation 4.21, there is no part of $\Delta_{int} S$ from which a term of the form $k_i \cdot k_j$ (with $k < 1 - t$) could arise. The only dependence on k_i that appears in the modified u_{2n} is through products of δ functions. The A_2 term involves k^2 , but this is evaluated on shell ($k^2 = 1$). There is no place from which momentum dependent Δ_{int} corrections can arise in a local, non-derivative theory. The only possible modification to the kinetic term is from momentum scaling. Applying Δ_{dim} to $u_{2:kin, k_1, k_2} = -\frac{1}{2} k_1 \cdot k_2$, we find

$$\Delta_{dim} u_{kin} = -(d+2) u_{kin}. \quad (4.31)$$

The requisite scaling is

$$S_{2n}^{norm}[\phi''] = S[\phi_s] \quad (4.32a)$$

$$\phi'' \equiv Z \phi_s \quad (4.32b)$$

$$Z = 1 - \frac{1}{2}(d+2)t, \quad (4.32c)$$

⁷From the transition relations 4.4 it is evident that this does not give rise to volume factors.

and using the form 4.5 for S_{2n} we have

$$S_{2n}^{norm}[\phi''] = [1 + n(d+2)t + O(t^2)]S_{2n}[\phi''], \quad (4.33)$$

from which we read off $\Delta_{norm}u_{2n}$

$$\Delta_{norm}u_{2n} = n(d+2)u_{2n}. \quad (4.34)$$

4.2.8 Propagation of Constraints

Before removing our bounding box, we should address a perplexing difficulty that arises. Suppose we perform an infinitesimal RG transformation on a local, non-derivative theory. Our calculations show that the integration correction to u_{2n} is momentum dependent. For example, there is a modification to $u_{4,k_1\dots k_4}$ of the form $\delta_{k_1+k_2}$ (this multiplies the overall $\delta_{k_1\dots k_4}$). In position space an example of a term that arises (at $O(t)$) is⁸

$$\frac{u_4^2}{V} \left[\int d^d x \phi^2(x) \right]^2, \quad (4.35)$$

which contributes to $\Delta_{int}u_{4,k_1\dots k_4}$.

Such a term is highly nonlocal and cannot be expanded in derivative interactions. In fact, no derivative interactions ever arise from a local, non-derivative theory. The delta function interactions represent constraints. In diagrammatic language, Wegner and Houghton's theorems demand that each vertex have vanishing total external momentum. Each vertex, rather than each diagram, carries a momentum conserving delta function. The implication is that the space of local, non-derivative interactions is not closed.

One might be tempted to broaden the initial theory to include such delta function interactions in the hope that the enlarged space is closed. It is not. Delta function interactions breed theta function interactions and these proliferate wildly. The sharp cutoff, through the Wegner-Houghton theorem, renders analysis feasible. At the same time it leads to a propagation of constraints (delta and theta interactions).

The situation, however, is worse than this. Not only is the smallest closed interaction space huge and its interactions uncategorizable, it is ambiguous as well. In the infinite volume limit, non-local terms of the type described are not well defined.

Many attempts have been made to extract the local part of the delta function interactions[6, 8, 16, 13]. Usually, the individual external momenta are set to zero (as if the $\Delta u_{2n,k_1\dots k_{2n}}$ are Taylor expandable) and the resulting expression employed. Unfortunately, in the infinite volume limit the "local" part, ill-defined to begin with, vanishes in this analysis.

It may seem that all is lost. The sharp cutoff that makes calculation tractable also appears to make it useless. Indeed, we cannot search for fixed points or study

⁸This arises from the $tr(A_{kin}^{-1}A_4)^2$ term in the expansion of equation 4.20 about the part of A from $S_{kinetic}$. See section A.3 for details.

general flows using this scheme. However, there is one significant calculation to which this method is well suited. To linear order near the Gaussian fixed point, the flow equations are unambiguous. It is this case that we now proceed to examine.

4.3 Near the Gaussian Fixed Point

4.3.1 Linearized equations

We treat the entire action—except the kinetic term— as small. For convenience, we denote⁹ linear order in all parameters ($u_2, u_4 \dots$) by $O(r)$ (recall that $r = 2u_2$). The third step of the RG procedure ensures that the kinetic term remains unaltered. Therefore we refer to this term as $O(1)$. From equation 4.21 we know that B_2 is zero. The only $O(1)$ contribution to A or B is from the kinetic term, so $B \sim O(r)$. A does receive a contribution from the kinetic term. We divide A into a kinetic part A_{kin} and a non-kinetic part A' (note that u_2 is contained in A').

$$(A_{kin})_{ij} = \frac{1}{V} \delta_{ij} \quad (4.36)$$

Both A and A^{-1} are $O(1)$. The $BA^{-1}B^*$ term in Δ_{int} is $O(r^2)$, and may be neglected. The logarithmic term can be expanded:

$$\begin{aligned} \text{tr} \ln A &= \text{tr} \ln A_{kin} + \text{tr} \ln(I + A_{kin}^{-1}A') \\ &\approx \text{tr} \ln A_{kin} + \text{tr}(A_{kin}^{-1}A') + O(r^2) \\ &= \text{tr} \ln A_{kin} + \text{tr}(VA') + O(r^2). \end{aligned} \quad (4.37)$$

The $(\text{tr} \ln A_{kin})$ term contains no ϕ dependence, and we ignore it. The integration correction to the action is

$$S^{int} - S = V \sum_{k \in \partial\Omega_+} \text{tr} A'_k. \quad (4.38)$$

Using equation 4.21, and noting that $\text{tr} I = N$,

$$\begin{aligned} S^{int} - S &= V \sum_{k \in \partial\Omega_+} \sum_{n=1}^{\infty} 2n V^{1-2n} \sum_{k_1 \dots k_{2n-2} < 1-t} \delta_{\Sigma k_i} \phi_{m_1, k_1} \phi_{m_1, k_2} \dots \\ &\quad \dots \phi_{m_{n-1}, k_{2n-3}} \phi_{m_{n-1}, k_{2n-2}} [(2n-2) + N] u_{2n}. \end{aligned} \quad (4.39)$$

The modification to S_{2n-2} comes from A_{2n} . Recalling that S_{2n-2} is defined as

$$S_{2n-2} = V^{3-2n} \sum_{k_1 \dots k_{2n-2}} \delta_{\Sigma k_i} \phi_{k_1} \dots \phi_{k_{2n-2}} u_{2n-2, k_1 \dots k_{2n-2}}, \quad (4.40)$$

⁹The parameters are all independent; this is just a convenient notation.

the effect of the integration step is

$$\Delta_{int} u_{2n} = \frac{1}{V} (2n+2)(2n+N) \left[\sum_{k \in \partial\Omega_+} \right] u_{2n+2}. \quad (4.41)$$

Combining this with the results of the second and third steps,

$$\begin{aligned} \Delta u_{2n} &= \frac{1}{V} (2n+2)(2n+N) \left[\sum_{k \in \partial\Omega_+} \right] u_{2n+2} - t(2nd - d - n(d+2)) u_{2n} \\ &= \frac{1}{V} (2n+2)(2n+N) \left[\sum_{k \in \partial\Omega_+} \right] u_{2n+2} + t(2n+d - nd) u_{2n}. \end{aligned} \quad (4.42)$$

4.3.2 Removing the Box

The bounding box has served its purpose, and we may now take the infinite volume limit using the transition relations 4.4:

$$\Delta u_{2n} = (2n+2)(2n+N) \left[\int_{\partial\Omega_+} \frac{d^d k}{(2\pi)^d} \right] u_{2n+2} + t(2n+d - nd) u_{2n}. \quad (4.43)$$

The integral is half the volume of a d -dimensional shell of radius 1 and width t .

$$\int_{\partial\Omega_+} \frac{d^d k}{(2\pi)^d} = t \frac{S_d}{2}, \quad (4.44)$$

where

$$S_d = \frac{2^{1-d} \pi^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (4.45)$$

Finally, we obtain

$$\left. \frac{du_{2n}}{dt} \right|_{t=0} = S_d(n+1)(2n+N) u_{2n+2} + (2n+d - nd) u_{2n}. \quad (4.46)$$

4.3.3 Closure

Local, non-derivative theories have been shown to give rise to delta function interactions under RG transformations. The space of local, non-derivative interactions is not closed in general. However, to linear order near the Gaussian fixed point it is closed. The delta function interactions arise from the compounding of vertices, each of which has its own momentum conserving delta function. At linear order, there are no products of vertices. Therefore, no nonlocal interactions arise. We have already seen that no derivative interactions arise in either case.

The closure of the space of local, non-derivative interactions at linear order near

the Gaussian fixed point means that we can perform calculations at this order without worrying about influences from “outside” directions that we neglected to incorporate. Any eigendirections we find in this restricted subspace are true eigendirections of the full theory. If the subspace weren’t closed at this order, outside interactions might affect the eigendirections and render our calculations invalid. Of course, regardless of whether the subspace is closed, there exist other eigendirections that do not lie in the subspace. These needn’t concern us, as they supplement rather than exclude any eigendirections we might find.

4.4 Eigenpotentials

4.4.1 Derivation

Having calculated the infinitesimal flow equations to linear order near the Gaussian fixed point, we are in a position to study the flow eigenstructure. As discussed in section 2.8.2, the object of interest is the matrix M , defined by

$$\left. \frac{du_{2n}}{dt} \right|_{t=0} = M_{nm}u_{2m} \quad (4.47)$$

From the flow equation 4.46 we observe that all elements of M are zero except for those which lie either on the diagonal or immediately to the right of the diagonal:

$$M = \begin{pmatrix} M_{11} & M_{12} & 0 & 0 & \cdots \\ 0 & M_{22} & M_{23} & 0 & \cdots \\ 0 & 0 & M_{33} & M_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.48a)$$

with

$$M_{n,n} = (2n + d - nd) \quad (4.48b)$$

$$M_{n,n+1} = (n + 1)(2n + N)S_d. \quad (4.48c)$$

The eigenvectors of this matrix can be determined recursively. For eigenvalue λ ,

$$(M_{nm} - \lambda\delta_{nm})u_{2m} = 0. \quad (4.49)$$

This constitutes an infinite set of simultaneous equations

$$(2n + d - nd - \lambda)u_{2n} + S_d(n + 1)(2n + N)u_{2n+2} = 0 \quad (4.50)$$

and, for any real eigenvalue λ , recursively defines an eigenvector

$$u_{2n+2}^\lambda = \left[\frac{\lambda + nd - d - 2n}{(n + 1)(2n + N)S_d} \right] u_{2n}^\lambda \quad (4.51)$$

We may solve this for u_{2n} in terms of an initial condition u_2 . In the interest of clarity, we employ $r = 2u_2$ as our initial parameter and define a new eigenvalue label

$u_4 = \frac{a(d-2)}{2(N+2)S_d} u_2$	$u_4 = \frac{a(d-2)}{4(N+2)S_d} r$	$u_4 = \frac{2\pi^2}{3} ar$
$u_6 = \frac{(a+1)(d-2)}{3(N+4)S_d} u_4$	$u_6 = \frac{a(a+1)(d-2)^2}{12(N+4)(N+2)S_d^2} r$	$u_6 = \frac{4\pi^4}{9} a(a+1)r$
$u_8 = \frac{(a+2)(d-2)}{4(N+6)S_d} u_6$	$u_8 = \frac{a(a+1)(a+2)(d-2)^3}{48(N+6)(N+4)(N+2)S_d^3} r$	$u_8 = \frac{8\pi^6}{45} a(a+1)(a+2)r$
a.	b.	c.

Figure 4-1: Examples of coefficients: a. Recursive definition, b. As functions of $r \equiv 2u_2$ for general d and N , and c. As functions of r for $d = N = 4$.

$$a \equiv \frac{\lambda - 2}{d - 2}. \quad (4.52)$$

The parameters a distance r along an eigendirection labeled by a are¹⁰

$$u_{2n}^a = \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{n-1} \left[\frac{(a+n-2)! \left(\frac{N}{2}\right)!}{(a-1)! \left(\frac{N}{2} + n - 1\right)! n!} \right]. \quad (4.53)$$

We note that complex eigenvalues yield complex eigenvectors and are unphysical. Some examples of coefficients are provided in figure 4.4.1.

The eigenpotential associated with an eigenvector is

$$U^a(\phi(x)) = \sum_{n=1}^{\infty} u_{2n}^a (\phi_i(x) \phi_i(x))^n. \quad (4.54)$$

These eigenpotentials have a closed form expression in terms of our vocabulary of special functions. Substituting equation 4.53 into equation 4.54, one finds that

$$U^a(\phi(x)) = r \frac{NS_d}{2(a-1)(d-2)} \left[M(a-1, N/2, \frac{(d-2)\phi^2}{2S_d}) - 1 \right], \quad (4.55)$$

where (see appendix D) $M(a, b, z)$ is Kummer's function, a type of confluent hypergeometric function defined by

$$M(a, b, z) = \frac{(b-1)!}{(a-1)!} \sum_{n=0}^{\infty} \frac{z^n (a+n-1)!}{n! (b+n-1)!}. \quad (4.56)$$

For easy reference, we recall the definitions of all parameters in the eigenpotential:

- N is the number of field components.
- d is the number of space-time dimensions.
- $r = 2u_2$ is the distance along the eigendirection.
- $S_d = \frac{2^{1-d} \pi^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.
- $a = \frac{\lambda-2}{d-2}$, where $-\infty < \lambda < \infty$ is the associated eigenvalue.

¹⁰ *with* $x!$ $\equiv \Gamma(x+1)$.

4.4.2 Truncations and Polynomials

The vast majority of eigenpotentials are nonpolynomial. They behave like exponentials for large field amplitudes. If we had truncated our eigen-calculation at any finite order, regardless of how large, we would not have found these solutions.

There is a small set of polynomial eigenpotentials. If there exists an integer $n > 0$ for which $\lambda + nd - d - 2n = 0$, the recursion relation 4.51 truncates and all higher coefficients vanish. The condition is that $\lambda = (-nd + d + 2n)$, or $a = (1 - n)$, for some $n > 0$. The case $a = 0$ ($\lambda = 2$) corresponds to a massive free field theory. If $a = 1 - n$, the eigenpotential is a polynomial in ϕ^2 of order n .

4.4.3 Relevant and Irrelevant Directions

An eigenpotential is proportional to r and scales in the same manner as does r under RG transforms. Along an eigendirection with eigenvalue λ , this scaling is ($t > 0$)

$$r(t) = r_0 e^{\lambda t}. \quad (4.57)$$

As discussed, an eigendirection with positive λ is relevant. Since renormalized parameters also scale like r , such a direction of approach to the Gaussian fixed point is asymptotically free.

The relevant directions correspond to $\lambda > 0$ or $a > \frac{-2}{d-2}$. In $d = 4$, this means $a > -1$. The lower bound on relevant a 's decreases toward $-\infty$ as $d \rightarrow 2+$.

Polynomial eigenpotentials correspond to nonpositive integers a . In $d = 4$ the only relevant polynomial theory is the free theory $a = 0$. As d decreases toward $d = 2$, the number of relevant polynomial theories increases. For example, in $d = 3$ directions with $a > -2$ are relevant, and the ϕ^4 polynomial theory with $a = -1$ is relevant. In $d = 4$ all relevant interacting eigenpotentials are nonpolynomial.

4.4.4 Marginal Direction and Wilson Fixed Point

The marginal direction $\lambda = 0$ ($a = \frac{-2}{d-2}$) corresponds to motion within or tangent to a dense region of fixed points. In general, the associated eigenpotential is nonpolynomial. However, in $d = 4$ it is a ϕ^4 theory and points in the direction of Wilson's fixed point[23] in the $u_2 - u_4$ plane. In $d = 4 - \epsilon$ dimensions, Wilson's fixed point is $O(\epsilon)$ away from the Gaussian fixed point. As $d \rightarrow 4$, it moves toward the Gaussian fixed point, merging with it in $d = 4$. The marginal eigendirection in $d = 4$ points along the path by which the two fixed points merge. The direction is

$$u_4 = \frac{-u_2}{(2 + N)S_d}. \quad (4.58)$$

4.4.5 Symmetry Breaking

Certain eigenpotentials exhibit symmetry breaking. No closed form expression exists for the zeros of the Kummer function, so we test for symmetry breaking by examining the low and high field behaviors.

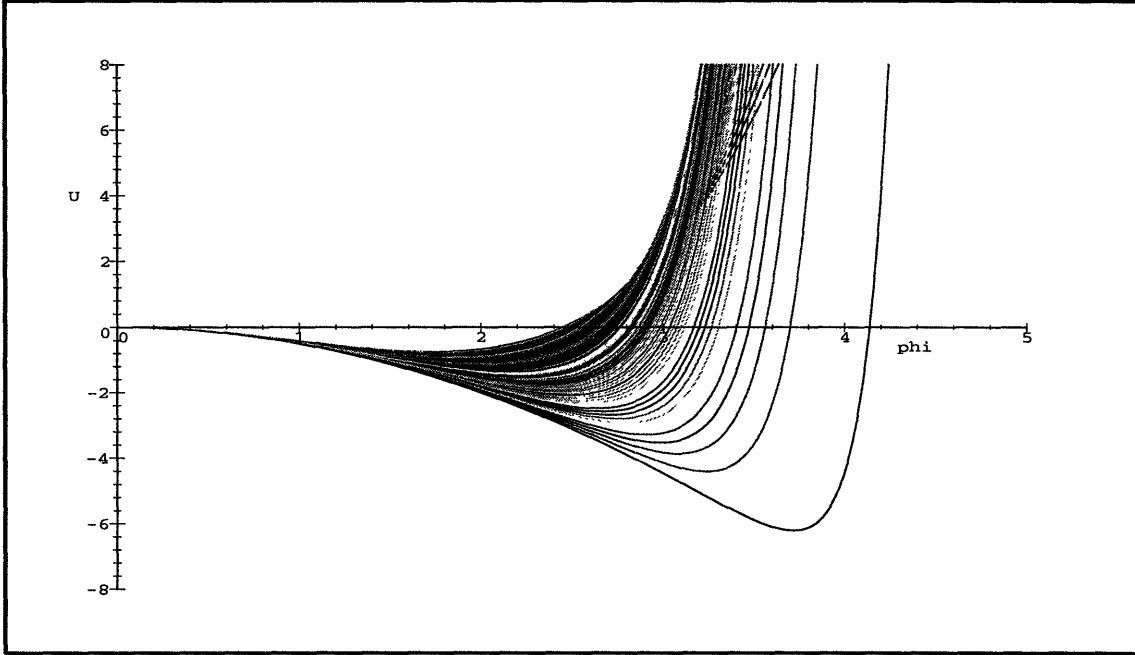


Figure 4-2: Plots of the eigenpotential for various a between -1 and 0 . The shallowest curve corresponds to $a = -1$.

The potential U is zero when $r = 0$. For small ϕ , U is dominated by its first derivative:

$$\left. \frac{dU^a}{d(\phi^2)} \right|_{\phi^2=0} = \frac{1}{2}r. \quad (4.59)$$

For $r < 0$ and ϕ small, U is a decreasing function of ϕ . Symmetry breaking occurs if U is an increasing function at large ϕ , because the global minimum will then be finite. Using the asymptotic formula for the Kummer function provided in appendix D, we find

$$U^a(\phi(x)) \sim r \frac{S_d}{(d-2)} \frac{\left(\frac{N}{2}\right)!}{(a-1)!} e^z z^{a-1-\frac{N}{2}}, \quad (4.60a)$$

with

$$z \equiv \frac{(d-2)\phi^2}{2S_d}. \quad (4.60b)$$

The power and exponential factors are positive. The potential is positive for $r < 0$ if $\frac{(N/2)!}{(a-1)!} < 0$, so symmetry breaking requires that $(a-1)!$ be negative. $(a-1)! = \Gamma(a)$ is negative iff $-2n-1 \leq a < -2n$ for non-negative integer n . Symmetry breaking is present when $a \in [-1, 0)$, $a \in [-3, -2)$, etc..

In $d = 4$, the only symmetry-broken relevant eigenpotentials are those with $-1 \leq a < 0$. A sequence of these potentials is plotted in figure 4-2.

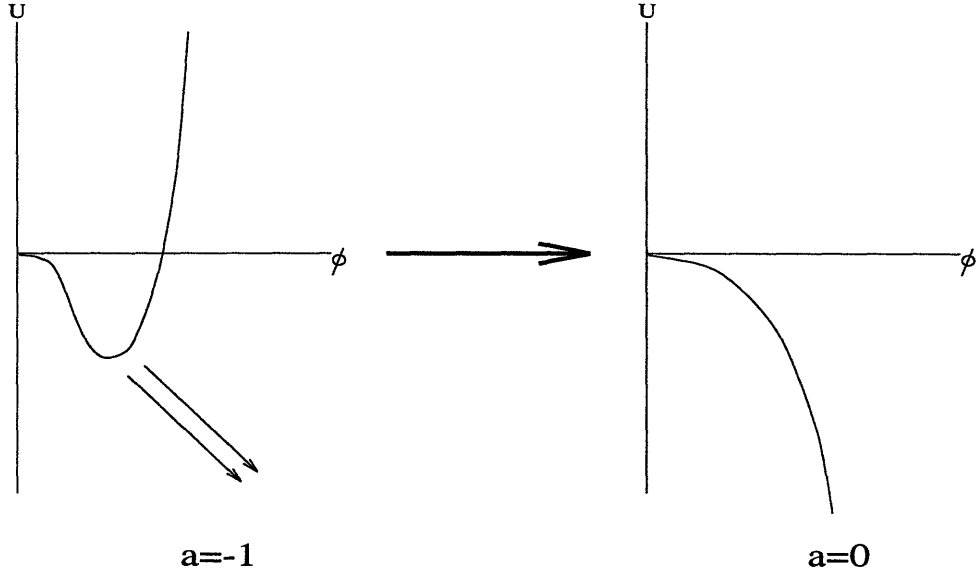


Figure 4-3: The progression of potentials in $d = 4$ from $a = -1$ to $a = 0$.

4.4.6 How the potentials vary from $a = -1$ to $a = 0$ in $d = 4$

In $d = 4$, there is a progression from a symmetry-broken ϕ^4 theory at $a = -1$ through a sequence of nonpolynomial symmetry-broken theories $-1 < a < 0$ to an unbroken free theory at $a = 0$. The ϕ^4 theory, $a = -1$, exhibits the shallowest symmetry breaking of the lot. As we increase a , the minimum moves downward and outward until $a = 0$, where the potential is a negative parabola¹¹. This progression is illustrated in figure 4-3.

4.4.7 Sine-Gordon Potentials in $d = 2$, $N = 1$

To treat¹² the case $d = 2$, we return to our original recursion formula 4.51

$$u_{2n+2}^\lambda = \left[\frac{\lambda - 2}{(n+1)(2n+N)S_d} \right] u_{2n}^\lambda, \quad (4.61)$$

with $S_2 = \frac{1}{2\pi}$. Solving for u_{2n}^λ ,

$$u_{2n}^\lambda = \left(\frac{\lambda - 2}{2S_2} \right)^{n-1} \left[\frac{(\frac{N}{2})!}{n!(\frac{N}{2} + n - 1)!} \right] u_2. \quad (4.62)$$

For one field component ($N = 1$), this is

$$u_{2n}^\lambda = \left(\frac{\lambda - 2}{2S_2} \right)^{n-1} \left[\frac{(\frac{1}{2})!}{n!(n - \frac{1}{2})!} \right] u_2. \quad (4.63)$$

¹¹It is negative because we chose $r < 0$, appropriate to symmetry breaking; for the free theory, $r > 0$.

¹²The ideas and results in this section were contributed by Immirzi[10].

Noting that $(2x)! = \frac{2^{2x}}{\sqrt{\pi}}(x - \frac{1}{2})!x!$ and $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$, we find (in terms of $r = 2u_2$)

$$u_{2n}^\lambda = r \frac{\left(\frac{2\lambda-4}{S_2}\right)^{n-1}}{(2n)!}. \quad (4.64)$$

The associated eigenpotential is

$$U^\lambda(\phi) = r \left(\frac{S_2}{2\lambda-4}\right) \sum_{n=1}^{\infty} \left(\frac{2\lambda-4}{S_2}\right)^n \frac{1}{(2n)!} \phi^{2n}, \quad (4.65)$$

which sums to a cosine

$$\begin{aligned} U^\lambda(\phi) &= r \left(\frac{S_2}{2\lambda-4}\right) \left[\cos\left(\phi \sqrt{\frac{2\lambda-4}{S_2}}\right) - 1 \right] \\ &= \frac{r}{4\pi(\lambda-2)} \left[\cos(\phi \sqrt{4\pi(\lambda-2)}) - 1 \right]. \end{aligned} \quad (4.66)$$

This is the Sine-Gordon potential. In $d = 2$, our eigenpotentials are Sine-Gordon potentials.

4.4.8 Broken Eigenpotentials

We observed that certain eigenpotentials exhibit symmetry breaking. In $d = 4$ these correspond to $-1 \leq a < 0$. For such theories, it is the broken potential expanded around the vacuum, rather than the eigenpotential itself, that is of physical interest. Calculation of the broken potentials is straightforward but involved, and we defer a detailed derivation to appendix B. Unlike the eigenpotential, the broken potential involves terms with odd powers of the field.

Let $\phi = \rho$ be the location of the minimum of the eigenpotential U . No analytic expression for ρ exists, so we define it implicitly by

$$\left. \frac{dU}{d\phi} \right|_{\rho} = 0. \quad (4.67)$$

Substituting equation 4.55 and using the derivative relations for Kummer functions described in appendix D, we find the defining condition to be

$$M\left(a, N/2 + 1, \frac{(d-2)\rho^2}{2S_d}\right) = 0. \quad (4.68)$$

To obtain the broken potential, we expand the eigenpotential in a new dynamical field, ϕ' , around the minimum ρ :

$$U(\phi) = U(\rho + \phi') \equiv V(\phi'). \quad (4.69)$$

We parameterize this in the same manner as we did U , but allowing for odd terms as well.

$$V(\phi') = \sum_{n=0}^{\infty} v_n \phi'^n. \quad (4.70)$$

The couplings v_n are obtained from U by Taylor expanding equation 4.69.

$$v_n = \frac{1}{n!} \frac{d^n U}{d\rho^n}. \quad (4.71)$$

The v_n may be computed by expanding U in u_{2n} , differentiating n times, and resumming. The Kummer function takes ρ^2 , not ρ , as its argument, so the differentiation relations of appendix D are not directly applicable. The v_n in the case $N = 1$ have a particularly simple form, and it is this which we calculate in appendix B. There, we find that

$$v_{2n} = u_{2n} M\left(a + n - 1, \frac{1}{2}, \frac{\rho^2(d-2)}{2S_d}\right) \quad (4.72a)$$

$$v_{2n+1} = \rho u_{2n} \left(\frac{2(d-2)}{S_d}\right) \left(\frac{a+n-1}{2n+1}\right) M\left(a+n, \frac{3}{2}, \frac{\rho^2(d-2)}{2S_d}\right). \quad (4.72b)$$

4.5 Summary

1. We showed that the space of local, non-derivative interactions, though not closed in general, is closed to linear order near the Gaussian fixed point.
2. We studied the flow structure near the Gaussian fixed point and found the eigenpotentials to be

$$U^a(\phi(x)) = r \frac{NS_d}{2(a-1)(d-2)} \left[M\left(a-1, N/2, \frac{(d-2)\phi^2}{2S_d}\right) - 1 \right] \quad (4.73)$$

where

- N is the number of field components.
 - d is the number of space-time dimensions.
 - $r = 2u_2$ is the distance along the eigendirection.
 - $S_d = \frac{2^{1-d}\pi^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.
 - $a = \frac{\lambda-2}{d-2}$, where $-\infty < \lambda < \infty$ is the associated eigenvalue.
 - M is Kummer's function.
3. We classified the eigendirections as follows for general dimension d (n is any odd positive integer and m is any positive integer):

a	λ	Behavior
0	2	Free theory
$\frac{-2}{d-2}$	0	Marginal
$> \frac{-2}{d-2}$	> 0	Asymptotically free
$[-n, -n+1)$	$[2(n+1) - nd, 2n - d(n-1))$	Symmetry Broken
$1 - m$	$2 + (d-2)(1-m)$	Polynomial

and for $d = 4$,

a	λ	Behavior
0	2	Free theory
-1	0	Marginal
> -1	> 0	Asymptotically free
$[-n, -n+1)$	$[2 - 2n, 4 - 2n)$	Symmetry Broken
$1 - m$	$4 - 2m$	Polynomial

4. In the case $d = 2$, $N = 1$, the eigenpotentials were shown to be Sine-Gordon potentials.
5. We outlined the procedure for determining symmetry-broken potentials, and stated the results for $N = 1$:

$$v_{2n} = u_{2n} M\left(a + n - 1, \frac{1}{2}, \frac{\rho^2(d-2)}{2S_d}\right) \quad (4.74a)$$

$$v_{2n+1} = \rho u_{2n} \left(\frac{2(d-2)}{S_d}\right) \left(\frac{a+n-1}{2n+1}\right) M\left(a+n, \frac{3}{2}, \frac{\rho^2(d-2)}{2S_d}\right). \quad (4.74b)$$

Chapter 5

Scattering

5.1 Overview

Our analysis of the flow eigenstructure near the Gaussian fixed point revealed the existence of asymptotically free scalar field theories corresponding to nonpolynomial potentials. In this chapter and the next, we calculate physical quantities within these theories[5]. In the present chapter, we compute scattering amplitudes and cross sections to lowest order in perturbation theory and analyze the high energy scaling behavior of the cross sections. Approaches to and difficulties with higher order calculations are discussed. We conclude with a summary of the results obtained.

Several of the calculations constitute lengthy digressions and are presented in appendix C. These include some of the scattering amplitude computations as well as a derivation of the kinematic part of the cross sections.

Throughout this chapter, the next chapter, and appendix C we work in $d = 4$ dimensions with a single component ($N = 1$) scalar field theory. Extension of our calculations to general dimension d and to $O(N)$ symmetric theories is straightforward but combinatorially involved.

5.2 Diagrammatic Difficulties

Even lowest order calculations require the evaluation of an infinite series of diagrams. Diagrammatic perturbation theory is adapted to potentials consisting of a finite number of monomial interactions. Nonpolynomial theories involve an infinite number of different vertices. An example of the consequent proliferation of diagrams is illustrated in figure 5-1. A single-vertex graph is shown, along with some of the other graphs that contribute at the same order. These graphs are obtained from the original by attaching self-contractions to the vertex. Worse types of sums appear at higher order. As will be discussed, the self-contracted sum of figure 5-1 can be dealt with by employing “resummed” vertices, rather than those obtained directly from the potential. However, the nastier higher order sums are not so easily circumvented.

To properly deal with nonpolynomial potentials, a new form of perturbation theory must be developed, as the standard diagrammatic techniques prove inadequate.

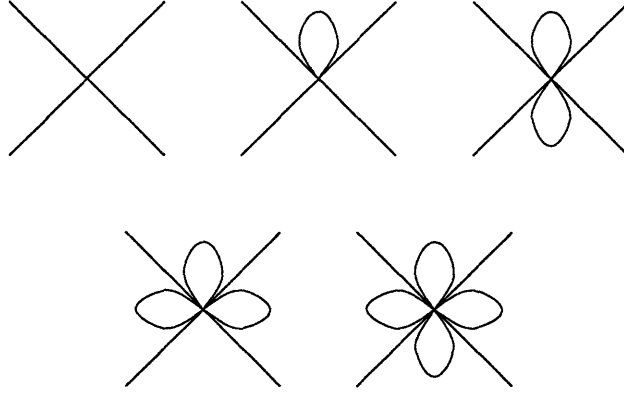


Figure 5-1: A single-vertex 4-point diagram and its self-contracted brethren.

However, at lowest order we can successfully compute scattering amplitudes using ordinary perturbation theory. Although involved, the calculation is tractable.

5.3 Scattering Amplitudes

To compute lowest scattering amplitudes we must evaluate the n -point analogue of the series of diagrams depicted in figure 5-1. We work in Euclidean momentum space and with dimensionless objects (the cutoff $\Lambda = 1$). The Euclidean space Feynman rules are provided in figure 5-2. As mentioned, we consider the case of one field component ($N = 1$) and $d = 4$ space-time dimensions.

5.3.1 Unbroken Potential

The scattering amplitudes for an unbroken theory are¹

$$A_{2n} = \sum_{j=0}^{\infty} u_{2j+2n} I^j \frac{(2j+2n)!}{2^j j!}, \quad (5.1)$$

where I is the self-contracted propagator. I is independent of the external momenta and can be computed (in terms of $r = 2u_2$):

$$I \equiv \int_0^1 \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + r} = \frac{S_4}{2} \left(1 + r \ln \frac{r}{1+r} \right). \quad (5.2)$$

We can substitute the eigenpotential parameters u_{2n} from formula 4.53 into equation 5.1 and evaluate the resultant series. This involves some tricky combinatorics and is performed in appendix C. The result is

$$A_{2n} = u_{2n} (2n)! \left(-r \ln \frac{r}{1+r} \right)^{1-a-n}. \quad (5.3)$$

¹The sum is over the number of self-contractions.

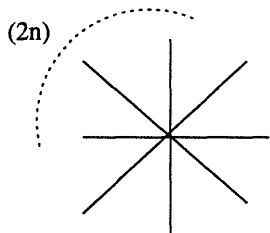
$\text{---} = \frac{1}{k^2 + 2u_2}$
<div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: center; margin-right: 10px;"> $(2n)$  </div> <div style="margin: 0 20px;">=</div> <div style="text-align: center;"> $u_{2n} \quad (n=2\dots)$ </div> </div>

Figure 5-2: The Euclidean space Feynman rules.

5.3.2 Broken Potential

The calculation of scattering amplitudes for symmetry broken potentials proceeds analogously to that for the unbroken potentials, but with three important differences: (i) there are odd vertices, so there are scattering amplitudes involving odd numbers of particles, (ii) the calculation is more difficult because the coefficients v_n are themselves computable as infinite series (Kummer functions), and (iii) the propagator is modified because the mass term in the broken potential is not r . From equation 4.72, we have

$$v_2 = \frac{r}{2} M\left(a, \frac{1}{2}, \frac{\rho^2}{S_4}\right). \quad (5.4)$$

For convenience we define \bar{r} , the broken analogue of r , as

$$\bar{r} = 2v_2 = r M\left(a, \frac{1}{2}, \frac{\rho^2}{S_4}\right). \quad (5.5)$$

The broken propagator is

$$I' \equiv \int_0^1 \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \bar{r}} = \frac{S_4}{2} \left(1 + \bar{r} \ln \frac{\bar{r}}{1 + \bar{r}}\right), \quad (5.6)$$

and the scattering amplitudes are

$$A_n = \sum_{j=0}^{\infty} v_{n+2j} I'^j \frac{(n+2j)!}{2^j j!}. \quad (5.7)$$

We must treat the even and odd amplitudes separately. Again, the calculation is combinatorially messy and we defer it to appendix C. There, the amplitudes are

found to be

$$A_{2n} = r \frac{S_4^{1-n} 2^{2n-2} (a+n-2)!}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{1-a-n} M\left(a+n-1, \frac{1}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1+\bar{r}}}\right)$$

$$A_{2n+1} = r \rho \frac{S_4^{-n} 2^{2n} (a+n-1)!}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{-a-n} M\left(a+n, \frac{3}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1+\bar{r}}}\right). \quad (5.8a)$$

5.4 Field Scaling

There appears to be a significant problem with the scattering amplitudes 5.3 and 5.8. The closer our approach to the Gaussian fixed point, the larger the amplitudes grow. As we let $t \rightarrow -\infty$, $r(t) \rightarrow 0$. We are examining relevant eigendirections, so the scattering amplitudes diverge in the high energy limit. Though this may seem to indicate a serious flaw in our programme, it is actually not a problem. A simple field renormalization removes the divergences. The field that appears in the action is not the physical field; to obtain the physical field, we must scale it. The renormalized field, potential, and parameters are

$$\phi_r \equiv Z \phi \quad (5.9a)$$

$$U_r(\phi_r) \equiv U(\phi) \quad (5.9b)$$

$$U_r(\phi_r) = \sum_{n=1}^{\infty} u_{2n}^r \phi_r^{2n} \quad (5.9c)$$

$$u_{2n}^r = Z^{-2n} u_{2n}. \quad (5.9d)$$

There is also a kinetic counterterm. The Greens functions for the two theories are related by²

$$G^{(n)}(x_1 \cdots x_n) = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle \quad (5.10a)$$

$$G_r^{(n)}(x_1 \cdots x_n) = \langle 0 | T \phi_r(x_1) \cdots \phi_r(x_n) | 0 \rangle \quad (5.10b)$$

$$G_r^{(n)}(x_1 \cdots x_n) = Z^n G^{(n)}(x_1 \cdots x_n). \quad (5.10c)$$

The propagator also scales, because it is the free two-field VEV. Denoting the propagator by $P(k)$ and the free field by ϕ^f ,

$$P(k) = \int d^d x e^{ik \cdot x} \langle 0 | T \phi^f(x) \phi^f(0) | 0 \rangle \quad (5.11a)$$

$$P_r(k) = Z^2 P(k). \quad (5.11b)$$

The scattering amplitudes are amputated Greens functions— ordinary Greens

²The actions are the same for the two theories. We have simply changed what we call “the interacting part.” However, the Greens functions differ because of the field scaling.

functions divided by external propagators. Therefore they scale differently from the Greens functions:

$$A_{2n}(k_1 \cdots k_{2n}) = \frac{1}{P(k_1)} \cdots \frac{1}{P(k_{2n})} G^{(2n)}(k_1 \cdots k_{2n}) \quad (5.12a)$$

$$A_{2n}^r(k_1 \cdots k_{2n}) = Z^{-2n} A_{2n}(k_1 \cdots k_{2n}). \quad (5.12b)$$

As we scale the cutoff to infinity, $r \rightarrow 0$ and Z diverges. Both u_{2n} and u_{2n}^r approach the Gaussian fixed point. However, the couplings are no longer of direct physical significance in a nonpolynomial theory. Rather, it is the scattering amplitudes that concern us. As $r \rightarrow 0$ the renormalized amplitudes vanish, demonstrating the expected asymptotic freedom.

5.4.1 Unbroken Potential

From equation 5.3 it is evident that the scale factor

$$Z = \frac{1}{\sqrt{-r \ln \frac{r}{1+r}}} \quad (5.13)$$

removes the divergences in an unbroken theory. The renormalized scattering amplitudes are then

$$A_{2n}^r = u_{2n}(2n)! \left(-r \ln \frac{r}{1+r} \right)^{1-a}, \quad (5.14)$$

which converge as $r \rightarrow 0$ for $a \leq 2$.

5.4.2 Broken Potential

The scale factor obtained from equation 5.8 is

$$Z = \frac{1}{\sqrt{-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}}}}, \quad (5.15)$$

where we recall that

$$\bar{r} = r M \left(a, \frac{1}{2}, \frac{\rho^2}{S_4} \right). \quad (5.16)$$

The renormalized scattering amplitudes from equation 5.8 are

$$A_{2n} = r \frac{S_4^{1-n} 2^{2n-2} (a+n-2)!}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{1-a} M \left(a+n-1, \frac{1}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1+\bar{r}}} \right) \quad (5.17a)$$

$$A_{2n+1} = r \rho \frac{S_4^{-n} 2^{2n} (a+n-1)!}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{-a} M \left(a+n, \frac{3}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1+\bar{r}}} \right). \quad (5.17b)$$

We may express these more concisely using the broken coefficients v_n . The v_n are functions of the minimum ρ . Let us write this dependence explicitly as $v_n(\rho)$. We recognize the Kummer function of equation 5.17 as that appearing in the expression for $v_n(\rho)$, but with ρ replaced by $\rho_r \equiv Z\rho$. Of course, ρ_r is not a minimum of U ; but, it is perfectly valid to evaluate $v_n(\rho_r)$.

Recognizing the coefficient in equation 5.17 to be $v_n(\rho_r)$,

$$A_n^r = v_n(\rho_r)(n!) \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{1-a}. \quad (5.18)$$

There remains one divergence. As $r \rightarrow 0$, $\rho_r \rightarrow \infty$. The Kummer function behaves like an exponential for large values of its argument, so the divergence has the form of an exponential. This divergence has an interesting physical explanation. A broken theory has two wells. In an ordinary symmetry broken theory (eg. ϕ^4), scattering amplitudes are calculated in one of the two wells. They possess no knowledge of the other well. Unfortunately, each scattering amplitude in our theory has contributions from v_n with arbitrarily large n . The v_n is the n^{th} derivative of the classical potential U at the minimum. Because our scattering amplitude involves derivatives of all orders, it is not “local” in field space. By contrast, the scattering amplitudes of symmetry broken ϕ^4 theory involve only a finite number of derivatives of the classical potential and are “local” in field space. Our scattering amplitudes see tunneling contributions from the other side of the hump. This is a direct consequence of the exponential nature of the walls of the well. To summarize, we cannot construct a symmetry broken theory in which the physics arising from the two vacua are isolated.

The location of a minimum remains fixed as $r \rightarrow 0$; it is independent of r . However, the depth of the minimum is proportional to r . Consequently, as $r \rightarrow 0$, the depth-width ratio vanishes. Tunneling effects become progressively more significant. This leads to strange behavior on the part of the scattering amplitudes. The inability of perturbation theory to account for the tunneling effects manifests itself as an exponential divergence in the broken theory’s scattering amplitudes.

5.4.3 Defining a Resummed Vertex

The series of graphs in figure 5-1, summed to compute the scattering amplitudes, gives us a clue as to how we might simplify higher order calculations should we attempt them. Associated with any diagram is a series of kindred diagrams that differ solely in the number of self-contractions on the vertices. If we replace our ordinary vertex with a “resummed” vertex, defined as the sum of all self-contracted vertices with the same number of external lines, the diagrammatic expansions simplify drastically. Unfortunately, higher order calculations still involve infinite sums, and convergence issues are difficult— if not impossible— to deal with.

An intuitive notion of these difficulties presents itself. In ordinary perturbation theory we think of the lowest order scattering amplitude as an individual vertex. For a single-vertex theory this makes sense, but for a nonpolynomial theory it is no longer true. It is the resummed vertex, rather than the plain vertex, that is the lowest order scattering amplitude. Self-contracted vertices of all orders affect lowest order

scattering, and we can no longer interpret an individual vertex in the usual manner.

In computing an n -point amplitude, we obtained a factor of $n!$ from counting the permutations of external lines. This is unnecessary if we are computing a resummed vertex. Therefore, we divide the n -point amplitude by $n!$ to obtain the associated resummed vertex. We denote the resummed vertices u' and v' , and incorporate the aforementioned field scaling.

In the unbroken case

$$u'_{2n} = u_{2n} \left(-r \ln \frac{r}{1+r} \right)^{1-a}, \quad (5.19)$$

and in the broken case

$$v'_n = v_n(\rho_r) \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{1-a}. \quad (5.20)$$

Although we incorporated the original two-point interaction into the propagator, an independent resummed two-point vertex arises. If we are calculating to higher order, we may also incorporate this in the propagator.

5.5 Scattering Cross Sections

The computation of scattering cross sections is straightforward. We are dealing with a single particle theory, so decays are kinematically disallowed³. In the unbroken theory, the simplest scattering is $2 \rightarrow 2$. In a symmetry broken theory, odd vertices are present, but the simplest kinematically allowed scattering is still $2 \rightarrow 2$. We therefore compute this cross section.

5.5.1 Kinematics

The cross section at energy E in the center of mass frame is (see, for example, Itzykson and Zuber[11])

$$\begin{aligned} \sigma_{2 \rightarrow 2}(E) = \frac{1}{2} \left(\frac{1}{2E\sqrt{E^2 - 4m^2}} \right) (A_4^r)^2 \int_0^1 \frac{d^3 p_1}{(2\pi)^3 2\omega_{p_1}} \frac{d^3 p_2}{(2\pi)^3 2\omega_{p_2}} \\ \cdot (2\pi)^4 \delta^3(\vec{p}_1 + \vec{p}_2) \delta(\omega_{p_1} + \omega_{p_2} - E). \end{aligned} \quad (5.21)$$

At lowest order, the scattering amplitudes are momentum independent, and the phase-space integral is purely kinematic:

$$\sigma_{2 \rightarrow 2}(E) = K_E (A_4^r)^2, \quad (5.22)$$

where K_E is the kinematic factor. In appendix C we find K_E to be

³In the center of mass frame, a particle cannot decay into multiple particles of the same mass.

$$K_E = \frac{1}{32\pi E^2}. \quad (5.23)$$

Noting that, from equation 4.53,

$$u_4 = \frac{ra}{6S_4} = \frac{4\pi^2 ra}{3}, \quad (5.24)$$

and substituting equations 5.23 and 5.14 into the cross section 5.22, we obtain

$$\sigma_{2 \rightarrow 2}(E) = \frac{32a^2 r^2 \pi^3}{E^2} \left(-r \ln \frac{r}{1+r} \right)^{2-2a}. \quad (5.25)$$

The scattering cross section in a symmetry broken theory is obtained analogously by employing the broken amplitude 5.18 instead of the unbroken one.

$$\sigma_{2 \rightarrow 2}(E) = \frac{32a^2 r^2 \pi^3}{E^2} M\left(a+1, \frac{1}{2}, \frac{\rho_r^2}{S_4}\right)^2 \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{2-2a}. \quad (5.26)$$

5.5.2 Scaling behavior

Consider an eigenpotential (broken or unbroken) with associated eigenvalue λ . We can choose a point at a distance r_0 along this direction and associate with it a physical energy scale E_0 . The choice of E_0 is arbitrary; we cannot constrain it within our theory. As discussed in section 2.5, the mass scale changes under the RG operation and the energy E scales as $E = E_0 e^t$. We can use this to obtain the scaling of r with energy⁴ E ,

$$r(E) = r_0 \left(\frac{E}{E_0} \right)^\lambda. \quad (5.27)$$

The cross section has units of inverse momentum squared and scales as E_0^2/E^2 . For r small (E large), we approximate

$$r \ln \frac{r}{1+r} \approx r \ln r. \quad (5.28)$$

The scaling behavior of the cross section as a function of the energy E for large energies is:

$$\sigma_{2 \rightarrow 2}(E) \approx [8(\lambda-2)^2 \pi^3 r_0^{6-\lambda} E_0^{4-\lambda} \lambda^{4-\lambda}] E^{\lambda^2-6\lambda-2} (\ln E)^{4-\lambda} \quad (5.29)$$

for the unbroken case, and

$$\sigma_{2 \rightarrow 2}(E) \approx [8(\lambda-2)^2 \pi^3 r_0^{6-\lambda} E_0^{4-\lambda} \lambda^{4-\lambda}] M\left(a+1, \frac{1}{2}, \frac{\rho^2}{S_4}\right)^2 M\left(a, \frac{1}{2}, \frac{\rho^2}{S_4}\right)^{4-\lambda} \cdot E^{\lambda^2-6\lambda-2} (\ln E)^{4-\lambda} \quad (5.30)$$

⁴ \bar{r} scales the same way.

for the broken case. The scaling behavior in both cases is given by (with c some constant)

$$\sigma_{2 \rightarrow 2}(E) \sim cE^{\lambda^2 - 6\lambda - 2}(\ln E)^{4 - \lambda}. \quad (5.31)$$

The cross section diverges at large energies unless $\lambda^2 - 6\lambda - 2 < 0$. We therefore require that $6.32 > \lambda > -.317$. Outside this range our calculation is invalid.

5.6 Summary

We have computed, at lowest order in perturbation theory, the scattering amplitudes for both symmetry broken and unbroken eigentheories. The nonpolynomial nature of the potentials makes higher order calculations intractable. Even at lowest order, an infinite series of diagrams had to be evaluated. The results required a field renormalization to be physically useful. The $2 \rightarrow 2$ scattering cross section was computed and its high energy scaling found to be (for both broken and unbroken theories, and with $6.32 > \lambda > -.317$)

$$\sigma_{2 \rightarrow 2}(E) \sim cE^{\lambda^2 - 6\lambda - 2}(\ln E)^{4 - \lambda}. \quad (5.32)$$

Chapter 6

Effective Potential

6.1 Overview

We noted in chapter 5 that perturbative calculations are intractable beyond lowest order. This severely limits our ability to compute quantities of physical interest. One of the few significant calculations we can perform is that of the 1-loop effective potential. This is rendered feasible by Jackiw's functional method[12]. In such functional approaches lies our greatest hope for understanding the physics of nonpolynomial theories.

Because no closed form expression exists for the zeros of a Kummer function, most of the results relating to the 1-loop effective potential must be obtained numerically. Nonetheless, certain physically interesting properties manifest themselves. Particularly, we can identify a phase boundary between symmetry broken and unbroken theories[5].

We begin by carrying analytic calculation of the effective potential as far as possible. We then examine the effect of radiative corrections upon the classically symmetry broken and unbroken eigenpotentials. We demonstrate that radiative corrections suppress classical symmetry breaking in a wide range of theories. However, we find no instances of radiatively induced symmetry breaking. After identifying a phase boundary between symmetry broken and unbroken theories (in $r - a$ parameter space), we discuss the possible implications for particle theory. As in the preceding chapter, we work in $d = 4$ dimensions and with $N = 1$ field component.

6.1.1 Jackiw's Method

With cutoff $\Lambda = 1$, Jackiw's functional formula[12] for the 1-loop effective potential V is (see, for example, [9])

$$V(\phi) = U(\phi) + \frac{1}{32\pi^2}U''(\phi) + \frac{U''(\phi)^2}{64\pi^2} \left(-\frac{1}{2} + \ln U''(\phi) \right). \quad (6.1)$$

An eigenpotential is proportional to the small parameter r . For convenience, we isolate this dependence by defining a function f that is independent of r ,

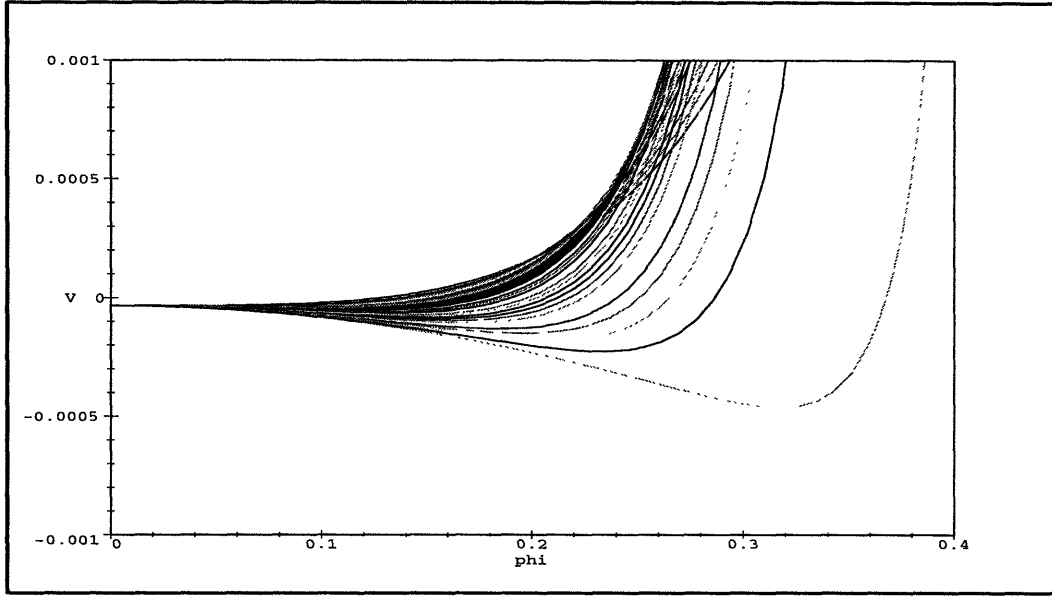


Figure 6-1: Some effective potentials with $-1 < a < 0$.

$$U(\phi) \equiv r f(\phi). \quad (6.2)$$

The 1-loop effective potential has three parts: $O(r)$, $O(r^2 \ln |r|)$, and $O(r^2)$. Again, we may isolate the r dependence by defining

$$V(\phi) \equiv r h_0(\phi) + (r^2 \ln |r|) h_1(\phi) + r^2 h_2(\phi), \quad (6.3)$$

with

$$h_0 \equiv f + \frac{1}{32\pi^2} f'' \quad (6.4a)$$

$$h_1 \equiv \frac{(f'')^2}{64\pi^2} \quad (6.4b)$$

$$h_2 \equiv \frac{(f'')^2}{64\pi^2} \left(-\frac{1}{2} + \ln |f''| \right). \quad (6.4c)$$

A sequence of effective-potentials, with a varying from -1 to 0 and r fixed, is plotted in figure 6-1.

6.1.2 Qualifiers

We note that the function $V(\phi)$ in equation 6.1 is only the effective potential for $|\phi| > |\phi_{min}|$, where ϕ_{min} is the location of the minimum of V . For $|\phi| < |\phi_{min}|$, the Legendre transform used to construct the effective potential must be defined

geometrically¹ and does not correspond to equation 6.1. The V of Jackiw's method, however, is fine for our purposes if we consider only the real part and are careful about the domain of applicability.

A more significant consideration is the range of validity of our calculation as a function of r . Our derivation of the eigenpotentials was valid to linear order in r . Any calculation of a physical quantity from such a potential should not be able to be carried beyond linear order. Beyond this order, nonlinear RG corrections might affect the results. Our 1-loop calculation is carried to $O(r^2)$. We do not expect the $O(r^2)$ part to be accurate, because the RG will generate nonlocal corrections at that order. However, there is no mechanism for the RG to generate logarithmic corrections. Therefore, we have reason to believe that the $O(r^2 \ln |r|)$ part of the 1-loop effective potential is valid. Although we include the $O(r^2)$ part in our discussion, it does not affect the results. The radiative corrections are primarily effected through the $O(1)$ and logarithmic terms.

6.2 Location and depth of minima

Radiative corrections affect both the location and depth (and sometimes existence) of the classical minimum. We can expand the location of the minimum of V in powers of r ,

$$\phi_{min} \approx \phi_0 + (r \ln |r|)\phi_1 + r\phi_2. \quad (6.5)$$

Note that ϕ_0 is the minimum of $h_0(\phi)$, not of $U(\phi)$. Because there is an $O(1)$ correction to the potential, we expand around the minimum of the $O(1)$ term rather than of the classical potential. The corrections are

$$\phi_1 = -\frac{h'_1}{h''_0} \quad (6.6a)$$

$$\phi_2 = -\frac{h'_2}{h''_0}. \quad (6.6b)$$

Noting that

$$h'_0 = f' + \frac{1}{32\pi^2} f''' \quad (6.7a)$$

$$h'_1 = \frac{f'' f'''}{32\pi^2} \quad (6.7b)$$

$$h'_2 = \frac{f'' f'''}{32\pi^2} \ln |f''|, \quad (6.7c)$$

this is (with f and its derivatives evaluated at ϕ_0)

¹See, for example, [3].

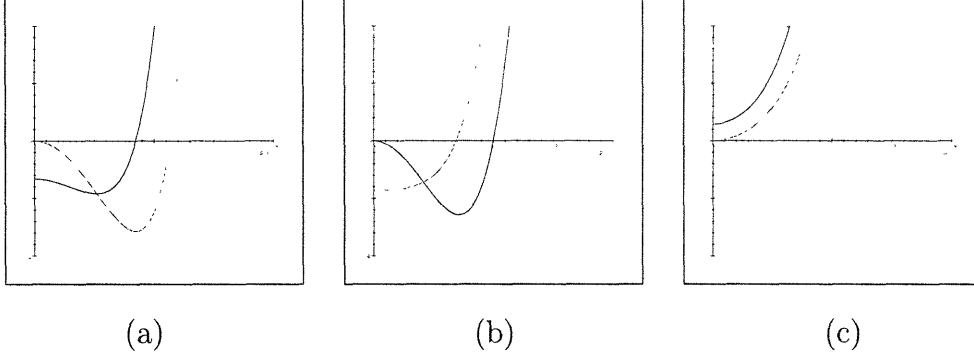


Figure 6-2: Comparison of classical potential U and effective potential V for (a) $r = -0.02$ and $a = -0.5$, (b) $r = -0.05$ and $a = -0.9$, and (c) $r = 0.05$ and $a = 0.5$.

$$\phi_1 = \frac{f' f''}{f'' - f'} \quad (6.8a)$$

$$\phi_2 = \frac{f' f''}{f'' - f'} \ln |f''|. \quad (6.8b)$$

The shift in location of the minimum affects the depth only at $O(r^3 \ln |r|^2)$, and we ignore it. The depth of interest is not $V(\phi_{min})$, because $V(0)$ is no longer zero. What we are really interested in is the difference

$$\Delta V \equiv V(\phi_{min}) - V(0) \approx V(\phi_0) - V(0). \quad (6.9)$$

From equation 6.3 we obtain

$$\Delta V = r[(h_0(\phi_0) - h_0(0)) + (r \ln |r|)(h_1(\phi_0) - h_1(0)) + r(h_2(\phi_0) - h_2(0))] + O(r^2 (\ln |r|)^2).$$

Using equation 6.4, this becomes (using f'_0 to denote $f'(0)$ and f' to denote $f'(\phi_0)$)

$$\Delta V = r \left[f + \frac{1}{32\pi^2} (f'' - f''_0) + (r \ln |r|) \frac{f''^2 - f''_0^2}{64\pi^2} + r \left(\frac{f''^2}{64\pi^2} \ln |f''| - \frac{f''_0^2}{64\pi^2} \ln |f''_0| \right) \right]. \quad (6.10)$$

This is as far as we can carry analytical calculation. To further study the properties of V , we must resort to numerical methods.

6.3 Phase Transition Boundary

Figure 6-2 compares the classical and effective potentials for three choices of (r, a) . In the first case, the effective potential maintains but dampens the broken nature of the classical potential. In the second, the classical potential exhibits symmetry breaking but the effective potential does not. In the third case, the classical potential

is unbroken, as is the effective potential. As is evident from the second case, radiative corrections can suppress classical symmetry breaking in our theories.

To determine whether symmetry breaking occurs, we examine ΔV . If a theory is classically broken, there remains a nontrivial minimum in the effective theory, but it may no longer be the global minimum. If $\Delta V > 0$, the nontrivial minimum is local, $\phi = 0$ is the global minimum, and no symmetry breaking is present. If $\Delta V < 0$, symmetry breaking is present.

Classical symmetry breaking occurs in the range of asymptotically free theories $-1 < a < 0$. Numerical analysis of this region for various small $r < 0$ reveals a boundary between broken and unbroken theories. A theory is parameterized by the coupling parameter r , and the choice of eigendirection a . In a classical theory, the r axis and a axis are boundaries between broken theories ($a < 0, r < 0$), unbroken but unphysical theories ($r < 0, a > 0$ and $r > 0, a < 0$), and unbroken theories ($a > 0, r > 0$). These boundaries remain; no radiatively induced symmetry breaking is observed in numerical calculations. However, a new boundary appears. For $-1 < a < a_c \approx -0.585$, there is an $r(a)$ such that symmetry breaking occurs if $0 > r > r(a)$ and does not if occur if $r < r(a)$. This boundary is plotted in figure 6-3. A comparison of the classical and 1-loop phase boundaries is provided in figure 6-4.

Our calculation is valid for small r , so we expect only the small band of theories close to the a -axis to be accurately described. Most of the phase boundary is beyond the scope of our theory. Our calculation has alerted us to its existence and provided clues as to its qualitative nature. In particular, one oddity is evident. If we choose an eigendirection with $-1 < a < a_c$, and send $r \rightarrow 0$ (raise the cutoff), we pass from an unbroken theory to a broken theory. Symmetry breaks as we go to higher energy. This is the opposite of the usual state of affairs. The significance of this is unclear, and we continue to explore the physical implications of this strange behavior.

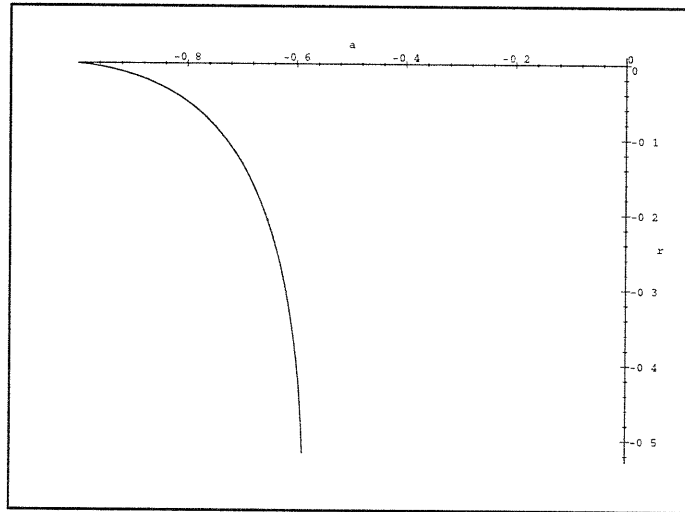
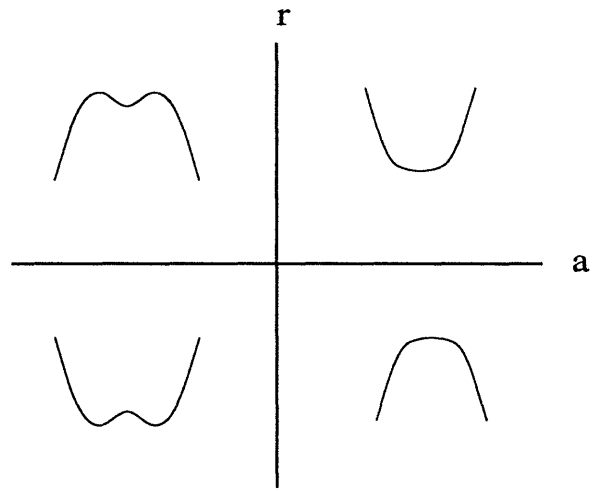
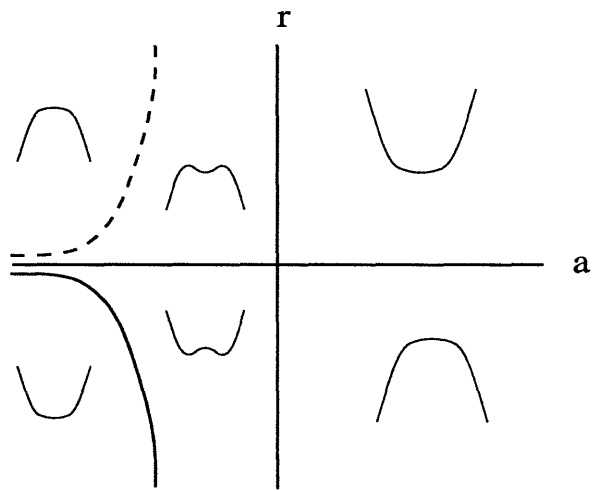


Figure 6-3: Symmetry broken and unbroken regions for $-1 < a < 0$ and $r < 0$. The region to the left of the curve is unbroken.



Classical



Effective

Figure 6-4: Symmetry broken and unbroken regions in classical and 1-loop effective theories.

Chapter 7

General Bose/Fermi Theories

All of our work thus far has been restricted to $O(N)$ symmetric scalar theories. While these are useful in modeling an approximate isolated Higgs sector, any realistic analysis of a physical theory must involve more general scalar interactions, fermion fields, and gauge fields. Gauge theories are beyond the scope of this thesis. A realistic analysis of them is wanting and promises to prove treacherous. As a first step toward understanding physical theories, we ignore gauge fields and only consider matter fields. We allow our theory to possess both Bose and Fermi fields, as well as arbitrary internal symmetries.

Realistically, there is no reason why gauge fields should not significantly influence the flow structure of parameter space. We have already seen that artificial restrictions to unclosed subspaces are not reliable. Why then should we consider the matter interactions in isolation? The reason is twofold. First, it is the only analysis we can perform. The sharp cutoff scheme, essential for calculation, explicitly violates gauge invariance. No analogous gauge invariant cutoff scheme is extant. Second, there is a small chance that gauge couplings are not as important as matter couplings. At any stage of our process, we can imbue our theory with gauge structure via the minimal coupling prescription. Covariant derivatives act like normal derivatives under the RG operation. No derivative interactions arise in a local, non-derivative $O(N)$ scalar theory, and a similar result holds for general local, non-derivative matter interactions. Therefore, it may be sensible to study a modified RG procedure that consists of the three RG steps preceded by deactivation of minimal coupling, and followed by its reinstatement. This is not meant to be a proof that we can ignore gauge fields; it is merely a plausibility argument.

In this chapter, we derive the linearized flow equations near the Gaussian fixed point. The eigenstructure depends on the internal symmetries of the particular theory under consideration. However, we are able to show that fermionic interactions act independently in theory space. The Bose field symmetry structure alone determines the significant aspects of the flow eigenbehavior near the Gaussian fixed point.

7.1 Overview

To study a general bose/fermi theory, we employ the same Wegner-Houghton flow equations as were used for $O(N)$ symmetric theories. Similar problems with constraint propagation obstruct any search for fixed points. We can avoid replicating the details of our previous calculation by noting that (1) the Wegner-Houghton theorems, rooted in the sharp cutoff, hold for arbitrary bose/fermi theories, and (2) volume factors arise and cancel one another in the same manner as they did for the $O(N)$ theory. Therefore, we work directly with infinite-volume fields.

We allow for two types of fields: real scalar and complex grassman. All spinor and internal indices are collected in a single index. It is implicit that the action and couplings obey whatever symmetries, internal or space-time, have been imposed. We allow the couplings and kinetic terms to be very general. As before, the flow equations are only unambiguous in linear approximation near the Gaussian fixed point. We will employ this approximation at the earliest opportunity.

7.2 General Wegner-Houghton Equation

Denoting the scalar fields ϕ and the grassman fields ψ and ψ^* , the most general action we consider is¹

$$S = \sum_{n,m=1}^{\infty} S_{2n,m}, \quad (7.1a)$$

with

$$\begin{aligned} S_{2n,m} = & \int_{\Omega} d^d k_1 \cdots d^d k_{2n} d^d p_1 \cdots d^d p_m d^d q_1 \cdots d^d q_m \delta(\sum k_i + \sum p_i - \sum q_i) \\ & \cdot \phi_{i_1}(k_1) \cdots \phi_{i_{2n}}(k_{2n}) \psi_{\beta_1}^*(q_1) \psi_{\alpha_1}(p_1) \cdots \psi_{\beta_m}^*(q_m) \psi_{\alpha_m}(p_m) \\ & \cdot u_{\substack{i_1 \cdots i_{2n} \\ \alpha_1 \cdots \alpha_m \\ \beta_1 \cdots \beta_m}}(k_1 \cdots k_{2n}, p_1 \cdots p_m, q_1 \cdots q_m). \end{aligned} \quad (7.1b)$$

Space-time homogeneity is implicit. The action is real and a Lorentz scalar. Accordingly, we only incorporate terms with equal numbers of ψ and ψ^* factors. The q 's appear in the momentum-conserving delta function with a minus sign because the associated fields are complex-conjugated.

As before, we split the fields into slow and fast components

$$\phi(k) = \phi_s(k) + f(k) \quad (7.2a)$$

$$\phi_s(k) = \phi(k) \theta(\Lambda' - |k|) \quad (7.2b)$$

$$f(k) = \phi(k) \theta(|k| - \Lambda') \quad (7.2c)$$

¹We will sometimes use $u_{(m)}^{(2n)}$ as an abbreviated notation for the full coupling function $u_{\substack{i_1 \cdots i_{2n} \\ \alpha_1 \cdots \alpha_m \\ \beta_1 \cdots \beta_m}}(k_1 \cdots k_{2n}, p_1 \cdots p_m, q_1 \cdots q_m)$.

$$\psi(k) = \psi_s(k) + g(k) \quad (7.3a)$$

$$\psi_s(k) = \psi(k)\theta(\Lambda' - |k|) \quad (7.3b)$$

$$g(k) = \psi(k)\theta(|k| - \Lambda'), \quad (7.3c)$$

and integrate over the fast components

$$S'[\phi_s, \psi_s, \psi_s^*] = -\ln \int [Df][Dg^*][Dg] e^{-S[\phi_s+f, \psi_s+g, \psi_s^*+g^*]}. \quad (7.4)$$

The Wegner-Houghton theorem is based on domains of integration and is independent of the symmetry structure of the theory. Therefore, the contributing terms in a Taylor expansion of the action are

$$\begin{aligned} S[\phi_s + f, \psi_s + g, \psi_s^* + g^*] \approx & S[\phi_s, \psi_s, \psi_s^*] + \int_{\partial\Omega} d^d k \left[f_i(k) \frac{\delta S}{\delta f_i(k)} \Big|_{f=g=0} + g_\alpha(k) \frac{\delta S}{\delta g_\alpha(k)} \Big|_{f=g=0} \right. \\ & + g_\alpha^*(k) \frac{\delta S}{\delta g_\alpha^*(k)} \Big|_{f=g=0} + \frac{1}{2} f_i(k) f_j(-k) \frac{\delta^2 S}{\delta f_i(k) \delta f_j(-k)} \Big|_{f=g=0} \\ & + \frac{1}{2} g_\alpha(k) g_\beta(-k) \frac{\delta^2 S}{\delta g_\beta(-k) \delta g_\alpha(k)} \Big|_{f=g=0} + \frac{1}{2} g_\alpha^*(k) g_\beta^*(-k) \frac{\delta^2 S}{\delta g_\beta^*(-k) \delta g_\alpha^*(k)} \Big|_{f=g=0} \\ & + g_\alpha(k) g_\beta^*(k) \frac{\delta^2 S}{\delta g_\beta^*(k) \delta g_\alpha(k)} \Big|_{f=g=0} + f_i(k) g_\alpha(-k) \frac{\delta^2 S}{\delta g_\alpha(-k) \delta f_i(k)} \Big|_{f=g=0} \\ & \left. + f_i(k) g_\alpha^*(k) \frac{\delta^2 S}{\delta g_\alpha^*(k) \delta f_i(k)} \Big|_{f=g=0} \right]. \quad (7.5) \end{aligned}$$

Let us define

$$A_i(k) \equiv \frac{\delta S}{\delta f_i(k)} \Big|_{f=g=0} \quad (7.6a)$$

$$B_\alpha(k) \equiv \frac{\delta S}{\delta g_\alpha(k)} \Big|_{f=g=0} \quad (7.6b)$$

$$C_{ij}(k) \equiv \frac{\delta^2 S}{\delta f_i(k) \delta f_j(-k)} \Big|_{f=g=0} \quad (7.6c)$$

$$D_{\beta\alpha}(k) \equiv \frac{\delta^2 S}{\delta g_\beta(-k) \delta g_\alpha(k)} \Big|_{f=g=0} \quad (7.6d)$$

$$E_{\beta\alpha}(k) \equiv \frac{\delta^2 S}{\delta g_\beta^*(k) \delta g_\alpha(k)} \Big|_{f=g=0} \quad (7.6e)$$

$$F_{\alpha i}(k) \equiv \left. \frac{\delta^2 S}{\delta g_\alpha(-k) \delta f_i(k)} \right|_{f=g=0}. \quad (7.6f)$$

We note that

- $A(-k) = A^*(k)$.
- $C(-k) = C^*(k)$.
- $C(k) = C^\dagger(k)$.
- $D(-k) = -D^\top(k)$.
- $E(k) = E^\dagger(k)$.
- A , C , D , and E are even in the number of grassman variables and commute (in the grassman sense) with everything.
- B and F are odd in the number of grassman variables and anticommute (in the grassman sense) with each other and with other grassman variables.

Applying the grassman relations of appendix D, equation 7.5 becomes

$$\begin{aligned} S[\phi_s + f, \psi_s + g, \psi_s^* + g^*] &\approx S[\phi_s, \psi_s, \psi_s^*] \\ &+ \int_{\partial\Omega} d^d k \left[f(k) \cdot A(k) + g(k) \cdot B(k) + B^*(k) \cdot g^*(k) \right. \\ &+ \frac{1}{2} f^\top(k) C(k) f^*(k) - \frac{1}{2} g^\top(-k) D(k) g(k) + \frac{1}{2} g^{*\top}(-k) D^*(k) g^*(k) \\ &\left. - g^{*\top}(k) E(k) g(k) + g^\top(-k) F(k) f(k) - g^{*\top}(k) F^*(-k) f(k) \right]. \quad (7.7) \end{aligned}$$

The fields at k and $-k$ interact, so we can only separate the k components if we group k and $-k$ together and integrate over a half-shell $\partial\Omega_+$. We define a composite field

$$\mathbf{g} \equiv \begin{bmatrix} g(k) \\ g(-k) \end{bmatrix}, \quad (7.8a)$$

and the composite matrices and vectors

$$M \equiv \begin{bmatrix} E(k) & 0 \\ 0 & E(-k) \end{bmatrix} \quad (7.8b)$$

$$N \equiv \frac{1}{2} \begin{bmatrix} 0 & D(-k) \\ D(k) & 0 \end{bmatrix} \quad (7.8c)$$

$$\eta \equiv \begin{bmatrix} B^*(k) + F^*(-k) f(k) \\ B^*(-k) + F^*(k) f(-k) \end{bmatrix}, \quad (7.8d)$$

which obey

$$M = M^\dagger \quad (7.9a)$$

$$N^\top = -N. \quad (7.9b)$$

Equation 7.7 is then

$$\begin{aligned} S[\phi_s + f, \psi_s + g, \psi_s^* + g^*] &\approx S[\phi_s, \psi_s, \psi_s^*] + \int_{\partial\Omega_+} d^d k \left[f(k) \cdot A(k) + f(-k) \cdot A(-k) \right. \\ &\quad \left. + \frac{1}{2} f^\top(k) C(k) f^*(k) + \frac{1}{2} f^\top(-k) C(-k) f^*(-k) \right. \\ &\quad \left. - \mathbf{g}^{*\top} M \mathbf{g} - \mathbf{g}^\top N \mathbf{g} + \mathbf{g}^{*\top} N^* \mathbf{g}^* - \eta^* \mathbf{g} - \mathbf{g}^* \eta \right]. \end{aligned} \quad (7.10)$$

The gaussian integral decouples in different components of k . We may consider this as a product of integrals, one for each k . Using the formulae of appendix D, we can evaluate the Gaussian integrals over $\mathbf{g}^* \mathbf{g}$. Equation 7.4 evaluates to

$$\begin{aligned} S'[\phi_s, \psi_s, \psi_s^*] &= S[\phi_s, \psi_s, \psi_s^*] - \ln \left[\int [Df][Df^*][Dg][Dg^*] \right. \\ &\quad \left. \left(e^{-\int_{\partial\Omega_+} d^d k [f \cdot A + f^* \cdot A^* + f^\top C f^*]} e^{\frac{1}{4} \int_{\partial\Omega_+} d^d k R^\top G^{-1} R} \sqrt{\det G} \right) \right], \end{aligned} \quad (7.11)$$

where

$$\prod_k \det G = e^{\int d^d k \operatorname{tr} \ln G}, \quad (7.12)$$

and (M_r denotes the real part of M , N^a is the antisymmetric part of N , etc.)

$$\begin{aligned} G &\equiv i \begin{bmatrix} M_I + 2N_I^a & M_r + 2N_r^a \\ -M_r + 2N_r^a & M_I - 2N_I^a \end{bmatrix} \\ R &= \begin{bmatrix} -2i\eta_I \\ 2i\eta_r \end{bmatrix}, \end{aligned}$$

and it is understood that the functional integral is over components of f and g that lie in the half-shell $\partial\Omega_+$.

To perform the Df integration, we must extract the f -dependent part of R . The $(\det G)$ does not depend on f so we may pull it outside the integral. Performing the f integral at this point is messy and leads to unenlightening results. We are interested in the flow eigenstructure, so it is appropriate to implement the linear approximation.

The most general local, non-derivative theory has parameters

$$u_{\alpha_1 \dots \alpha_m}^{i_1 \dots i_{2n}}(k_1 \dots k_{2n}, p_1 \dots p_m, q_1 \dots q_m) = u_{\beta_1 \dots \beta_m}^{i_1 \dots i_{2n}} \quad (7.13a)$$

for $n > 1$ and $m > 1$, and quadratic terms

$$u^{ij} \equiv a_{ij} k^2 + b_{ij} \quad (7.13b)$$

$$u_{\alpha\beta} \equiv \zeta_{\alpha\beta}^\mu k_\mu + \sigma_{\alpha\beta}. \quad (7.13c)$$

As mentioned, all the spinor indices and internal field indices are combined in α and β . In the cases of physical interest, $a_{ij} = \frac{1}{2}\delta_{ij}$, and ζ^μ is the direct product of γ^μ with some matrix in the internal indices. However, we have no need to specialize the equations at this point. It is notationally cleaner to maintain our present generality.

Using $O(r)$ to denote linear order in all couplings, the kinetic coefficients ζ and a are $O(1)$, and all other terms are $O(r)$. We implement the same approximate translation and Lorentz symmetries as in earlier chapters. In order for the action to be a scalar we require that the grassman variables occur in conjugate pairs. Only even powers of the scalar fields are included. As always, we must retroactively justify our restrictions by proving closure of the subspace that they define.

One consequence of our restrictions is that the parameters $A \cdots F$ can be classified by order in the couplings. The only possible $O(1)$ contribution to any of these derivatives is from the kinetic terms. The kinetic terms do not contribute to A or B because differentiation leaves one fast field factor, which vanishes when we set the fast fields to zero. The kinetic terms are purely f^*f and g^*g . Therefore they cannot contribute to D or F either. The categorization is as follows:

- $A, B, D,$ and F are $O(r)$.
- C and E are $O(1)$.

The $RG^{-1}R$ contribution from the grassmanian integration is $O(r^2)$ and may be discarded. This simplifies calculation dramatically because this term would have contributed to the exponent in the scalar integration. The remaining integrals are independent of the grassman result. Using the formulae of appendix D, we obtain

$$\int [Df] e^{-\int d^d k [f \cdot A + f^* \cdot A^* + f^{*\top} C f]} = \frac{\pi^n}{\det C} e^{\frac{1}{4} A^{*\top} C^{-1} A}. \quad (7.14)$$

The $A^{*\top} C^{-1} A$ term is $O(r^2)$, and we discard it. Plugging our result into equation 7.11 and ignoring constants, we have (to $O(r)$)

$$\begin{aligned} S'[\phi_s, \psi_s, \psi_s^*] &= S[\phi_s, \psi_s, \psi_s^*] - \prod_k \frac{\sqrt{\det G}}{\det C} \\ &= S[\phi_s, \psi_s, \psi_s^*] - \int_{\partial\Omega_+} d^d k \left[\frac{1}{2} \text{tr} \ln G - \text{tr} \ln C \right]. \end{aligned} \quad (7.15)$$

We separate G and C into kinetic parts and non-kinetic parts, and further subdivide the non-kinetic part by parent term in the action:

$$G = G_{kin} + G' \quad (7.16a)$$

$$G' = \sum_{n,m=1}^{\infty} G_{n,m} \quad (7.16b)$$

$$C = C_{kin} + C' \quad (7.16c)$$

$$C' = \sum_{n,m=1}^{\infty} C_{n,m}. \quad (7.16d)$$

Where appropriate, we designate with $'$ the non-kinetic part of other matrices as well. We expand the logarithms as²

$$\begin{aligned} \text{tr} \ln C &= \text{tr} \ln C_{kin} + \text{tr} \ln(I + C_{kin}^{-1} C') \\ &\approx \text{tr} \ln C_{kin} + \text{tr}(C_{kin}^{-1} C') + O(r^2) \end{aligned} \quad (7.17a)$$

$$\begin{aligned} \text{tr} \ln G &= \text{tr} \ln G_{kin} + \text{tr} \ln(I + G_{kin}^{-1} G') \\ &\approx \text{tr} \ln G_{kin} + \text{tr}(G_{kin}^{-1} G') + O(r^2). \end{aligned} \quad (7.17b)$$

The $\text{tr} \ln C_{kin}$ and $\text{tr} \ln G_{kin}$ terms contain no slow-field dependence. They contribute constants to the action, and we ignore them. The modified action from the integration step of the RG procedure is

$$S'[\phi_s, \psi_s, \psi_s^*] = S[\phi_s, \psi_s, \psi_s^*] - \int_{\partial\Omega_+} d^d k \left[\frac{1}{2} \text{tr}(G_{kin}^{-1} G') - \text{tr}(C_{kin}^{-1} C') \right]. \quad (7.18)$$

We are not finished. There is a crucial simplification that follows. The trace is over internal indices. The only k dependence is in C_{kin} and G_{kin} because there are no derivative interactions and the other parts of C and G arise from constant, local-interaction terms. We may explicitly perform the integration. The shell is of infinitesimal width t and radius 1, so

$$\int_{\partial\Omega_+} d^d k = t \int d\hat{k}, \quad (7.19)$$

where $\int d\hat{k}$ denotes an angular integral. While it is natural to use the half shell $\partial\Omega_+$ for the f 's ($f(-k) = f^*(k)$), it is not natural to do so for the g 's. This is partly what led to our messy expression for G . We can derive G_{kin} from equation 7.13. Unraveling the definition³,

$$D_{kin} = 0 \quad (7.20a)$$

$$E_{kin} = \zeta^\mu k_\mu \quad (7.20b)$$

$$N_{kin} = 0 \quad (7.20c)$$

$$M_{kin} = \begin{bmatrix} \zeta^\mu k_\mu & 0 \\ 0 & -\zeta^\mu k_\mu \end{bmatrix}$$

²We needn't worry about ordering since the G 's commute (in the grassman sense).

³Note that ζ is real.

$$= \zeta^\mu k_\mu \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (7.20d)$$

$$\begin{aligned} M_{kin}^{-1} &= (\zeta^\mu k_\mu)^{-1} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \\ &\equiv \frac{1}{k^2} \theta^\mu k_\mu \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \end{aligned} \quad (7.20e)$$

$$\begin{aligned} G_{kin} &= \begin{bmatrix} 0 & M_{kin} \\ -M_{kin} & 0 \end{bmatrix} \\ &= M_{kin} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \end{aligned} \quad (7.20f)$$

$$G_{kin}^{-1} = M_{kin}^{-1} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad (7.20g)$$

$$\begin{aligned} \text{tr}(G_{kin}^{-1} G') &= 2 \text{tr}(M_{kin}^{-1} M') \\ &= \frac{1}{k^2} k_\mu \text{tr} \begin{bmatrix} \theta^\mu E'(k) & 0 \\ 0 & -\theta^\mu E'(-k) \end{bmatrix}. \end{aligned} \quad (7.20h)$$

Because all non-kinetic terms are non-derivative, $E'(k)$ is independent of k , and $E'(k) = E'(-k)$. Therefore the trace vanishes.

The point of this digression is that the $O(r)$ contribution from the $\text{tr} \ln G$ term vanishes because the fermion kinetic term is odd in the momentum. We integrate over the shell, and contributions from opposite points cancel. This was masked in our calculation since it was necessary to group the k and $-k$ together for the Gaussian integrals to decouple. Evaluating⁴ $C_{kin} = 2a_{ij}$, the final result of the integration step is:

$$S'[\phi_s, \psi_s, \psi_s^*] = S[\phi_s, \psi_s, \psi_s^*] + \frac{1}{2} t S_d \text{tr}(a^{-1} C'). \quad (7.21)$$

We now assume that the kinetic matrix a is (as appropriate to scalar fields)

$$a_{ij} = \frac{1}{2} \delta_{ij}. \quad (7.22)$$

After differentiating equation 7.1 to obtain C , the integration contribution to the u 's is

$$\Delta_{int} u_{\substack{\alpha_1 \dots \alpha_m \\ \beta_1 \dots \beta_m}}^{i_1 \dots i_{2n}} = 2S_d \sum_{q < r = 1 \dots 2n+2} u_{\substack{\alpha_1 \dots \alpha_m \\ \beta_1 \dots \beta_m}}^{i_1 \dots i_{q-1}, l, \dots, i_{r-1}, l, \dots, i_{2n+2}}. \quad (7.23)$$

⁴There is a volume factor that we omit here. A careful analysis of the discretized case shows that, as in earlier calculations, the volume factors eventually cancel.

The contribution to $u_{(m)}^{(2n)}$ is from $u_{(m)}^{(2n+2)}$ and involves a sum over all possible contractions of a pair of indices.

Dimensional scaling and field scaling are straightforward. The calculation proceeds analogously to our earlier development. However, there is one important difference. The canonical dimension associated with a Fermi field is $\frac{(d+1)}{2}$, whereas that associated with a scalar field is $\frac{(d+2)}{2}$. The fermion kinetic term has one k factor, so the fields must have different dimensions to make the action dimensionless. $u_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_m}^{i_1 \dots i_{2n}}$ is associated with $2n$ factors of the scalar field and $2m$ factors of the fermi fields, so after appropriate analysis, we find

$$\Delta^{norm+dim} u_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_m}^{i_1 \dots i_{2n}} = t[-nd - md + d + 2n + m] u_{\alpha_1 \dots \alpha_m \beta_1 \dots \beta_m}^{i_1 \dots i_{2n}} \quad (7.24)$$

7.3 Fermionic Decoupling

A very important simplification follows from equation 7.23. Because the $\text{tr} \ln G$ term vanishes at $O(r)$, the only contributions to $u_{(m)}^{(2n)}$ are from $u_{(m)}^{(2n+2)}$ and itself. In a theory with potential (with indices implicit)

$$U(\phi, \psi, \psi^*) = \sum_m U_m(\phi) \psi^* \psi \dots \psi^* \psi, \quad (7.25)$$

the different m terms decouple. The bosonic interactions govern the eigenstructure of the theory. The fermi interactions decouple. Each U_m acts like an independent scalar field theory. The presence of fermion fields is unimportant. At linear order, the scalar fields, and the scalar fields alone, determine the flow eigenstructure.

7.4 Role of Symmetries and Invariants

As is evident from the renormalization group equations, the flow eigenstructure near the Gaussian fixed point depends heavily on the internal symmetry structure of the theory. To classify the independent parameters in action space, we must determine the set of invariants of the internal symmetry group. Given a theory with a set of fields that transform according to a certain set of representations of a group, we may construct the invariant field polynomials by using a Clebsch-Gordon expansion. Consider a product of n fields. An invariant polynomial of degree n will transform as a singlet representation of the group. We take a direct product of the representations by which the fields transform, and decompose it into a direct sum of irreducible representations. A few examples of such Clebsch-Gordon decompositions are provided in appendix D. We must also include the $SO(3,1)$ spinor structure of fermion fields in the couplings.

7.4.1 Pure scalar $O(N)$ symmetric theory

To make contact with our previous results, we consider the case of a pure scalar field theory with an $O(N)$ symmetry. The couplings (the invariant polynomials of $O(N)$) are then

$$u^{i_1 \cdots i_{2n}} = \frac{1}{2^n n!} \sum_{p(i_1 \cdots i_{2n})} \delta_{i_{p_1}, i_{p_2}} \cdots \delta_{i_{p_{2n-1}}, i_{p_{2n}}}, \quad (7.26)$$

where $p(i_1 \cdots i_{2n})$ is a permutation of the indices. The only fundamental invariant polynomial is $\phi_i \phi_i$, so we could just as well choose

$$u^{i_1 \cdots i_{2n}} = \frac{(2n)!}{2^n n!} \delta_{i_1, i_2} \cdots \delta_{i_{2n-1}, i_{2n}}. \quad (7.27)$$

7.5 Remarks

Although further calculation is beyond the scope of this thesis, some comments about possible continuations are in order. The exact dependence of the eigenstructure on the particular symmetries and field representations of a theory has yet to be determined. There are a number of simple models to which the results of such a study may be applied. Finally, it would be very interesting to see whether a fruitful analysis of the Standard Model could be carried out along these lines.

Chapter 8

Conclusions

8.1 Concise summary of results

- We have shown that, though not closed in general, the space of local, non-derivative theories is closed to linear order near the Gaussian fixed point.
- We have discovered a set of asymptotically free eigendirections, corresponding to nonpolynomial potentials, near the Gaussian fixed point in $O(N)$ symmetric scalar field theory. Some of these theories exhibit symmetry breaking.
- Our eigenpotentials are Sine-Gordon Potentials in $d = 2$, $N = 1$.
- In $d = 4$, $N = 1$, we computed $2 \rightarrow 2$ scattering cross sections and found that they scale as $E^{\lambda^2 - 6\lambda - 2} (\ln E)^{4 - \lambda}$ at high energies for an eigentheory with eigenvalue λ .
- Radiative corrections at one loop are found to suppress symmetry breaking in some cases, giving rise to a phase boundary between broken and unbroken theories in $r - a$ parameter space. No cases of radiatively induced symmetry breaking are observed.
- In theories possessing Bose and Fermi fields with arbitrary internal symmetries, the eigenstructure near the Gaussian fixed point is found to depend only on the Bose symmetries. The Fermi interactions decouple from one another.

8.2 Comprehensive summary of results

We have used the Wegner-Houghton infinitesimal renormalization group equations to study the flow eigenstructure near the Gaussian fixed point. The following are our results:

8.2.1 Local, Non-derivative Field Theories

- The space of local, non-derivative field theories (Bose or Fermi) is not closed under the RG operation in general. Nonlocal interactions arise, but no derivative interactions arise (sections 4.2.8 and 4.3.3).
- The space of local, non-derivative field theories (Bose or Fermi) is closed at linear order near the Gaussian fixed point. Any eigenvectors that lie within this subspace are exact eigenvectors of the complete theory (section 4.3.3).
- The flow eigenstructure near the Gaussian fixed point is governed entirely by the Bose symmetry group composition. Different Fermi interactions do not influence one another under the RG flow at linear order (section 7.3).

8.2.2 $O(N)$ Symmetric Scalar Theory

All results are from section 4.4.

- The eigenpotentials of $O(N)$ symmetric local, non-derivative scalar field theory near the Gaussian fixed point are

$$U^a(\phi(x)) = r \frac{NS_d}{2(a-1)(d-2)} \left[M(a-1, N/2, \frac{(d-2)\phi^2}{2S_d}) - 1 \right], \quad (8.1)$$

where (see appendix D) $M(a, b, z)$ is Kummer's function, a type of confluent hypergeometric function defined by

$$M(a, b, z) = \frac{(b-1)!}{(a-1)!} \sum_{n=0}^{\infty} \frac{z^n (a+n-1)!}{n! (b+n-1)!}, \quad (8.2)$$

and

- ▷ N is the number of field components.
 - ▷ d is the number of space-time dimensions.
 - ▷ $r = 2u_2$ is the distance along the eigendirection.
 - ▷ $S_d = \frac{2^{1-d}\pi^{-\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.
 - ▷ $a = \frac{\lambda-2}{d-2}$, where $-\infty < \lambda < \infty$ is the associated eigenvalue.
- The associated coefficients are

$$U(\phi(x)) = \sum_{n=1}^{\infty} u_{2n} (\phi_i(x) \phi_i(x))^n \quad (8.3)$$

$$u_{2n}^a = \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{n-1} \left[\frac{(a+n-2)! (\frac{N}{2})!}{(a-1)! (\frac{N}{2} + n - 1)! n!} \right]. \quad (8.4)$$

- The eigenpotentials are classified as follows for general dimension d (n is any odd positive integer and m is any positive integer):

a	λ	Behavior
0	2	Free theory
$\frac{-2}{d-2}$	0	Marginal
$> \frac{-2}{d-2}$	> 0	Asymptotically free
$[-n, -n+1)$	$[2(n+1) - nd, 2n - d(n-1))$	Symmetry Broken
$1 - m$	$2 + (d-2)(1-m)$	Polynomial

and for $d = 4$,

a	λ	Behavior
0	2	Free theory
-1	0	Marginal
> -1	> 0	Asymptotically free
$[-n, -n+1)$	$[2 - 2n, 4 - 2n)$	Symmetry Broken
$1 - m$	$4 - 2m$	Polynomial

- For the case $d = 2$, $N = 1$, the eigenpotentials are Sine-Gordon potentials.
- The symmetry-broken potentials for $a < 0$ and $N = 1$ are

$$v_{2n} = u_{2n} M\left(a + n - 1, \frac{1}{2}, \frac{\rho^2(d-2)}{2S_d}\right) \quad (8.5a)$$

$$v_{2n+1} = \rho u_{2n} \left(\frac{2(d-2)}{S_d}\right) \left(\frac{a+n-1}{2n+1}\right) M\left(a+n, \frac{3}{2}, \frac{\rho^2(d-2)}{2S_d}\right). \quad (8.5b)$$

8.2.3 Scalar Theories with $N = 1$, $d = 4$

- The physical scattering amplitudes for the unbroken theory, after the requisite field renormalization, are (section 5.4)

$$A_{2n}^r = u_{2n} (2n)! \left(-r \ln \frac{r}{1+r}\right)^{1-a}. \quad (8.6)$$

- The analogous amplitudes for the symmetry-broken theories are (section 5.4)

$$A_n^r = v_n(\rho_r) (n)! \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}}\right)^{1-a}. \quad (8.7)$$

- At high energies, the cross sections scale like (for both broken and unbroken theories, but with $6.32 > \lambda > -.317$) (section 5.5.2)

$$\sigma_{2\rightarrow 2}(E) \sim cE^{\lambda^2-6\lambda-2}(\ln E)^{4-\lambda}. \quad (8.8)$$

- Radiative corrections to the one-loop effective potential have not been found to induce symmetry breaking, but have been found to suppress it. A phase boundary $r(a)$ in $r - a$ parameter space has been shown to exist. For $r < r(a)$, symmetry breaking is suppressed. For $0 > a > -0.585$, symmetry breaking is never suppressed.

8.3 Discussion

The main result of this thesis is the existence of a set of relevant approaches to the Gaussian fixed point of scalar field theory. Previous studies had restricted themselves to ϕ^4 theory, arguing that higher interactions were non-renormalizable and therefore unphysical. Little research focused on nonpolynomial interactions— more for practical reasons than due to concerns regarding renormalizability.

As mentioned, the space of ϕ^4 theories is not closed in general. To linear order near the Gaussian fixed point it is closed, and it is this regime that research has typically focused on. In any such restricted analysis, the only eigendirections are the mass-axis and the marginal direction. Scalar theories appear to be trivial since perturbative calculation yields a bare coupling that diverges at finite cutoff unless the renormalized coupling is chosen to be zero. This triviality limits the utility of scalar theories in describing physics.

Our research has demonstrated the existence of relevant eigendirections to the Gaussian fixed point. These eigendirections correspond to nonpolynomial potentials. However, the renormalizability condition 3.3 is satisfied¹. Because the eigendirections are relevant, they represent high energy limits. Triviality is no longer a problem. There exist renormalizable scalar field theories with well-defined high-energy limits.

There is no obvious feature that allows us to identify a particular eigendirection as special. Each eigentheory describes certain physics. This has implications for the Higgs sector of the Standard Model, which may be described approximately by an isolated scalar field theory. The physics of the Higgs sector must be recomputed using the new eigentheories. No longer is pure ϕ^4 the only candidate for the physical Higgs theory.

8.4 Future Directions

Our original intention in pursuing the present course of research was to sufficiently restrict the class of continuum theories to be able to determine some of the parameters *ab initio*. Unfortunately, practical aspects of calculation made such a determination infeasible in the class of theories we studied. However, it is possible that a similar

¹This relation is obtained directly from the RG mapping between Greens functions at different effective cutoffs.

analysis of the Standard Model will reveal the existence of a discernable continuum limit. As mentioned, it is the possibility of such a case that endows the renormalization group method with the potential to determine the fundamental masses and couplings.

Even if one entirely restricts oneself to the cases we have studied, there is interesting physics. We have demonstrated the existence of non-trivial scalar theories. The implications for the Higgs sector of the Standard Model must be explored. The physical significance of the decoupling of Fermi interactions near the Gaussian fixed point is also of interest. We have only begun to explore a field that is vast and rich, a field that has the potential to provide a deeper understanding of field theory than has previously been possible. We have demonstrated that, even in the simplest theories, the renormalization group procedure provides unexpected insights. When first disembarking, an explorer cannot chart the mysterious realms before him. At best, he can suggest those directions that seem most promising. Here are our suggestions for future directions:

- Extend the analysis to gauge theories. Attempt to study the Standard Model in its entirety.
- Calculate the physics of a Higgs sector governed by one of the nonpolynomial potentials. This entails constructing a new form of diagrammatic perturbation theory.
- Study the case of two dimensions, examining the role of vortices.
- Examine the nonlocal delta-interactions as correlated particles. In particular, look for an analogy with Cooper pairs in superconductivity.
- Try to examine the RG equations beyond linear order.
- Study the role of the sharp cutoff and determine whether there are ways to avoid the nonlocal ambiguities that arise.
- Understand the dependence of the eigenstructure near the Gaussian fixed point on the group structure of the underlying theory.
- Rigorously examine the issues relating the RG to renormalization theory that are discussed in chapter three.

Appendix A

Wegner-Houghton Theorem

Our ability to derive infinitesimal renormalization group equations rests heavily on a result proved by Wegner and Houghton. They demonstrated that to $O(t)$ we need only consider a part of the action that is quadratic in the fast fields. This is a direct consequence of the use of a sharp cutoff. Although Wegner and Houghton originally proved their claims using combinatoric and cumulant arguments[22], there is a simple heuristic justification for their theorem using Feynman diagrams. We first explain the basis for a Feynman expansion in the context of the RG analysis, and then proceed to expatiate on the Wegner-Houghton results.

A.1 Diagrammatic Language

The integration step in the RG procedure modifies the action:

$$S'[\phi_s] = -\ln \int [Df] e^{-S[\phi_s+f]}. \quad (\text{A.1})$$

Divide S into a quadratic part S_0 and an “interacting” part S_1 :

$$S[\phi_s + f] = S_0[\phi_s + f] + S_1[\phi_s + f]. \quad (\text{A.2})$$

The quadratic part S_0 has the form (requiring translation invariance)

$$S_0[\phi] = \int d^d k (\phi_k^* \phi_k) u_2(k). \quad (\text{A.3})$$

$\phi_s(k)$ and $f(k)$ are defined in disjoint domains of momentum space, and the quadratic part of the action decouples in ϕ_s and f ,

$$S_0[\phi_s + f] = S_0[\phi_s] + S_0[f]. \quad (\text{A.4})$$

We can interpret the integral in equation A.1 as the expectation value of a function over a probability distribution. The appropriate distribution is

$$p[f] \equiv \frac{e^{-S_0[f]}}{\int [Df] e^{-S_0[f]}}, \quad (\text{A.5})$$

with the expectation value of a function $A[f]$ defined by

$$\langle A[f] \rangle \equiv \int [Df] p[f] A[f], \quad (\text{A.6})$$

and normalization

$$Z_0 \equiv \int [Df] e^{-S_0[f]}. \quad (\text{A.7})$$

In this language, equation A.1 can be written

$$S'[\phi_s] = -\ln \left[Z_0 \langle e^{-S_0[\phi_s] - S_1[\phi_s + f]} \rangle \right], \quad (\text{A.8})$$

the important (ϕ_s -dependent) part of which is

$$S'[\phi_s] = S_0[\phi_s] - \ln \langle e^{-S_1[\phi_s + f]} \rangle. \quad (\text{A.9})$$

The logarithmic term is the same as that present in the usual Wick expansion in perturbation theory. There, we also consider a quadratic distribution in the fields. A corresponding diagrammatic expansion is applicable here. The couplings from S_1 , Taylor expanded in f , are the vertices. The ϕ_s are external lines and the f are internal (contracted) lines. Note that single- f vertices do arise. The modified action S' is the sum over connected diagrams. S'_{2n} is the sum over connected diagrams with $2n$ external lines. This is entirely analogous to calculation of the $2n$ -point amputated connected Greens function of ordinary perturbation theory, except that the propagators are constrained to lie in the fast momentum shell $\partial\Omega$.

The diagrammatic scheme we have described holds for finite as well as infinitesimal RG transforms. If we restrict ourselves to the latter, the propagators must lie on an infinitesimal shell in momentum space. This is the basis for the Wegner-Houghton claims. In diagrammatic language, the claims are:

1. Diagrams containing vertices that possess more than two internal lines do not contribute to the infinitesimal flow equations. It follows that no diagrams with more than one loop contribute.
2. The only diagrams with one loop that do contribute to the infinitesimal flow equations are those with uniform propagator momentum throughout the loop. This is equivalent to requiring that the external momenta at each vertex sum to zero.

Our strategy for justifying these claims is to show that those diagrams which are excluded contribute to equation A.9 at $O(t^2)$ or higher.

A.2 Verification of the Claims

Consider a diagram with n vertices and L loops. Let p_i be the total external momentum at the i^{th} vertex, and let k_i be the i^{th} propagator momentum. An example is provided in figure A-1.

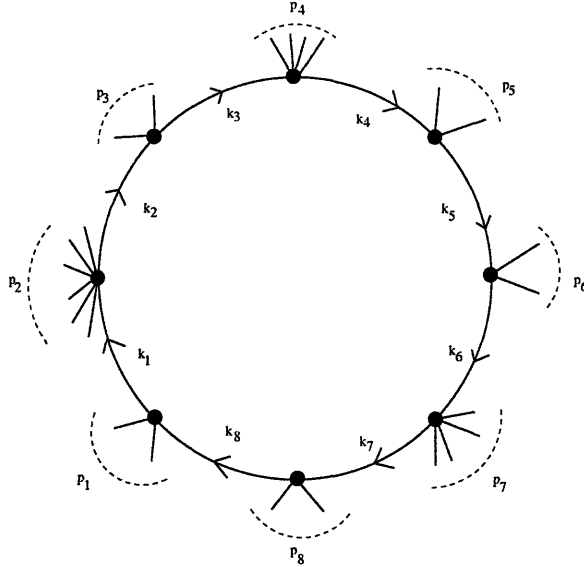


Figure A-1: An example of a one-loop diagram.

Let us denote by $P(k)$ the propagator in our theory. The contribution from an n -point diagram is proportional to

$$\int_{\partial\Omega} d^d k_1 \cdots d^d k_n P(k_1) \cdots P(k_n) \delta(k_1 + p_1 - k_2) \cdots \delta(k_n + p_n - k_1). \quad (\text{A.10})$$

This expression involves n propagator integrals and should be $O(t^n)$. However, a δ -function can cause an integral over a domain of $O(t)$ to evaluate as $O(1)$. The integral over a volume, however small, of a delta function that lies within that volume, is 1. Therefore, to determine whether it is possible that a given diagram contributes at $O(t)$, we must count the number of integrals minus the number of delta functions. Any diagram with more integrals than delta functions cannot contribute at $O(t)$. One of the delta functions must correspond to overall momentum conservation, so at least two integrals are not canceled and such a diagram is at best $O(t^2)$.

Associated with every vertex, there is a momentum conserving delta function and at least one propagator (internal line). No diagram can have fewer propagator integrals than delta functions. The only diagrams possessing the same number of integrals as delta functions are those with exactly one internal line per vertex. Each vertex must have either two internal lines and lie inside a chain, or have one internal line and terminate a chain. Therefore, only vertices with one or two internal lines contribute. An immediate corollary is that no diagrams with more than one loop can contribute at $O(t)$.

The one loop diagrams that have not been excluded have no tree extensions. Such diagrams have the same number of delta functions as propagator integrals. For an n -point diagram to contribute at $O(t)$, we require that $n - 1$ of the delta functions

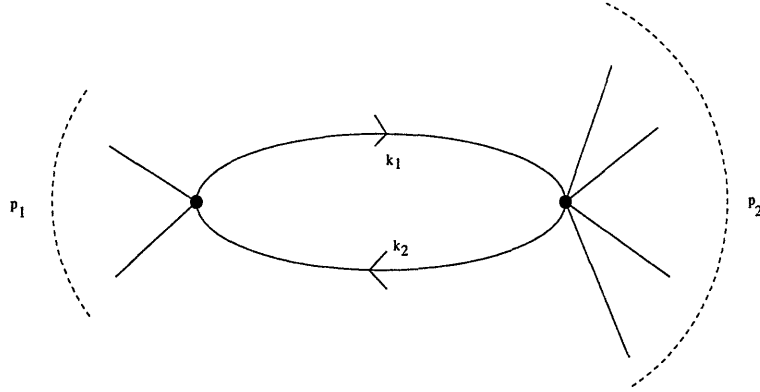


Figure A-2: A simple two-point one-loop diagram.

lie within the domains of the integrals¹. The remaining delta function corresponds to overall momentum conservation.

For simplicity, consider a two-vertex diagram, as depicted in figure A-2. The associated integral is

$$\int_{\partial\Omega} d^d k_1 d^d k_2 P(k_1) P(k_2) \delta(k_1 - k_2 + p_2) \delta(k_2 - k_1 + p_1) \quad (\text{A.11})$$

The integral only contributes if $p_2 = -p_1$ and p_2 lies in the domain of $k_2 - k_1$. Let us fix k_2 . If $p_2 - k_2 \in \partial\Omega$, the integral over k_1 yields

$$\delta(p_1 + p_2) \int_{\partial\Omega} d^d k_2 P(k_2 - p_2) P(k_2) \quad (\text{A.12})$$

which is an integral over the shell and obviously $O(t)$. We now determine which values of k_2 satisfy $p_2 - k_2 \in \partial\Omega$. The range of values of k_2 is $O(t)$, so the double integral is at best $O(t^2)$. It would seem that one-loop diagrams do not contribute at $O(t)$. However, there is one exception. If $p_1 = p_2 = 0$, the integral A.11 is obviously $O(t)$. The situation is illustrated in figure A-3.

Our analysis can be extended to n -point graphs. A one-loop graph will contribute only if the total external momentum at each vertex is zero. This is equivalent to demanding that the propagator momentum throughout the loop be uniform.

A.3 Example of Nonlocal Term

In this section we derive the form of the sample nonlocal term 4.35 that arises under the RG operation. We assume that our initial theory is pure ϕ^4 . The integration modification to the action is given by equation 4.20:

$$S'[\phi_s] = S[\phi_s] + \sum_{k \in \partial\Omega_+} [\text{tr} \ln A_k - B_k A_k^{-1} B_k^*]. \quad (\text{A.13})$$

¹The explanations of this section are really a heuristic justification for our sloppiness in multiplying distributions.

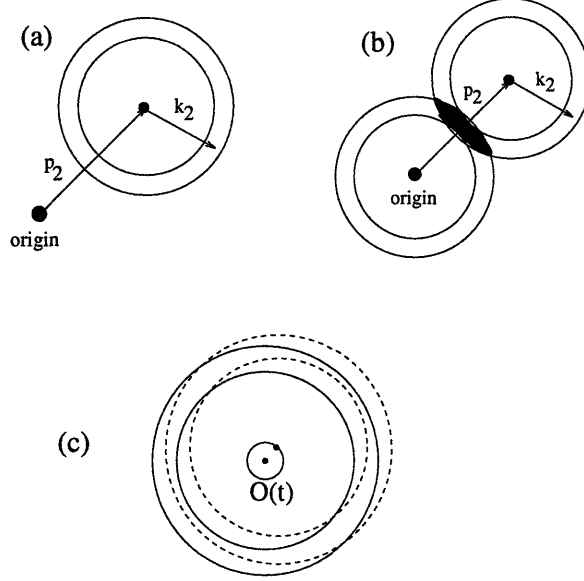


Figure A-3: a. The domain of $p_2 - k_2$. b. The overlap of domains of $p_2 - k_2$ and k_1 . c. The overlap of the domains when $p_2 \sim O(t)$.

We only wish to examine a particular nonlocal contribution from the \ln term. Let us denote the part of A obtained from the kinetic piece of the action by A_{kin} . The remainder of A we will denote A' and expand in a series $A' \equiv \sum_{n=1}^{\infty} A_{2n}$, where A_{2n} is the contribution to A from S_{2n} (except for A_2 which is from the non-kinetic part of S_2). The kinetic part of A is

$$(A_{kin})_{ij} = \frac{1}{V} \delta_{ij}. \quad (\text{A.14})$$

The modification to S from the \ln term is

$$\begin{aligned} \Delta S[\phi_s] &= \sum_{k \in \partial\Omega_+} \text{tr} \ln(A_{kin} + A') \\ &= \sum_{k \in \partial\Omega_+} \text{tr}[\ln(A_{kin}) + \ln(I + A_{kin}^{-1} A')] \\ &= \sum_{k \in \partial\Omega_+} \text{tr}[-\ln V + \ln(I + V A')] \\ &= \sum_{k \in \partial\Omega_+} \text{tr}[-\ln V + V A' - \frac{1}{2} V^2 A'^2 + \dots]. \end{aligned} \quad (\text{A.15})$$

One of the contributions from the $\text{tr} A'^2$ term is

$$\Delta_{example} S \propto V^2 \sum_{k \in \partial\Omega_+} \text{tr}(A_4^2). \quad (\text{A.16})$$

The matrix A_4 is given by

$$(A_4)_{ij} = V^{-3} u_4 \sum_{k_1, k_2} \delta_{k_1+k_2, 0} [4\delta_{ij} \phi_{k_1, m} \phi_{k_2, m} + 8\phi_{k_1, i} \phi_{k_2, j}], \quad (\text{A.17})$$

from which we find

$$\begin{aligned} \text{tr}(A_4^2) &= V^{-6} u_4^2 \sum_{k_1 \dots k_4} \delta_{k_1+k_2, 0} \delta_{k_3+k_4, 0} \\ &\quad [\phi_{k_1, m} \phi_{k_2, m} \phi_{k_3, n} \phi_{k_4, n} (16N + 64) \\ &\quad + \phi_{k_1, m} \phi_{k_3, m} \phi_{k_2, n} \phi_{k_4, n} (64)] \end{aligned} \quad (\text{A.18})$$

We recall that

$$V^{-1} \sum_{k_1, k_2} \delta_{k_1+k_2, 0} \phi_{k_1, i} \phi_{k_2, i} = \int d^d x \phi_i(x) \phi_i(x) \quad (\text{A.19})$$

and that

$$\left[\sum_{k \in \partial\Omega_+} \right] \propto V t \quad (\text{A.20})$$

to obtain

$$\Delta_{example} S \propto \frac{t}{V} \left[\int d^d x \phi_i(x) \phi_i(x) \right]^2. \quad (\text{A.21})$$

This is the nonlocal contribution mentioned in the text.

Appendix B

Derivation of Broken Potentials

In this section we derive the symmetry broken potentials for the case $N = 1$, as described in section 4.4.8. The broken potential can be parameterized as

$$V(\phi') = \sum_{n=0}^{\infty} v_n \phi'^n. \quad (\text{B.1})$$

The coefficients v_n are given by

$$v_n = \frac{1}{n!} \left. \frac{d^n U(\phi' + \rho)}{d\rho^n} \right|_{\phi'=0}, \quad (\text{B.2})$$

where ρ is the location of the minimum of the eigenpotential U . The eigenpotential evaluated at the minimum is

$$U(\rho) = \sum_{n=1}^{\infty} u_{2n} \rho^{2n}, \quad (\text{B.3})$$

with the coefficients u_{2n} given by equation 4.53:

$$u_{2n} = \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{n-1} \frac{(a+n-2)! (\frac{1}{2})!}{(a-1)! (n-\frac{1}{2})! n!}. \quad (\text{B.4})$$

Substituting the expansion B.3 into equation B.2, we obtain

$$v_n = \frac{1}{n!} \sum_{m=\lfloor \frac{n+1}{2} \rfloor}^{\infty} u_{2m} \frac{(2m)!}{(2m-n)!} \rho^{2m-n}, \quad (\text{B.5})$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Substituting expression B.4 for the u_{2n} , we get

$$v_n = \sum_{m=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{m-1} \left[\frac{(a+m-2)! (\frac{1}{2})! (2m)!}{(a-1)! (m-\frac{1}{2})! m! (2m-n)! n!} \right] \rho^{2m-n}. \quad (\text{B.6})$$

Noting that

$$(2m)! = \frac{2^{2m}}{\sqrt{\pi}} (m - \frac{1}{2})! m! \quad (\text{B.7})$$

and

$$(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}, \quad (\text{B.8})$$

we find

$$\begin{aligned} v_n &= \sum_{m=\lceil \frac{n+1}{2} \rceil}^{\infty} \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{m-1} \left[\frac{(a+m-2)! 2^{2m-1}}{(a-1)!(2m-n)!n!} \right] \rho^{2m-n} \\ &= \sum_{m=\lceil \frac{n+1}{2} \rceil}^{\infty} \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{m-1} \left[\frac{(a+m-2)! 2^{n-1} \sqrt{\pi}}{(a-1)!(m-\frac{n}{2})!(m-\frac{n}{2}-\frac{1}{2})!n!} \right] \rho^{2m-n}. \end{aligned} \quad (\text{B.9})$$

To proceed, we must distinguish between the case of n even and of the case n odd.

B.1 Even n

We consider $n = 2j$ even, so equation B.9 becomes

$$v_{2j} = \sum_{m=j}^{\infty} \frac{r}{2} \left(\frac{d-2}{2S_d} \right)^{m-1} \left[\frac{(a+m-2)! 2^{2j-1} \sqrt{\pi}}{(a-1)!(m-j)!(m-j-\frac{1}{2})!(2j)!} \right] \rho^{2m-2j}. \quad (\text{B.10})$$

We apply formula B.7 to $(2j)!$ and shift the index from m to $p = m - j$, to obtain

$$\begin{aligned} v_{2j} &= \sum_{p=0}^{\infty} \frac{r}{4} \left(\frac{d-2}{2S_d} \right)^{p+j-1} \left[\frac{(a+p+j-2)! \pi}{(a-1)! p! (p-\frac{1}{2})! j! (j-\frac{1}{2})!} \right] \rho^{2p} \\ &= \frac{\pi r}{4} \left(\frac{d-2}{2S_d} \right)^{j-1} \left[\frac{1}{(a-1)! j! (j-\frac{1}{2})!} \right] \sum_{p=0}^{\infty} \rho^{2p} \left(\frac{d-2}{2S_d} \right)^p \frac{(a+p+j-2)!}{p! (p-\frac{1}{2})!}. \end{aligned} \quad (\text{B.11})$$

The sum is now in a form that we recognize as the familiar Kummer function. Recalling the definition

$$M(a, b, z) = \frac{(b-1)!}{(a-1)!} \sum_{n=0}^{\infty} \frac{z^n (a+n-1)!}{n! (b+n-1)!}, \quad (\text{B.12})$$

the coefficients v_{2j} can be expressed as

$$v_{2j} = \frac{\sqrt{\pi} r}{4} \left(\frac{d-2}{2S_d} \right)^{j-1} \left[\frac{(a+j-2)!}{(a-1)! j! (j-\frac{1}{2})!} \right] M\left(a+j-1, \frac{1}{2}, \frac{\rho^2 (d-2)}{2S_d}\right). \quad (\text{B.13})$$

The factors to the left of the Kummer function we recognize as the expression for u_{2j} from formula B.4. So

$$v_{2j} = u_{2j} M\left(a + j - 1, \frac{1}{2}, \frac{\rho^2(d-2)}{2S_d}\right). \quad (\text{B.14})$$

B.2 Odd n

We can derive v_{2j+1} easily from v_{2j} using equation B.2

$$v_{2j+1} = \frac{1}{2j+1} \frac{d}{d\rho} v_{2j} \quad (\text{B.15})$$

Applying the Kummer function differential relation from appendix D, we obtain

$$v_{2j+1} = \rho u_{2j} \left[\frac{2(a+j-1)(d-2)}{(2j+1)S_d} \right] M\left(a + j, \frac{3}{2}, \frac{\rho^2(d-2)}{S_d}\right). \quad (\text{B.16})$$

Appendix C

Scattering Calculations

C.1 Scattering Amplitudes

C.1.1 Useful Information

Before computing scattering amplitudes from our eigentheories, we reiterate some of our earlier results and provide some useful mathematical relations:

- The coefficients from the unbroken eigenpotentials in the case $d = 4$, $N = 1$ are

$$\begin{aligned} u_{2n} &= \frac{r}{4} S_4^{1-n} \frac{(a+n-2)! \sqrt{\pi}}{(a-1)! n! (n-\frac{1}{2})!} \\ &= r S_4^{1-n} 2^{2n-2} \frac{(a+n-2)!}{(a-1)! (2n)!}. \end{aligned} \tag{C.1}$$

- $x! \equiv \Gamma(x+1)$.
- $(2x)! = \frac{2^{2x}}{\sqrt{\pi}} (x-\frac{1}{2})! x!$.
- $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$.
- $(-\frac{1}{2})! = \sqrt{\pi}$.
- $\binom{x}{k} = (-1)^k \binom{k-x-1}{k}$.
- $\sum_{j=0}^{\infty} \alpha^j \binom{z}{j} = (1+\alpha)^z$ if $|\alpha| < 1$.
- Kummer's function:

$$M(a, b, z) \equiv \sum_{n=0}^{\infty} \frac{(a+n-1)! (b-1)! z^n}{(a-1)! (b+n-1)! n!}$$

C.1.2 Unbroken case

The scattering amplitudes for an unbroken eigentheory are

$$A_{2n} = \sum_{j=0}^{\infty} u_{2j+2n} I^j \frac{(2j+2n)!}{2^j j!}. \quad (\text{C.2})$$

We substitute the expression for u_{2n} provided in section C.1.1 to obtain

$$A_{2n} = \frac{r\sqrt{\pi}}{4} \sum_{j=0}^{\infty} S_4^{1-j-n} I^j \frac{(2j+2n)!(a+j+n-2)!}{(a-1)!(n+j)!(n+j-\frac{1}{2})!2^j j!}. \quad (\text{C.3})$$

Expanding $(2j+2n)!$, this simplifies to

$$\begin{aligned} A_{2n} &= \frac{r}{4} \sum_{j=0}^{\infty} S_4^{1-j-n} I^j \frac{(a+j+n-2)!}{(a-1)!2^j j!} 2^{2j+2n} \\ &= \frac{r S_4^{1-n} 2^{2n-2}}{(a-1)!} \sum_{j=0}^{\infty} \left(\frac{2I}{S_4}\right)^j \frac{(a+j+n-2)!}{j!}. \end{aligned} \quad (\text{C.4})$$

Noting that

$$\frac{(a+j+n-2)!}{j!} = (a+n-2)! \binom{a+j+n-2}{j}, \quad (\text{C.5})$$

we can apply the binomial relations of section C.1.1 to reduce the scattering amplitudes to the form

$$A_{2n} = \frac{r S_4^{1-n} 2^{2n-2} (a+n-2)!}{(a-1)!} \sum_{j=0}^{\infty} \left(\frac{-2I}{S_4}\right)^j \binom{1-a-n}{j} \quad (\text{C.6})$$

and perform the summation

$$A_{2n} = \frac{r S_4^{1-n} 2^{2n-2} (a+n-2)!}{(a-1)!} \left(1 - \frac{2I}{S_4}\right)^{1-a-n}. \quad (\text{C.7})$$

We recognize this to be

$$A_{2n} = u_{2n}(2n)! \left(1 - \frac{2I}{S_4}\right)^{1-a-n}. \quad (\text{C.8})$$

Recalling that

$$I = \frac{S_4}{2} \left(1 + r \ln \frac{r}{1+r}\right), \quad (\text{C.9})$$

we obtain

$$A_{2n} = u_{2n}(2n)! \left(-r \ln \frac{r}{1+r}\right)^{1-a-n}. \quad (\text{C.10})$$

Note that the binomial series has unit radius of convergence. In order for our

manipulations to be valid, we require

$$\left| \frac{2I}{S_4} \right| = \left| 1 + r \ln \frac{r}{1+r} \right| < 1. \quad (\text{C.11})$$

This in turn requires that

$$-2 < r \ln \frac{r}{1+r} < 0. \quad (\text{C.12})$$

Our eigenpotentials were derived by requiring r to be small. As $r \rightarrow 0$, the radius of convergence is approached. This is the reason that we have a divergent field scaling factor Z in section 5.4.

C.1.3 Broken case

In the case of a symmetry-broken potential, we must compute even and odd scattering amplitudes separately. In either case we have

$$A_m = \sum_{j=0}^{\infty} v_{m+2j} I^j \frac{(m+2j)!}{2^j j!}. \quad (\text{C.13})$$

For $m = 2n$ even,

$$A_{2n} = \sum_{j=0}^{\infty} v_{2n+2j} I^j \frac{(2n+2j)!}{2^j j!}. \quad (\text{C.14})$$

The v_{2n} , calculated in appendix B, are Kummer functions. For our purposes, their series form, equation B.11, is more useful. Adapted to $d = 4$ dimensions and with index $2n + 2j$, the formula is

$$v_{2n+2j} = \frac{\pi r}{4} S_4^{1-n-j} \frac{1}{(a-1)!(n+j)!(n+j-\frac{1}{2})!} \sum_{p=0}^{\infty} \rho^{2p} S_4^{-p} \frac{(a+p+n+j-2)!}{p!(p-\frac{1}{2})!}. \quad (\text{C.15})$$

Recalling that

$$(2n+2j)! = (n+j)!(n+j-\frac{1}{2})! \frac{2^{2n+j}}{\sqrt{\pi}}, \quad (\text{C.16})$$

the amplitude is

$$\begin{aligned} A_{2n} &= \frac{\sqrt{\pi} r}{4} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} 2^{2n+2j} S_4^{1-n-j} \frac{1}{(a-1)!} \rho^{2p} S_4^{-p} \frac{(a+p+n-j-2)!}{p!(p-\frac{1}{2})!} \frac{I^j}{2^j j!} \\ &= \sqrt{\pi} r 2^{2n-2} S_4^{1-n} \frac{1}{(a-1)!} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \left(\frac{2I'}{S_4} \right)^j \frac{1}{j!} \frac{(a+p+n-j-2)!}{p!(p-\frac{1}{2})!} \left(\frac{\rho^2}{S_4} \right)^p \end{aligned} \quad (\text{C.17})$$

We perform the sum over j first. Noting that

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\frac{2I'}{S_4}\right)^j \frac{(a+p+n-j-2)!}{j!} &= (a+p+n-2)! \sum_{j=0}^{\infty} \binom{a+p+n-j-2}{j} \left(\frac{2I'}{S_4}\right)^j \\
&= (a+p+n-2)! \sum_{j=0}^{\infty} \binom{1-a-p-n}{j} \left(\frac{-2I'}{S_4}\right)^j \\
&= (a+p+n-2)! \left(1 - \frac{2I'}{S_4}\right)^{1-a-p-n}, \tag{C.18}
\end{aligned}$$

the amplitude reduces to

$$\begin{aligned}
A_{2n} &= \sqrt{\pi r} 2^{2n-2} S_4^{1-n} \frac{1}{(a-1)!} \left(1 - \frac{2I'}{S_4}\right)^{1-a-n} \\
&\quad \cdot \sum_{p=0}^{\infty} \left(\frac{\rho^2}{S_4(1 - \frac{2I'}{S_4})}\right)^p \frac{(a+p+n-2)!}{p!(p - \frac{1}{2})!}. \tag{C.19}
\end{aligned}$$

Noting that the sum is a Kummer function and that

$$1 - \frac{2I'}{S_4} = -\bar{r} \ln \frac{\bar{r}}{1 + \bar{r}}, \tag{C.20}$$

we obtain

$$\begin{aligned}
A_{2n} &= \sqrt{\pi r} 2^{2n-2} S_4^{1-n} \frac{1}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1 + \bar{r}}\right)^{1-a-n} \frac{(a+n-2)!}{(-\frac{1}{2})!} \\
&\quad \cdot M\left(a+n-1, \frac{1}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1 + \bar{r}}}\right) \tag{C.21} \\
&= r 2^{2n-2} S_4^{1-n} \frac{(a+n-2)!}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1 + \bar{r}}\right)^{1-a-n} M\left(a+n-1, \frac{1}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1 + \bar{r}}}\right),
\end{aligned}$$

which is the result quoted in the text.

The odd amplitudes can be directly derived from these. The general formula for an odd amplitude is

$$A_{2n+1} = \sum_{j=0}^{\infty} v_{2n+2j+1} I'^j \frac{(2n+2j+1)!}{2^j j!}. \tag{C.22}$$

The odd vertices $v_{2n+2j+1}$ are easily obtained from the even ones by

$$v_{2n+2j+1} = \frac{1}{(2n+2j+1)} \frac{d}{d\rho} v_{2n+2j}. \tag{C.23}$$

Therefore, we may write

$$A_{2n+1} = \frac{d}{d\rho} \left[\sum_{j=0}^{\infty} v_{2n+2j} I'^j \frac{(2n+2j)!}{2^j j!} \right]. \quad (\text{C.24})$$

Note that the differentiation does not apply to I' . It is simply a mechanism for properly counting the factors of ρ in the v_{2n+2j} . The sum is just A_{2n} , so

$$A_{2n+1} = \frac{d}{d\rho} A_{2n}. \quad (\text{C.25})$$

Using the formula from appendix D for differentiation of a Kummer function, we obtain the result stated in the text:

$$A_{2n+1} = r\rho \frac{S_4^{-n} 2^{2n} (a+n-1)!}{(a-1)!} \left(-\bar{r} \ln \frac{\bar{r}}{1+\bar{r}} \right)^{-a-n} M\left(a+n, \frac{3}{2}, \frac{\rho^2}{-S_4 \bar{r} \ln \frac{\bar{r}}{1+\bar{r}}}\right). \quad (\text{C.26})$$

C.2 Cross Section Kinematic Factor

We need to calculate

$$K_E \equiv \frac{1}{2} \left(\frac{1}{2E\sqrt{E^2 - 4m^2}} \right) \int_0^1 \frac{d^3 p_1}{(2\pi)^3 2\omega_{p_1}} \frac{d^3 p_2}{(2\pi)^3 2\omega_{p_2}} (2\pi)^4 \delta^{(3)}(\vec{p}_1 + \vec{p}_2) \delta(\omega_{p_1} + \omega_{p_2} - E). \quad (\text{C.27})$$

Performing the integral over p_1 , and using $\omega_p = \sqrt{p^2 + m^2}$, we find

$$\begin{aligned} K_E &= \frac{1}{32\pi^2} \left(\frac{1}{2E\sqrt{E^2 - 4m^2}} \right) \int_0^1 \frac{d^3 p}{\omega_p^2} \delta(2\omega_p - E) \\ &= \frac{1}{32\pi^2} \left(\frac{1}{2E\sqrt{E^2 - 4m^2}} \right) 4\pi \int_0^1 dp \frac{p^2}{p^2 + m^2} \delta(2\sqrt{p^2 + m^2} - E). \end{aligned} \quad (\text{C.28})$$

We can use the relation

$$\int \delta(f(x)) g(x) dx = \sum_{x_i | f(x_i)=0} \frac{g(x_i)}{f'(x_i)} \quad (\text{C.29})$$

to obtain

$$\begin{aligned} K_E &= \frac{1}{32\pi^2} \left(\frac{1}{2E\sqrt{E^2 - 4m^2}} \right) 4\pi \frac{\bar{p}^2}{\bar{p}^2 + m^2} \frac{\sqrt{\bar{p}^2 + m^2}}{2\bar{p}} \\ &= \frac{1}{16\pi} \left(\frac{1}{2E\sqrt{E^2 - 4m^2}} \right) \frac{\bar{p}}{\sqrt{\bar{p}^2 + m^2}} \end{aligned} \quad (\text{C.30})$$

where

$$\bar{p} = \sqrt{E^2/4 - m^2} \tag{C.31}$$

is the solution to the delta function constraint. Substituting this, we obtain

$$\begin{aligned} K_E &= \frac{1}{32\pi} \frac{1}{E\sqrt{E^2 - 4m^2}} \frac{2}{E} \sqrt{\frac{E^2}{4} - m^2} \\ &= \frac{1}{32\pi E^2}. \end{aligned} \tag{C.32}$$

Appendix D

Compendium of Useful Mathematical Relations

We present here some mathematical relations relevant to the calculations performed in the thesis or of importance in reproducing or extending our work. The topics covered are (1) Grassman Algebra, (2) Gaussian integration, (3) Results involving matrices, (4) Combinatoric relations, (5) Confluent hypergeometric functions, (6) Irreducible representations and invariants of Lie groups, (7) Fourier transforms, and (8) Finite vs. infinite volume objects.

D.1 Grassman Algebra

We use θ and ϕ to denote Grassman variables. All other symbols represent commuting numbers. Many of the relations in this section are from Zinn-Justin[24].

D.1.1 Basic properties

- $\theta_i^2 = 0$.
- A Grassman polynomial in an algebra with N generators contains at most $N+1$ terms.
- Every Grassman polynomial is multilinear in the generators
- Polynomials containing only even numbers of generators in each term commute with all polynomials.
- Polynomials containing only odd numbers of generators in each term anticommute with each other.
- Any real function of a finite number of Grassman variables is a polynomial.

D.1.2 Complex Conjugation

- θ “real” means $\theta^* = \theta$.
- $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$.
- A product of n real Grassman variables is not real in general. It is only real if $\frac{n(n-1)}{2}$ is even. If not, the product is pure imaginary.

D.1.3 Differentiation

The left derivative of a monomial with respect to a Grassman variable is 0 if the monomial does not contain the variable, and is obtained by anticommuting the variable to the leftmost position and removing it if the monomial does contain it. The right derivative is defined analogously.

- $\frac{d}{d\theta_k L}(\theta_1 \cdots \theta_n) = (-1)^{k-1} \theta_1 \cdots \theta_{k-1} \theta_{k+1} \cdots \theta_n$.
- $\frac{d}{d\theta_k R}(\theta_1 \cdots \theta_n) = (-1)^{n-k} \theta_1 \cdots \theta_{k-1} \theta_{k+1} \cdots \theta_n$.
- Differentiation does not commute with complex conjugation. For a real monomial f , of n Grassman variables,

$$\left(\frac{df}{d\theta_i}\right)^* = (-1)^{n-1} \left(\frac{df}{d\theta_i^*}\right).$$

- When taking multiple derivatives, one must consider whether the later variables would have had to be anticommutated through the earlier ones.

$$\frac{d^n(\theta_1 \cdots \theta_n)}{d\theta_n \cdots d\theta_1} = 1.$$

- Second derivatives anticommute with complex conjugation.

$$\left(\frac{d^2 f}{d\theta_1 d\theta_2}\right)^* = -\left(\frac{d^2 f^*}{d\theta_1^* d\theta_2^*}\right).$$

D.1.4 Integration

- For Grassman variables (indefinite) integration and differentiation are the same.

•

$$\int (d\theta)\theta = 1$$

$$\int (d\theta) = 0.$$

- Change of variables:

$$\int d\theta_1 \cdots d\theta_n f(\theta_1 \cdots \theta_n) = \int d\phi_1 \cdots d\phi_n \left(\det \frac{\partial \theta_i}{\partial \phi_j} \right)^{-1} P(\theta(\phi)).$$

D.2 Gaussian Integrals

D.2.1 C-number

- If a is a matrix with positive definite real part and x is real,

$$\int_{-\infty}^{\infty} e^{-x^\top a x + b \cdot x} dx = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det a_s}} e^{\frac{1}{4} b^\top a_s^{-1} b}, \quad (\text{D.1})$$

where a_s is the symmetric part of a .

- Let

$$z \equiv x + iy$$

and

$$\int dz^* dz \equiv \int_{-\infty}^{\infty} dx dy$$

Then, if a is a hermitian matrix with positive definite real part,

$$\int dz^* dz e^{-z^{*\top} a z - z^\top b z - z^{*\top} b^* z^* + c^* \cdot z^* + c \cdot z} = \frac{\pi^n}{\sqrt{\det M_s}} e^{\frac{1}{4} N^\top M_s^{-1} N}, \quad (\text{D.2})$$

where (a_r denotes the real part of a , b^s is the symmetric part of b , etc.)

$$M_s \equiv \begin{pmatrix} a_r + 2b_r^s & -a_I - 2b_I^s \\ a_I - 2b_I^s & a_r - 2b_r^s \end{pmatrix}$$

$$N = \begin{pmatrix} 2c_r \\ -2c_I \end{pmatrix}.$$

- When $b = 0$ and a is hermitian, the previous case reduces to

$$\int dz^* dz e^{-z^{*\top} a z + c^* \cdot z^* + c \cdot z} = \frac{(2i\pi)^n}{\det a} e^{\frac{1}{4} c^{*\top} a^{-1} c}. \quad (\text{D.3})$$

D.2.2 Grassman

- Let

$$\psi \equiv x + iy$$

and

$$\int d\psi^* d\psi \equiv \int dx dy$$

with ψ , ψ^* , x , and y Grassman variables. Then, if a is a hermitian matrix that commutes with Grassman variables and η is a vector that anticommutes with Grassman variables,

$$\int d\psi^* d\psi e^{\psi^{*\top} a \psi + \psi^\top b \psi - \psi^{*\top} b^* \psi^* + \eta^* \cdot \psi + \psi^* \cdot \eta} = e^{\frac{1}{4} R^\top G_a^{-1} R} \sqrt{\det G_a}, \quad (\text{D.4})$$

where (a_r denotes the real part of a , b^a is the antisymmetric part of b , etc.)

$$G_a \equiv i \begin{pmatrix} a_I + 2b_I^a & a_r + 2b_r^a \\ -a_r + 2b_r^a & a_I - 2b_I^a \end{pmatrix}$$

$$R = \begin{pmatrix} -2i\eta_I \\ 2i\eta_r \end{pmatrix}.$$

D.3 Square Matrices

Much of the information in this section is from the books by Richards[19] and Wedderburn[21].

D.3.1 General

- If a is a $2n \times 2n$ matrix composed of $n \times n$ submatrices of the form $a = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}$, $\det a = (\det b)(\det d)$.
- The inverse of a $2n \times 2n$ matrix composed of $n \times n$ submatrices and of the form $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is (assuming the requisite matrices are invertible)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} [a - bd^{-1}c]^{-1} & [c - db^{-1}a]^{-1} \\ [b - ac^{-1}d]^{-1} & [d - ca^{-1}b]^{-1} \end{pmatrix} \quad (\text{D.5})$$

- The following identity for the matrix A holds under appropriate conditions:

$$(I - A)^{-1} + (I - A^{-1})^{-1} = I. \quad (\text{D.6})$$

D.3.2 Hermitian

- If $A = a + ib$ is hermitian, the real part a is symmetric and the imaginary part b is antisymmetric.
- The eigenvalues of a hermitian matrix are real.
- If $A = a + ib$ is an $n \times n$ hermitian matrix, it behaves like the real symmetric $2n \times 2n$ matrix $P = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. There is a one-to-one correspondence between

the real eigenvalues of A and those of P ([21],101). It follows that $\det P = (\det A)^2$.

- The similarity transformation necessary to bring a hermitian matrix to diagonal form is unitary.
- The transpose and the complex conjugate of a hermitian matrix are equal and hermitian themselves.
- The inverse of a nonsingular hermitian matrix is hermitian.
- The product of two hermitian matrices is hermitian iff the matrices commute.
- An $n \times n$ hermitian matrix consists of n^2 real parameters.

D.3.3 Symmetric

- The eigenvalues of a real symmetric matrix are real.
- The similarity transformation necessary to bring a symmetric matrix to diagonal form is orthogonal.
- The inverse of a nonsingular symmetric matrix is symmetric.
- The product of two symmetric matrices is symmetric iff the matrices commute.
- An $n \times n$ real symmetric matrix consists of $\frac{n(n+1)}{2}$ real parameters.

D.3.4 Real Antisymmetric

- The eigenvalues of a real antisymmetric matrix are imaginary and exist in conjugate pairs.
- The similarity transformation necessary to bring an antisymmetric matrix to diagonal form is orthogonal.
- The product of two antisymmetric matrices is antisymmetric iff the matrices anticommute.
- An $n \times n$ real antisymmetric matrix consists of $\frac{n(n-1)}{2}$ real parameters.

D.4 Combinatorics

The relations in this section are mostly from Abramowitz and Stegun[1] and Gradshteyn and Ryzhik[4].

D.4.1 Gamma function

- $x! \equiv \Gamma(x + 1)$.
- $x! = x(x - 1)!$ and $\Gamma(x + 1) = x\Gamma(x)$.
- $(2x)! = \frac{2^{2x}}{\sqrt{\pi}}(x - \frac{1}{2})!x!$.
- $\Gamma(\frac{1}{2}) = (-\frac{1}{2})! = \sqrt{\pi}$.
- $\Gamma(\frac{3}{2}) = (\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$.
- $\Gamma(1) = 0! = 1$.

D.4.2 Binomials

- $\binom{x}{k} = (-1)^k \binom{k-x-1}{k}$.
- $\sum_{j=0}^{\infty} \alpha^j \binom{z}{j} = (1 + \alpha)^z$ if $|\alpha| < 1$.

D.5 Confluent Hypergeometric Functions

- Kummer's function:

$$M(a, b, z) \equiv \sum_{n=0}^{\infty} \frac{(a+n-1)!(b-1)!z^n}{(a-1)!(b+n-1)!n!}.$$

•

$$\frac{d^n}{dz^n} M(a, b, z) = \frac{(a+n-1)!(b-1)!}{(a-1)!(b+n-1)!} M(a+n, b+n, z).$$

•

$$\frac{d}{dz} M(a, b, z) = \frac{a}{b} M(a+1, b+1, z).$$

- As $|z| \rightarrow \infty$ with $\Re z > 0$,

$$M(a, b, z) \sim \frac{(b-1)!}{(a-1)!} e^z z^{a-b}.$$

•

$$\sum_{n=0}^{\infty} z^n \frac{(\alpha+n)!}{(2n)!} = \alpha! M\left(\alpha+1, \frac{1}{2}, \frac{z}{4}\right).$$

D.6 Some Lie Groups

We provide the Clebsch Gordon rules for decomposing a direct product of two irreducible representations into a direct sum of irreducible representations for several Lie groups of interest.

D.6.1 $U(1)$

The irreducible representations are labeled by one real number α as (α) . The C-G rule is

$$(\alpha) \otimes (\beta) = (\alpha + \beta).$$

D.6.2 $SU(2)$

The irreducible representations are labeled by a non-negative integer n as $(\frac{n}{2})$. The C-G rule is

$$\left(\frac{n}{2}\right) \otimes \left(\frac{m}{2}\right) = \bigoplus_{k=\frac{|m-n|}{2}}^{\frac{m+n}{2}} (k).$$

D.6.3 $SO(3,1)$

The irreducible representations are labeled by two non-negative integers n and m as $(\frac{n}{2}, \frac{m}{2})$ and behave like a direct product of $SU(2)$ groups. The C-G rule is

$$\left(\frac{n}{2}, \frac{i}{2}\right) \otimes \left(\frac{m}{2}, \frac{j}{2}\right) = \bigoplus_{k=\frac{|m-n|}{2}}^{\frac{m+n}{2}} \bigoplus_{l=\frac{|i-j|}{2}}^{\frac{i+j}{2}} (k, l).$$

D.7 Fourier Transforms

Here, we describe our Fourier transform conventions and provide some useful relationships.

D.7.1 Continuum

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d} \phi(k) e^{ik \cdot x} \quad (\text{D.7a})$$

$$\phi(k) = \int d^d x \phi(x) e^{-ik \cdot x} \quad (\text{D.7b})$$

D.7.2 Momentum space cutoff

$\phi(k) = 0$ if $|k| > \Lambda$. The sharp cutoff imposes a lattice-like structure in x -space. However, we still use Fourier integrals since these make life easier. The relations are identical to the infinite cutoff case.

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d} \phi(k) e^{ik \cdot x} \quad (\text{D.8a})$$

$$\phi(k) = \int d^d x \phi(x) e^{-ik \cdot x} \quad (\text{D.8b})$$

D.7.3 Spatially bounded with momentum space cutoff

Space sits in a box of side L with periodic boundary conditions.

$$\phi(x) = \frac{1}{L^d} \sum_k \phi_k e^{ik \cdot x} \quad (\text{D.9a})$$

$$\phi_k = \int d^d x \phi(x) e^{-ik \cdot x} \quad (\text{D.9b})$$

D.7.4 Useful relations

These are several relations useful in determining Fourier normalizations.

$$\int_{-\infty}^{\infty} dx e^{ikx} = (2\pi)\delta(k) \sim L\delta_{k,0} \quad (\text{D.10})$$

$$\sum_{n=-\infty}^{\infty} e^{\frac{2\pi i n x}{L}} = L\delta(x) \quad (\text{D.11})$$

$$\int_{-L}^L dx e^{ikx} = (2\pi)\delta(k) + O\left(\frac{1}{L}\right) \sim L\delta_{k,0} + O\left(\frac{1}{L}\right) \quad (\text{D.12})$$

$$\sum_{n=-N}^N e^{\frac{2\pi i n x}{L}} = L\delta(x) + O\left(\frac{1}{N}\right) \quad (\text{D.13})$$

D.8 Infinite Volume Limit

Here we describe the transitions between finite volume objects and infinite volume objects.

We use \sim to denote equivalence up to the next order in $1/V$. k_{disc} denotes the discrete k 's ($\frac{2\pi\vec{n}}{L}$) and k_{cont} the infinite-volume k 's.

$$V\delta_{k,p} \sim (2\pi)^d \delta^{(d)}(k-p) \quad (\text{D.14a})$$

$$\frac{1}{V} \sum_k \sim \int \frac{d^d k}{(2\pi)^d} \quad (\text{D.14b})$$

$$k_{disc} \sim k_{cont} \quad (\text{D.14c})$$

$$\frac{\partial}{\partial k_{disc}} \sim \frac{\partial}{\partial k_{cont}} \quad (\text{D.14d})$$

$$\phi_k \sim \phi(k) \quad (\text{D.14e})$$

$$V \frac{\partial}{\partial \phi_k} \sim \frac{\delta}{\delta \phi(k)}. \quad (\text{D.14f})$$

Appendix E

Glossary of Symbols

Symbol	Description	Page
A	Intermediate vector.	68
$A, A_{2n}, A_{2,k}$, etc.	Second derivative matrices.	35
A_{kin}, A'	Kinetic and non-kinetic 2^{nd} derivatives.	40
A_{2n}, A_n	Scattering amplitudes.	51
A_n^r	Field-scaled Amplitudes.	54
a	Eigenvalue parameter.	42
a_c	Phase boundary eigenvalue.	63
a_{ij}	Kinetic matrix.	70
B	Intermediate vector.	68
B, B_2, B_{2n} , etc.	First derivative vectors.	35
b	RG scaling factor.	14
b_{ij}	Mass matrix term.	70
C	Intermediate matrix.	68
$C_{kin}, C', C_{n,m}$	Intermediate matrices.	71
d	Space-time dimension.	10
D, D_{kin}, D'	Intermediate matrix.	68
δ_k	Kronecker delta $\delta_{k,0}$.	33
$\Delta_{int}, \Delta_{dim}, \Delta_{norm}$	Operators that effect RG steps.	34
ΔV	Depth of valley in effective potential.	62
η	Intermediate vector.	69
E, E_{kin}, E'	Intermediate matrix.	68
E, E_0 , etc.	Physical energies.	57
$\langle \rangle$	Expectation over distribution.	82
f	Eigenpotential divided by r .	59
F	Intermediate matrix.	68
(g, h)	Point in parameter space.	25
$g_0\Lambda, g\lambda$	Bare parameter sequence.	25
\mathbf{g}	Composite fast fermi field.	69
G, H	Subspaces of parameter space.	25
G	Intermediate matrix.	70
$G_{kin}, G', G_{n,m}$	Intermediate matrices.	71
$G^{(n)}(k_1 \dots k_n)$, etc.	Greens function.	15

Symbol	Description	Page
$G_r^{(n)}$, etc.	Field-scaled Greens functions	53
h_0, h_1, h_2	Parts of effective potential.	60
$\int d\hat{k}$	Angular integral.	72
I	Integration over propagator.	51
I'	Integration over broken propagator.	52
k, p , etc.	Momenta.	Ubiquitous
k_{disc}	Finite volume k .	32
k_{cont}	Infinite volume k .	32
K_E	Kinematic part of cross section.	56
L	Length of bounding box side.	10
Λ	Momentum space cutoff.	10
λ	Eigenvalue.	18
m	Number of bare parameters.	28
$M(a, b, z)$	Kummer's function.	43
M	Flow matrix.	18
M	Intermediate matrix.	69
n	Number of relevant eigendirections.	27
N	Number of field components.	10
N	Intermediate matrix.	69
$o(E/\Lambda)$	Greatest vanishing order as $\Lambda \rightarrow \infty$.	25
$O(r)$	Linear order in couplings.	40
Ω	Momentum-space domain.	10
$\partial\Omega$	Shell of integration.	14
$\partial\Omega_+$	Half-shell of integration.	35
$p(i_1 \cdots i_{2n})$	Permutation of indices.	75
p	Point in parameter space.	13
p_Λ, p_E	Theories obeying norm. conditions.	26
p^*	Fixed point in parameter space.	17
P	Parameter space.	25
$P(k)$	Propagator.	53
$P_r(k)$	Field-scaled propagator.	53
$\phi(x)$, etc.	Scalar field.	10
$\phi_s(k), f(k)$, etc.	Slow and fast scalar fields.	14
ϕ^2	Sum over components of $\phi_i\phi_i$.	32
ϕ^f	Free field.	53
ϕ_{min}	Minimum of effective potential.	60
ϕ_r	Renormalized Field.	53
ϕ_0	Minimum of $O(1)$ part of effective potential.	61
ϕ_1, ϕ_2	Radiative corrections to location of min. of eff. pot..	61
ϕ'	Dynamical symmetry-broken field.	47
$\psi(x), \psi^*(x)$, etc.	Fermion Field.	10
$\psi_s(k), g(k)$, etc.	Slow and fast fermi fields.	68
r	Mass parameter.	33
\bar{r}	Symmetry broken mass parameter.	52

Symbol	Description	Page
$r(a)$	Phase boundary curve.	63
ρ	Location of Minimum of classical potential.	47
ρ_r	Scaled location of Minimum.	55
R	Intermediate vector.	70
S	Action.	10
S'	Intermediate form of action.	15
$S^{dim}, S^{norm}, S^{int}$.	Intermediate forms of action.	38
S_{2n}	Term in S with $2n$ field factors.	32
$S_{2n,m}$	Action term with $2n$ bose and $2m$ fermi fields.	67
S_d	Surface area of d -dim. sphere with Fourier factor.	41
$\sigma_{\alpha\beta}$	Mass matrix term.	70
$\sigma_{2 \rightarrow 2}$	Cross section.	56
\sim	Equivalence in infinite volume limit.	32
t	Infinitesimal RG scaling increment.	14
$\theta(x)$	Theta function.	14
θ^μ	Inverse kinetic matrix.	73
$u_{i_1 \dots i_n}^n(k_1 \dots k_n)$, etc.	General couplings.	10
$u_{2n}(k_1 \dots k_{2n})$	$O(N)$ symmetric couplings.	32
u_{2n}	Local, non-derivative couplings.	33
u_{2n}^λ, u_{2n}^a	Eigenvector.	42
$u_2(k)$	Local, non-deriv. quadratic term.	33
u_2	Local, non-deriv mass term.	33
u'_{2n}	Resummed vertex.	56
u_{kin}	Kinetic part of $u_2(k)$.	38
u_{2n}^r	Field-scaled parameters.	53
$u_{\alpha_1 \dots \alpha_m}^{i_1 \dots i_{2n}}(k_1 \dots q_m)$	General bose/fermi coupling.	67
$u_{\alpha_1 \dots \alpha_m}^{i_1 \dots i_{2n}}$	Local, non-deriv. bose/fermi coupling.	67
$u_{\alpha_1 \dots \alpha_m}^{(2n)}$	A fermi/bose coupling.	67
$u^{ij}, u_{\alpha\beta}$	Quadratic couplings.	70
U	Potential	11
U^a, U^λ	Eigenpotential.	43
U_r	Field-scaled potential.	53
V	Effective Potential.	59
V, V^a , etc.	Symmetry broken potential.	47
V	Volume of bounding box.	32
v_n	Symmetry broken coupling.	48
$v_n(\rho)$	Broken parameter with ρ dependence explicit.	55
v'_n	Resummed broken vertex.	56
$[x]$	Greatest integer $\leq x$.	87
$x!$	$\Gamma(x + 1)$.	43
Z	Field scaling factor.	53
Z	Normalization.	15
$\zeta_{\alpha\beta}^\mu$	Kinetic matrix.	70

Appendix F

Glossary of Terms

Closed Subspace: A subspace of parameter space that is closed under the renormalization group procedure. The trajectory through a point in the subspace lies entirely within the subspace.

Cutoff: A sphere of unit radius in Euclidean momentum space outside of which the fields are identically zero.

Effective Cutoff: The radius of the sphere in Euclidean momentum space outside of which the fields are zero in units of the initial cutoff. After the renormalization group procedure, the effective cutoff is lowered, while the cutoff is restored to unity by dimensional scaling.

Fixed Point: A point in parameter space corresponding to a theory on which the renormalization group procedure has no effect.

Flow equations: The infinitesimal RG equations corresponding to the tangent to a trajectory.

FP: Acronym for fixed point.

Gaussian Fixed Point: The origin in parameter space. All dimensionless parameters are zero.

Irrelevant Direction: In our usage, an eigendirection to a fixed point with negative eigenvalue. As the effective cutoff is lowered, the flow moves toward the fixed point along such a direction.

Local, Non-derivative Theory: An action containing no non-local or derivative interactions.

Marginal Direction: An eigendirection with zero eigenvalue. Such a direction corresponds to motion within a dense region of fixed points.

Parameter Space: A linear vector space of parameters that define the action. The sum of the non-kinetic part of two actions corresponds to vector addition.

Relevant Direction: In our usage, an eigendirection to a fixed point with positive eigenvalue. As the effective cutoff is lowered, the flow moves away from the fixed point along such a direction.

RG: Acronym for renormalization group.

Trajectory: The set of points in parameter space corresponding to renormalization group transformations of a theory with varying values of the cutoff scaling factor $b > 1$.

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