The Fractional Quantum Hall Effect, Chern-Simons Theory, and Integral Lattices

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"There's so much fun to be had.... I don't want you to take this stuff too seriously. I think we should just have fun imagining it, and not worry about it — there's no teacher going to ask you questions at the end."(R. P. Feynman)

1 Chern-Simons theory

Chern-Simons theory has come to play an important rôle in three-dimensional topology because of its connections with Ray-Singer analytic torsion [47], the Gauss linking number [25], [14], [57], the Jones polynomial in knot theory [35] and its generalizations [63], [23], and three-manifold invariants [63], [12]. Recently, Chern-Simons forms and actions over noncommutative spaces [7] have been defined [45], [6] and turn out to provide a unifying perspective for topological gauge theories in odd *and* even dimensions [6].

The comparatively trivial abelian pure Chern-Simons theories (which reproduce the Gauss linking number and analytic torsion) have turned out to be fundamental building blocks for a theory of the fractional quantum Hall effect [61], [31], [59], [20], [29], [49]. This effect is one of the more exciting effects in condensed matter physics, discovered and explored between 1980 and the present [58], [54], [9], [44]. It has also been observed that $SU(2)$-Chern-Simons theories come up in problems of condensed matter physics connected with the theory of spin liquids; see e.g. [26].

Thus, it is well justified to start this report with a short review of the definition and some mathematical properties of Chern-Simons theory.

Let M be an oriented, framed three-manifold (the framing of M corresponds to a choice of a trivialization of the tangent bundle of M). Below, we shall consider the example where $M = \mathbb{R}^3$. Let G be a compact Lie group, or let $G = \mathbb{R}^N$. Let E denote the total space of a principal G-bundle with base space M, and let ∇ be a connection on E. Locally, we may describe ∇ in terms of its components, A (the "gauge potential"), in some local trivialization of E. These components are 1-forms on M with values in Lie G (the Lie algebra of G). The Chern-Simons

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3-form on M is defined, locally, by the formula

$$CS^{(3)}(A) \;=\; \mathrm{tr}\,(A \wedge dA \,+\, \frac{2}{3}\, A \wedge A \wedge A), \tag{1.1}$$

where $\mathrm{tr}(\cdot)$ is a trace on Lie G that is invariant under the adjoint action of G on Lie G. The Chern-Simons action functional S is defined, formally, by

$$S(A) \;=\; \frac{1}{4\pi} \int\limits_M CS^{(3)}(A). \tag{1.2}$$

Unfortunately, this definition does not make sense in general. To understand the problems with (1.2), we consider the example where $M = S^3$ and $G = SU(N)$. We choose an orthonormal basis $\{T_\alpha\}_{\alpha=1}^{D_N}$, $D_N = N^2 - 1$, in $A_{N-1} = \text{Lie } SU(N)$ and choose $\mathrm{tr}(\cdot)$ such that

$$\mathrm{tr}\,(T_\alpha\, T_\beta) \;=\; -\,\frac{k}{2}\,\delta_{\alpha\beta}, \tag{1.3}$$

$k \in \mathbb{R}$. Because $\pi_3(G) = \mathbb{Z}$, the action $S(A)$ in eq. (1.2), with $\mathrm{tr}(\cdot)$ as in (1.3), is defined only modulo $2\pi k\mathbb{Z}$. It follows that $\exp i\; S(A)$ is a well-defined, single-valued functional of the connection ∇ if and only if $k \in \mathbb{Z}$. Similar remarks apply to general compact Lie groups.

Assuming now that $\mathrm{tr}(\cdot)$ has been chosen such that $\exp i\; S(A)$ is a well-defined functional of ∇, quantized Chern-Simons theory is defined as a mathematically precise interpretation of the formal Feynman "functional measure"

$$dP\,(A) \;:=\; Z^{-1}\; \exp i\, S(A)\, \mathcal{D}A, \tag{1.4}$$

where $\mathcal{D}A$ is a formal Lebesgue measure on the affine space of connections on E, and the normalization factor Z (the partition function) is chosen such that $\int dP(A) = 1$. One would hope to extract from (1.4) a precise definition of $dP(A)$ as a complex measure on the space $\mathcal{A}$ of orbits of gauge potentials under the action of the group of gauge transformations.

The functional $\exp i\, S(A)$ does not require choosing a metric on M, and one might expect, therefore, that $dP(A)$ is independent of a choice of a metric on M. Unfortunately, this is a wrong expectation. The definition of "$\mathcal{D}A$" involves the choice of a metric on M, and, in order to eliminate dependence of $dP(A)$ on that metric, one must add to $S(A)$ a "counterterm", which is given by the Chern-Simons action of the Levi-Civita spin connection [63], [5]. One may then hope to arrive at a definition of $dP(A)$ that depends only on the framing of M and hence yields what is called a topological gauge theory [63], [62].

The kinds of functionals on $\mathcal{A}$ one would like to integrate with the "measure" $dP(A)$ are *Wilson loops*: let $\mathcal{L}$ be a loop in M (i.e., a smooth embedding of S^1 in M), and let R be an irreducible, unitary representation of G. We define

$$W_R[\mathcal{L}] \;:=\; Tr_R R\;[P \exp\, \zeta \int\limits_{\mathcal{L}} A], \tag{1.5}$$

where P indicates path ordering, and ζ is some positive constant ("field strength renormalization" constant) to be determined. For a smooth Lie G-valued 1-form A, the R.S. of (1.5) can be defined via Chen's iterated integrals, i.e., through its *Dyson series.*

As it stands, the expression on the R.S. of eq. (1.4) is nonsense. A conventional strategy used to make sense of (1.4) is to *fix a gauge* and apply the Faddeev-Popov procedure [10] to interpret $\mathcal{D}A$. "Fixing a gauge" consists in choosing connection-dependent, local trivializations of E in such a way that the gauge potentials A satisfy certain constraints. We wish to exemplify gauge fixing in a special case, following [23]: we choose $G = SU(N)$ and $M = \mathbb{R}^3$. Points $x \in M$ are represented by (Cartesian) coordinates (x^+, x^-, t), with x^+, x^-, t in $\mathbb{R}$. We expand the gauge potential A in the basis $\{dx^+, dx^-, dt\}$ of 1-forms:

$$A(x) \;=\; a_+(x)dx^+ + a_-(x)dx^- + a_0(x)dt, \tag{1.6}$$

where $a_i(x) \in A_{N-1}$, $i = +, -, 0$. We choose a basis $\{T_\alpha\}_{\alpha=1}^{D_N}$ in A_{N-1} and a trace $\mathrm{tr}(\cdot)$ on A_{N-1} as specified in (1.3). Then

$$a_i(x) \;=\; \sum_{\alpha=1}^{D_N} a_i^\alpha(x)\, T_\alpha,$$

where $a_i^\alpha(x)$ is a function on M, $\forall i, \alpha$. One easily shows that the condition

$$a_-(x) \;=\; 0 \tag{1.7}$$

fixes a gauge (called "light-cone" or "axial" gauge). In this gauge, the Chern-Simons action S of eq. (1.2) takes the form

$$S(A) \;=\; \frac{1}{4\pi} \int \mathrm{tr}\,(a_+ \partial_- a_0)\; dx^+ \wedge dx^- \wedge dt. \tag{1.8}$$

This action is *quadratic* in A. One may therefore attempt to interpret the measure $dP(A)$ in (1.4) as a "complex Gaussian measure". Well, it actually is a "complex Gaussian", but it isn't a measure. However, all we really need to be able to do is to calculate *moments* of $dP(A)$. Let $\langle(\cdot)\rangle$ denote formal integration $\int dP(A)(\cdot)$ with respect to $dP(A)$. The first moments $\langle a_i^\alpha(x)\rangle$ *vanish* and the second moments $\langle a_i^\alpha(x)\, a_j^\beta(y)\rangle$ can be expressed in terms of the partial derivative of a Green function of the d'Alembertian $\partial_+\partial_-$ with respect to x^+. Together, they determine all higher moments ("Wick's theorem"). It is advantageous to complexify the planes $\{t = \text{const.}\}$, use complex coordinates, $z = x^+ \in \mathbb{C}$, $\bar{z} = x^- \in \mathbb{C}$, and analytically continue the moments of $dP(A)$ in x^+. The physicists call this "Wick rotation". Wick rotation is convenient, but not indispensable, in the following calculations. The Wick-rotated second moments are:

$$\begin{aligned}
\langle a_-^\alpha(x)\; a_j^\beta(y)\rangle &= 0, \quad \text{for all } j, \alpha, \beta,\\
\langle a_+^\alpha(x)\; a_+^\beta(y)\rangle &= 0, \quad \text{for all } \alpha, \beta,\\
\langle a_0^\alpha(x)\; a_0^\beta(y)\rangle &= 0, \quad \text{for all } \alpha, \beta,
\end{aligned}$$

and

$$\langle a_+^\alpha(z,t)\, a_0^\beta(w,s)\rangle \;=\; 2\lambda\, \delta^{\alpha\beta}\, \delta(t-s)\, \frac{1}{z-w}, \tag{1.9}$$

with $\lambda = -1/k$. Expectations $\langle(\cdot)\rangle$ of more complicated functionals of A can be calculated from (1.9) by using Wick's theorem. In particular, we may calculate expectations of "Wilson lines" and Wilson loops from (1.9) (e.g. by expanding them in a Dyson series).

Let $I_1, \ldots, I_m$ be a partition of $\{1,\ldots,n\}$, $m = 1,2,\ldots$, $n = 1,2,\ldots$. To every index set I_ℓ we assign a representation R_ℓ of $SU(N)$. Each index $j \in I_\ell$ labels a smooth curve

$$\gamma_j(t) \;=\; \{z_j(t') \in \mathbb{C} \;:\; t_0 \leq t' \leq t\}$$

in the complex plane that determines a smooth curve $\sigma_j(t)$ in $\mathbb{R}^3$ given by

$$\sigma_j(t) \;=\; \{(\mathrm{Re}\; z_j(t'),\; \mathrm{Im}\; z_j(t'), t') \;:\; z_j(t') \in \gamma_j(t),\; t_0 \leq t' \leq t\}. \tag{1.10}$$

We define a "Wilson line operator" $w_j(t)$ by setting

$$w_j(t) \;:=\; R_\ell\big[P \exp \zeta \int_{\sigma_j(t)} A\big], \tag{1.11}$$

where $\zeta > 0$ is a field strength renormalization constant. This operator is a holonomy matrix of the connection ∇ with components A and acts on the representation space V_{R_ℓ} of $SU(N)$. It is easy to see that

$$dw_j(t) \;=\; \zeta\; d\alpha_j(t)\; w_j(t), \tag{1.12}$$

where

$$\alpha_j(t) \;:=\; dR_\ell\Big[\int_{t_0}^{t} \big\{a_+\big(z_j(t'),t'\big)\dot{z}_j(t') + a_0\big(z_j(t'),t'\big)\big\}\, dt'\Big],$$

with $\dot{z}(t) = dz(t)/dt$, and dR_ℓ the representation of A_{N-1} determined by R_ℓ; $j \in I_\ell$, $\ell = 1,\ldots,m$.

The *basic object* in a mathematically precise definition of $SU(N)$ pure Chern-Simons theory on $\mathbb{R}^3$ is

$$\phi_n(t,t_0) \;:=\; \big\langle w_1(t)\; \otimes \cdots \otimes\; w_n(t)\big\rangle\,, \tag{1.13}$$

which is an endomorphism of the vector space

$$\mathcal{V}_n \;:=\; V_{R^{(1)}} \otimes \cdots \otimes V_{R^{(n)}}, \tag{1.14}$$

with $R^{(j)} = R_\ell$, for $j \in I_\ell$, $n = 1,2,3,\ldots$. One may attempt to calculate $\phi_n(t,t_0)$ by deriving a differential equation for it. We define

$$\Omega_{ij} \;:=\; \sum_{\alpha=1}^{D_N} \mathbb{I} \otimes \cdots \otimes dR^{(i)}(T_\alpha) \otimes \cdots \otimes dR^{(j)}(T_\alpha) \otimes \cdots \otimes \mathbb{I}, \tag{1.15}$$

for all i, j, with $1 \leq i < j \leq n$. Using (1.12), one shows — see [23] — that

$$\dot{\phi}_n(t, t_0) = \kappa \sum_{1 \leq i < j \leq n} \frac{\dot{z}_i(t) - \dot{z}_j(t)}{z_i(t) - z_j(t)} \, \Omega_{ij} \phi_n(t, t_0), \tag{1.16}$$

where $\kappa = \zeta^2 \lambda$. Eq. (1.16) is the celebrated *Knizhnik-Zamolodchikov equation*[38]. An alternative method to calculate $\phi_n(t, t_0)$ would be to expand all Wilson line operators $w_j(t)$ in their Dyson series and to calculate the resulting terms by using Wick's theorem and (1.9) [16].

Let M_n denote the subset of $\mathbb{C}^n$ consisting of n-tuples, $\underset{\sim}{z} = (z_1, \ldots, z_n)$, of complex numbers, with $z_i \neq z_j$, for $i \neq j$, and let $\widetilde{M_n}$ be the universal cover of M_n. Let K be the space of $\mathcal{V}_n$-valued functions on $\widetilde{M_n}$. On K we may define a connection 1-form ω by setting

$$\omega = \kappa \sum_{1 \leq i < j \leq n} d \, \log(z_i - z_j) \, \Omega_{ij}. \tag{1.17}$$

This connection is called the Knizhnik-Zamolodchikov connection. It is easy to verify that ω is *flat*, i.e.,

$$d\omega + \omega \wedge \omega = 0.$$

This is a consequence of the infinitesimal pure braid relations

$$[\Omega_{ij}, \Omega_{k\ell}] = 0, \quad [\Omega_{ij}, \Omega_{jk} + \Omega_{ki}] = 0, \tag{1.18}$$

where i, j, k, and ℓ are all distinct. Eq. (1.16) may now be written as

$$d \, \phi_n = \omega \, \phi_n, \tag{1.19}$$

which is the equation for a parallel transporter.

Let $(z_1, \ldots, z_n)$ be a point in M_n, and let π be an arbitrary permutation of $\{1, \ldots, n\}$ leaving the subsets $I_1, \ldots, I_m$ invariant. Let $\sigma_j = \sigma_j(t_1)$ be a curve in $\mathbb{R}^3$, as in (1.10), starting at the point (Re z_j, Im z_j, t_0) and ending at (Re $z_{\pi(j)}$, Im $z_{\pi(j)}, t_1$), for $j = 1, \ldots, n$. The family of all n-tuples $\{\sigma_1, \ldots, \sigma_n\}$ of such curves that do not intersect each other is a union of disjoint homotopy classes of curves labelled by elements b of a subgroup $B_n(I_1, \ldots, I_m)$ of the braid group, B_n, on n strands defined by the property that the cosets of elements of $B_n(I_1, \ldots, I_m)$ modulo the normal subgroup of pure braids are permutations π of $\{1, \ldots, n\}$ leaving $I_1, \ldots, I_m$ invariant. Let $b \in B_n(I_1, \ldots, I_m)$, and let $\{\sigma_1, \ldots, \sigma_n\}$ be n curves in the homotopy class b. Let $\phi_n(b; t_1, t_0)$ be a solution of the Knizhnik-Zamolodchikov eq. (1.16) for the curves $\{\sigma_1, \ldots, \sigma_n\}$, with initial condition $\phi_n(b; t_0, t_0) = \mathbb{1} \big|_{\mathcal{V}_n}$. Then

$$b \mapsto \phi_n(b; t_1, t_0) \tag{1.20}$$

defines a representation ϕ_n of $B_n(I_1, \ldots, I_m)$ on $\mathcal{V}_n$. This is a consequence of the identity

$$\phi_n(b_2 \circ b_1; t_2, t_0) = \phi_n(b_2; t_2, t_1) \, \phi_n(b_1; t_1, t_0)$$

(representation property) and the flatness of the Knizhnik-Zamolodchikov connection ω.

Let

$$g \mapsto R_{(n)}(g) := R^{(1)}(g) \otimes \cdots \otimes R^{(n)}(g), \ g \in SU(N),$$

be the representation of $SU(N)$ on $\mathcal{V}_n$. Because the Knizhnik-Zamolodchikov connection ω is $SU(N)$-invariant, the representation ϕ_n of $B_n(I_1, \dots, I_m)$ on $\mathcal{V}_n$ commutes with the representation $R_{(n)}$ of $SU(N)$ on $\mathcal{V}_n$. Let $\mathcal{I}_n$ be the subspace of $\mathcal{V}_n$ consisting of $SU(N)$-invariant tensors, i.e., for $\xi \in \mathcal{I}_n$, $R_{(n)}(g)\xi = \xi$, $\forall g \in SU(N)$. The space $\mathcal{I}_n$ inherits the scalar product of $\mathcal{V}_n$. It is an invariant subspace for ϕ_n. It is interesting to ask whether the representation ϕ_n of $B_n(I_1, \dots, I_m)$ on $\mathcal{V}_n$, or its subrepresentation $\phi_n \big|_{\mathcal{I}_n}$, are *unitary* in the scalar product of $\mathcal{V}_n$. The answer is that they are *not* unitary. However, ϕ_n may contain a *unitary subrepresentation*: suppose that

$$\kappa = \pm \frac{1}{k + c_2}, \quad k = 1, 2, 3, \dots, \tag{1.21}$$

where c_2 is the eigenvalue of the quadratic Casimir operator in the adjoint representation of the group G, normalized such that $c_2 = N$, for $G = SU(N)$. Let $U_q(\text{Lie } G)$ denote the usual quantum deformation of the universal enveloping algebra of Lie G with deformation parameter $q = \exp i\pi\kappa$ [34]. We assume that the representations R_ℓ, $\ell = 1, \dots, m$, have positive q-dimensions; see e.g. [21]. One may then define a certain quotient $\mathcal{I}_n^{(q)}$ of $\mathcal{V}_n$ of U_q (Lie G)-invariant tensors, which is expected to be invariant under the representation ϕ_n of $B_n(I_1, \dots, I_m)$; see e.g. Chapter 6 of [21]. The miracle is that $\phi_n \big|_{\mathcal{I}_n^{(q)}}$ appears to define a *unitary* representation of $B_n(I_1, \dots, I_m)$ on $\mathcal{I}_n^{(q)}$. For $G = SU(2)$, proofs have been sketched in [52], [39]. More details can be inferred from the explicit formulas in [23], [11] and the general results in Chapters 5 and 6 of [21]. For $G = SU(N)$, $N \geq 3$, a proof may, perhaps, be constructed on the basis of the results in [23], [21], [60], [37], but has apparently not appeared in the literature. The result described above is expected to hold for *arbitrary* compact, simple Lie groups G, but proofs are not available yet. The mathematical setting within which a proof might be constructed is that of braided tensor categories (more precisely "quantum categories" [21]) and of generalized hypergeometric functions [46]; see also the contributions of Felder and Wasserman to these proceedings, and references given there. A mathematically precise definition of quantized pure Chern-Simons theory on $M = \mathbb{R}^3$, with κ as in (1.21), would consist of converting the conjectures just described into theorems. Quantum-mechanical state vectors of this theory would be vectors in the spaces $\mathcal{I}_n^{(q)}$, $n = 0, 1, 2, \dots$ $\left(\mathcal{I}_0^{(q)} := \mathbb{C}\right)$, and it would determine unitary representations ϕ_n of the groups $B_n(I_1, \dots, I_m)$ on $\mathcal{I}_n^{(q)}$, for all $I_1, \dots, I_m$, and all n. The "physics-inspired" literature on these matters is somewhat confusing, with many incomplete proofs for fairly obvious conjectures.

The analysis sketched above for $G = SU(N)$ becomes very simple when $G = \mathbb{R}^N$, $N = 1, 2, \dots$ (abelian pure Chern-Simons theory). See Section 3. Chern-Simons theories with $G = \mathbb{R}^N$ are the basic building blocks in the theory of the fractional quantum Hall effect. (It will turn out that G is actually given by $\mathbb{R}^N/\Gamma$, where Γ is an integral Euclidian lattice.)

Chern-Simons theory becomes a more interesting, *dynamical* quantum field theory if the manifold M is a full cylinder (and $k = 1, 2, 3, \dots$). In this situation, it is equivalent to Lie G Kac-Moody algebra at level k and its representation category. See [63], and [43], [24], [16] for more details. In the context of the quantum Hall effect, the Kac-Moody currents acquire physical significance as "edge currents".

But let us return to the representations ϕ_n of the braid groups $B_n(I_1, \dots, I_m)$ on the spaces $\mathcal{I}_n$, for generic values of the parameter κ, and sketch their connection with polynomial invariants of knots and links. We choose $n = 2p$ to be an even integer and assume that

$$R^{(j+p)} = R^{(j)^{\vee}}, \quad j = 1, \dots, p, \tag{1.22}$$

where $R^{\vee}$ is the representation of $SU(N)$ conjugate to R. Let π be a permutation of $\{1, \dots, 2p\}$ with $\pi(j+p) = j+p$, $R^{(\pi(j))} = R^{(j)}$ (j and $\pi(j)$ are in the same subset I_ℓ of $\{1, \dots, 2p\}$) for $j = 1, \dots, p$. Let $\{e_\alpha^{(R)}\}$ be an orthonormal basis of the representation space V_R. We define vectors $\xi(\pi) \in \mathcal{I}_{2p}$ by setting

$$\xi(\pi) = \sum_{\alpha_1, \dots, \alpha_p} e_{\alpha_{\pi(1)}}^{(R^{(1)})} \otimes \cdots \otimes e_{\alpha_{\pi(p)}}^{(R^{(p)})} \otimes e_{\alpha_1}^{(R^{(1)})} \otimes \cdots \otimes e_{\alpha_p}^{(R^{(p)})}. \tag{1.23}$$

Let b be an element of the braid group B_{2p} with the property that the coset of b modulo pure braids on $2p$ strands is given by the permutation π. We consider the scalar products

$$\langle \xi(\pi), \quad \phi_{2p}(b; t_1, t_0)\, \xi(id.) \rangle. \tag{1.24}$$

These numbers are invariants of framed links. Quotients of these scalar products by analogous scalar products, with $SU(N)$ replaced by $\mathbb{R}$, yield the evaluation of an invariant of oriented links on the oriented link determined by the element $b \in B_{2p}$ and colored by the representations $R^{(1)}, \dots, R^{(p)}$. The special case where $R^{(1)} = \cdots = R^{(p)} = R$ is the N-dimensional, fundamental representation of $SU(N)$ has been analyzed in detail in [23], with generalizations appearing in Section 6.3 of [24].

The scalar products (1.24) can be calculated *perturbatively*, by expanding $\phi_{2p}(b; t_1, t_0)$ in a Taylor series in κ. The Taylor coefficients can be found by either solving the Knizhnik-Zamolodchikov equation for ϕ_{2p} iteratively (see the appendix in [23]) or, equivalently, by expanding the Wilson line operators $w_j(t)$ defined in (1.11) in their Dyson series, plugging the Dyson series into the R.S. of (1.13) and using Wick's theorem and (1.9). These Taylor coefficients are given in terms of multiple integrals along the curves $\sigma_1(t), \dots, \sigma_{2p}(t)$. They are special cases of what has become known under the name of Vassiliev invariants [56]: If, in eq. (1.19), a specific Knizhnik-Zamolodchikov connection ω is replaced by the "universal flat connection" defined by (1.17), with $\{\Omega_{ij}\}$ the "universal solution" of (1.18), one obtains the Vassiliev invariants of links.

It is natural to conjecture that the invariants built from (1.24) depend on the choice of the gauge group G in a nontrivial and interesting way. For a review of recent results concerning this topic see e.g. [2].

Now it is time to shift gears and talk about physics.

2 Quantum Hall effect and integral lattices

Experimentally, the quantum Hall effect is observed in two-dimensional systems of electrons confined to a planar region Ω and subject to a strong, uniform magnetic field $\vec{B}_c$ transversal to Ω, as indicated in *Figure 1.*

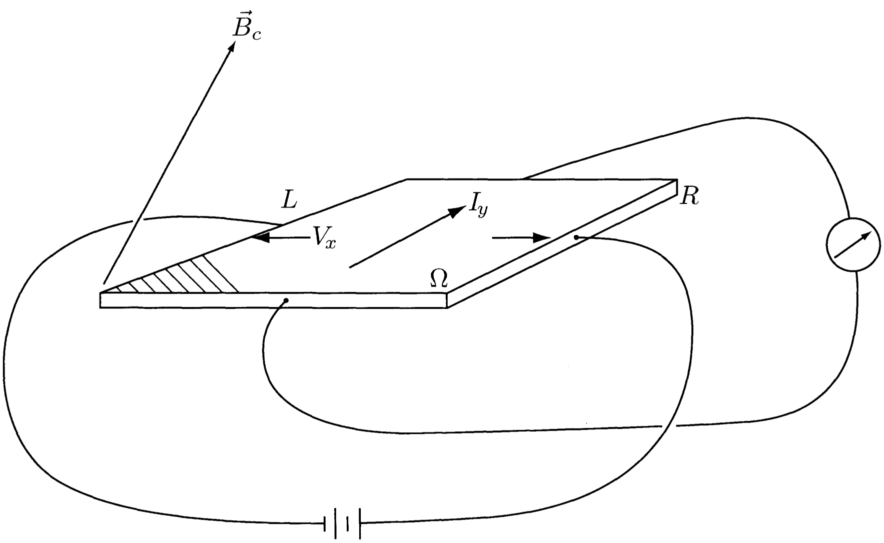


Figure 1

By tuning the y-component I_y of the total electric current to some value and then measuring the voltage drop V_x in the x-direction of the plane of the system, i.e., the difference in the chemical potentials of the electrons at the two edges R and L, one can calculate the *Hall resistance*

$$R_H = -\frac{V_x}{I_y}, \tag{2.1}$$

and finds that, for a fixed density n of electrons and at temperatures close to 0 K (absolute 0), the value of R_H is independent of the current I_y. It depends only on the external magnetic field $\vec{B}_c$. If the electrons are treated classically one finds, by equating the electrostatic to the Lorentz force, that

$$R_H = \frac{B_c}{ecn}, \tag{2.2}$$

where B_c is the z-component of $\vec{B}_c$ perpendicular to the plane of the system, e is the elementary electric charge, and c is the velocity of light.

By also measuring the voltage drop V_y in the y-direction, one can determine the longitudinal resistance, R_L, from the equation

$$R_L = \frac{V_y}{I_y}.$$

Neither classical nor quantum theory makes simple predictions about the behavior of R_L, but $R_L > 0$ means that there are dissipative processes in the system.

Two-dimensional systems of electrons are realized, in the laboratory, as *inversion layers* that form at the interface between an insulator and a semiconductor when an electric field (gate voltage) perpendicular to the interface, the plane of the system, is applied. An example of a material is a sandwich (a "heterojunction") made from GaAs and $\mathrm{Ga}_x\mathrm{Al}_{1-x}\mathrm{As}$. The quantum-mechanical motion of the electrons in the z-direction perpendicular to the interface (identified with the x-y plane) is then constrained by a deep potential well with a minimum on the interface. Quantum theory predicts that electrons of sufficiently low energy, i.e., at low enough temperatures, remain bound to the interface and form a very nearly two-dimensional system.

In a theoretical analysis of the Hall effect it is advantageous to consider the connection between the electric current density $\mathbf{j}(\mathbf{x}) = (j^1(\mathbf{x}), j^2(\mathbf{x}))$ and the electric field $\mathbf{E}(\mathbf{x}) = (E_1(\mathbf{x}), E_2(\mathbf{x}))$ at an arbitrary point $\mathbf{x} = (x^1, x^2) \equiv (x, y)$ of Ω which is given by the *Ohm-Hall law*

$$\mathbf{E}(\mathbf{x}) \;=\; \rho\,\mathbf{j}(\mathbf{x}), \quad \rho \;=\; \begin{pmatrix} \rho_{xx} & -\rho_H \\ \rho_H & \rho_{yy} \end{pmatrix}, \tag{2.3}$$

where $\rho_{xx} = R_L(\ell_y/\ell_x)$, $\rho_{yy} = R_L(\ell_x/\ell_y)$ are the two longitudinal resistivities, $\rho_H = R_H$ is the Hall resistivity, and ℓ_x, ℓ_y are the widths of the system in the x- and y-directions, respectively. This is a phenomenological law valid on macroscopic distance scales and at low frequencies.

It is convenient to introduce a dimensionless quantity, the so-called *filling factor* ν, by setting

$$\nu \;=\; n/(eB_c/hc), \tag{2.4}$$

where $\frac{hc}{e}$ is the quantum of magnetic flux. Then the classical Hall law (2.2) says that R_H^{-1} rises *linearly* in ν, $R_H^{-1} = \frac{e^2}{h}\nu$, the constant of proportionality being given by a constant of nature, $\frac{e^2}{h}$. Because, experimentally, B_c can be varied and n can be varied (by varying the gate voltage), this prediction of classical theory can be put to experimental tests. Experiments at very low temperatures and for rather pure inversion layers yield the following very surprising data shown in Figure 2 [58], [54], [9].

These data tell us the following:

(1) $\sigma_H := \frac{h}{e^2}\, R_H^{-1}$ (the dimensionless Hall conductivity) has *plateaux at certain rational heights.* The plateaux at integer height occur with an astronomical precision of 1:10^8 (defining a new standard for conductivity and yielding perhaps the most precise experimental value for the fine structure constant $\alpha = 2\pi\, e^2/hc \simeq 1/137$). The plateau quantization is insensitive to sample preparation and geometry.

(2) When (ν, σ_H) belongs to a plateau the longitudinal resistance R_L very nearly *vanishes.* This means that, for such values of ν and σ_H, there are no dissipative processes in the system.

The remarkable nature of these facts has been expressed by Laughlin [41] as follows: "The exactness of these results and their apparent insensitivity to the type

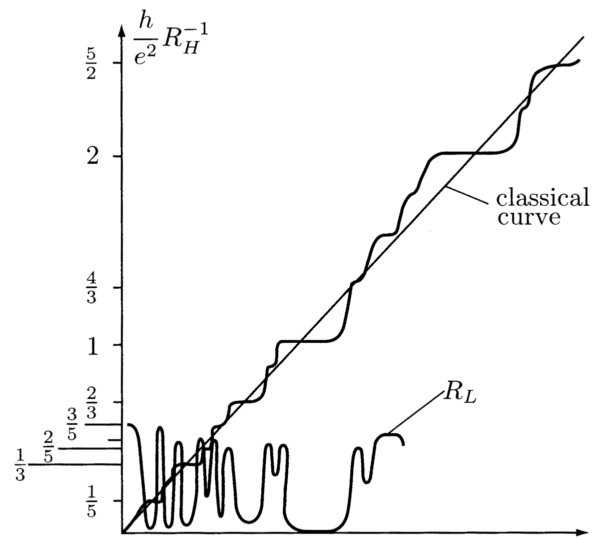


Figure 2

or location of impurities suggest that the effect is due, ultimately, to a fundamental principle."

It is the main purpose of this lecture to uncover some aspects of that principle. We shall be modest and focus our attention on the explanation of why σ_H must be a rational number when R_L vanishes, which rational numbers may occur, and what properties the system has when $R_L = 0$ and σ_H takes an allowed rational value.

As a first step, we formulate the classical electrodynamics of a two-dimensional system of electrons in an external electromagnetic field $(\vec{E}, \vec{B}_{\text{tot.}})$ when $R_L = 0$, and for an arbitrary value of σ_H. Here $\vec{E}$ is an external electric field, and $\vec{B}_{\text{tot.}} = \vec{B}_c + \vec{B}$, where $\vec{B}_c$ is a constant, external magnetic field transversal to the plane of the system, and $\vec{B}$ is a small, nonconstant perturbation of $\vec{B}_c$. As long as we do not describe the dynamics of the spins of the electrons — which are quantum-mechanical degrees of freedom — the laws of electromagnetism in such a system only involve $\mathbf{E} = (E_1, E_2)$, the component of $\vec{E}$ parallel to the plane of the system, and $B_{\text{tot.}} = B_c + B$, the component of $\vec{B}_{\text{tot.}}$ perpendicular to the plane of the system. Because R_L is assumed to vanish, eq. (2.3) can be rewritten as

(i) *Hall's law.*
$j^k(x) = \sigma_H\, \varepsilon^{k\ell} E_\ell(x)$, $x = (\mathbf{x}, t)$, with $k, \ell = 1, 2$, and $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, in units where $e = h = 1$.

More fundamental are the following two laws:

(ii) *Charge conservation.*
$\frac{\partial}{\partial t}\, j^0(x) + \boldsymbol{\nabla} \cdot \mathbf{j}(x) = 0$ (continuity equation for the electric charge density j^0 and the electric current density $\mathbf{j}$).

(iii) *Faraday's induction law.*
$\frac{\partial}{\partial t}\, B(x) + \boldsymbol{\nabla} \wedge \mathbf{E}(x) = 0$.

Combining (i), (ii), and (iii), we find that
$\frac{\partial}{\partial t}\, j^0(x) = \sigma_H\, \frac{\partial}{\partial t}\, B(x).$
Defining j^0 to be the difference between the total electric charge density and the uniform background density, n, we obtain the following result [20].
(iv) *Charge-flux relation.*
$j^0(x) = \sigma_H\, B(x).$
The laws (i)–(iv) are generally covariant and metric independent (topological) [20]. Integrating (iv) over all of space Ω, we conclude that

$$q_{\text{el}} \;=\; \sigma_H\, \Phi, \tag{2.5}$$

where $q_{\text{el}} = \int_\Omega d^2\mathbf{x}\; j^0(\mathbf{x},t)$ is the total (excess) electric charge of the system, and $\Phi = \int_\Omega d^2\mathbf{x}\; B(\mathbf{x},t)$ is the total (excess) magnetic flux passing through the system.

These simple, beautiful laws, (i)–(iv), are the starting point of our analysis. They remain valid in a quantum-mechanical treatment of the electrons, see Section 3, that leads to rather remarkable conclusions. Let me anticipate the main results of our analysis and discuss their consequences. To do this, I must recall what *integral Euclidian lattices* are.

Let V be a vector space over the rational number field equipped with a positive-definite inner product $\langle\cdot,\cdot\rangle$. In V we choose a basis $\{\mathbf{e}_i\}_{i=1}^N$, $N = \dim V$, with integral Gram matrix K, where

$$K_{ij} \;=\; K_{ji} \;=\; \langle \mathbf{e}_i, \mathbf{e}_j \rangle \;\in\; \mathbb{Z}, \tag{2.6}$$

for all $i,j = 1,\ldots,N$. The basis $\{\mathbf{e}_i\}_{i=1}^N$ generates an *integral Euclidian lattice* Γ defined by

$$\Gamma \;=\; \Big\{ \mathbf{q} \;=\; \sum_{i=1}^{N} q^i\, \mathbf{e}_i \;:\; q^i \;\in \mathbb{Z},\;\; \forall\, i \Big\}\;. \tag{2.7}$$

The lattice Γ^* *dual* to Γ, i.e., the lattice of integer-valued linear forms on Γ, is given by

$$\Gamma^* \;=\; \Big\{ \mathbf{n} \;=\; \sum_{i=1}^{N} n_i\, \boldsymbol{\varepsilon}^i \;:\; n_i \in \mathbb{Z}, \forall\, i \Big\}, \tag{2.8}$$

where $\{\varepsilon^i\}_{i=1}^N$ is the basis of V dual to $\{\mathbf{e}_i\}_{i=1}^N$, i.e.,

$$\boldsymbol{\varepsilon}^i \;=\; \sum_{j=1}^{N} (K^{-1})^{ij}\, \mathbf{e}_j\;, \tag{2.9}$$

and

$$(K^{-1})^{ij} \;=\; \left\langle \boldsymbol{\varepsilon}^i, \boldsymbol{\varepsilon}^j \right\rangle \;=\; \frac{1}{\Delta}\, \widetilde{K}^{ij}, \tag{2.10}$$

where

$$\Delta \;=\; \det K \;=\; \big|\, \Gamma^*/\Gamma\, \big| \tag{2.11}$$

is the discriminant of Γ, and $\widetilde{K}$ is the matrix of cofactors (Kramer's rule).

The matrix K is positive-definite, with rank $(K) = N$, if and only if $\langle\cdot,\cdot\rangle$ is positive-definite. The lattice Γ is called *odd* iff it contains an element $\mathbf{q}$, with $\langle \mathbf{q},\mathbf{q}\rangle \in 2\,\mathbb{Z}+1$. Thus, Γ is odd iff K_{ii} is odd, for at least one $i \in \{1,\ldots,N\}$.

We are now in a position to state our *main contention.* Consider a two-dimensional system of electrons in a uniform, external magnetic field $\vec{B}_c$ at a temperature $T \approx 0$ K, with the property that R_L vanishes. Following Laughlin, we call such a system an *incompressible quantum Hall fluid*, abbreviated as IQHF. We claim that the physics of an IQHF on very large distance scales and at very low frequencies (i.e., in the so-called *scaling limit*) is coded into the data $(\Gamma_e, \mathbf{Q}_e)$ and $(\Gamma_h, \mathbf{Q}_h)$, where

(i) Γ_e and Γ_h are two integral, odd Euclidian lattices, and
(ii) for $x = e, h$, $\mathbf{Q}_x$ is a *primitive, odd* vector in Γ^*_x.

A vector $\mathbf{Q} \in \Gamma^*$ is called *primitive*, or visible, iff g.c.d. $\big(\langle \mathbf{Q}, \mathbf{e}_j\rangle\big)_{j=1}^N = 1$, and $\mathbf{Q}$ is called *odd* iff

$$\big\langle \mathbf{Q},\mathbf{q}\big\rangle \equiv \big\langle \mathbf{q},\mathbf{q}\big\rangle \mod 2, \quad \forall \mathbf{q} \in \Gamma\,. \tag{2.12}$$

The dimensionless Hall conductivity σ_H is then given by

$$\sigma_H = \sigma_e - \sigma_h, \tag{2.13}$$

where

$$\sigma_x = \big\langle \mathbf{Q}_x, \mathbf{Q}_x\big\rangle, \quad \text{for } x = e, h. \tag{2.14}$$

This proves immediately that σ_H is a *rational number.* We shall denote it by

$$\sigma_H = \frac{n_H}{d_H}\,, \quad \text{with g.c.d. } (n_H, d_H) = 1.$$

At this point, there is the danger that our theory predicts far too many possible rational values of σ_H. However, what our theory really says is that if $R_L = 0$ then σ_H must belong to a certain subset $\mathbb{S}$ of the rational numbers, and that if $R_L = 0$ at some value of σ_H belonging to $\mathbb{S}$ then the properties of the system are encoded in some pair, $(\Gamma_e, \mathbf{Q}_e)$ and $(\Gamma_h, \mathbf{Q}_h)$, of integral Euclidian lattices and primitive vectors in their duals. Typically it happens that there are *many* pairs, $(\Gamma_e, \mathbf{Q}_e)$ and $(\Gamma_h, \mathbf{Q}_h)$, corresponding to a given value of σ_H in $\mathbb{S}$. Whether a pair $(\Gamma_e, \mathbf{Q}_e)$, $(\Gamma_h, \mathbf{Q}_h)$ describes an incompressible quantum Hall fluid that can be realized in a laboratory is a complicated analytical problem of quantum mechanics to which our theory can only give a tentative answer! Thus, it is very likely that many points in $\mathbb{S}$ do *not* correspond to the Hall conductivity σ_H of a *real* IQHF.

The subscripts "e" and "h" refer to the following *physics*: the basic charge carriers in a quantum Hall fluid (QHF) can be mobile *electrons* of electric charge $-e$. If $R_L \doteq 0$ the fluid is then described by a pair $(\Gamma_e, \mathbf{Q}_e)$. They could also be mobile *holes* ("missing electrons") of charge $+e$, in which case the IQHF is described by $(\Gamma_h, \mathbf{Q}_h)$. Finally, an IQHF could be composed of *two* fluids, one consisting of mobile electrons, the other one consisting of mobile holes. It is a natural, *physical* idea that, for small values of the filling factor, these two fluids do not mix. We shall assume this henceforth (but see [22], [27] for a more general analysis also involving (indecomposable) Lorentzian lattices). The IQHF is then described by a pair $(\Gamma_e, \mathbf{Q}_e)$, $(\Gamma_h, \mathbf{Q}_h)$. As the electric charge of an electron is $-e$ and the one of a hole is $+e$, there is a relative minus sign between σ_e and σ_h in eq. (2.13)[1]. As there is a complete symmetry between electrons and holes, it is sufficient to develop the theory of QH fluids composed of electrons, and we set $\sigma_H := \sigma_e$ and drop the subscript "e" henceforth.

A pair $(\Gamma, \mathbf{Q})$, where Γ is an integral, odd Euclidian lattice and $\mathbf{Q}$ is a primitive, odd vector in Γ^* satisfying (2.12), is called a *chiral quantum Hall lattice*(cQHL). Our task is to classify cQHL's and to compare the predictions of the theory with experimental data.

The success of the theory is quite impressive: In Figure 3 we display measured values of σ_H when $R_L \approx 0$ (i.e., for IQHF's) in the range $0 < \sigma_H \leq 1$ that have been reported in the literature [9], [53] (for so-called single-layer, narrow-well IQHF's). We divide the data into separate "windows", Σ_p, $p = 1, 2, 3, \dots$, and each window Σ_p is the union of a left window $\Sigma_p^<$ and a right window $\Sigma_p^>$. Well-established plateau values of σ_H (i.e., values of σ_H corresponding to some IQHF) are indicated as a $\bullet$. Values of σ_H where R_L has a clearly visible local minimum ≈ 0, and σ_H has an inflection point as a function of the filling factor ν are indicated as a $\circ$. Very weak, or controversial data are indicated by **..** Finally, the symbol *p.t.* indicates that, to such a value of σ_H, there correspond several distinct IQHF's, i.e., there are *phase transitions* between two or more different IQHF's with the same σ_H.

The remarkable fact is that these data — in particular the *absence* of data points — are very accurately reproduced by our theory of cQHL, see [28], [22], [27], if a *heuristic principle of stability* of a cQHL is introduced: the stability of a cQHL is intended to be a measure for the stability of the corresponding quantum-mechanical state of an IQHF under small perturbations, such as changes of the filling factor ν, see (2.4), or of the density of "impurities" in the system, etc. In order to formulate our stability principle for cQHL's mathematically, we must introduce some numerical *invariants* of cQHL's. The most primitive invariant of a cQHL $(\Gamma, \mathbf{Q})$ is the *dimension* N of the lattice Γ. Next, let

$$\Gamma = \bigoplus_{j=1}^{k} \Gamma_j \tag{2.15}$$

be the finest decomposition of the lattice Γ into an orthogonal direct sum of sublattices $\Gamma_j, j = 1, \dots, k$, and let

$$\mathbf{Q} = \sum_{j=1}^{k} \mathbf{Q}^{(j)}, \quad \mathbf{Q}^{(j)} \in \Gamma_j^*, \tag{2.16}$$

be the decomposition of $\mathbf{Q}$ associated to (2.15). We say that a cQHL $(\Gamma, \mathbf{Q})$ is *primitive* iff $\mathbf{Q}^{(j)}$ is a (nonvanishing) primitive vector of Γ_j^*, for all $j = 1, \dots, k$. This

[1]Historically, the existence of holes in semiconductors was first discovered in measurements of the sign of R_H!

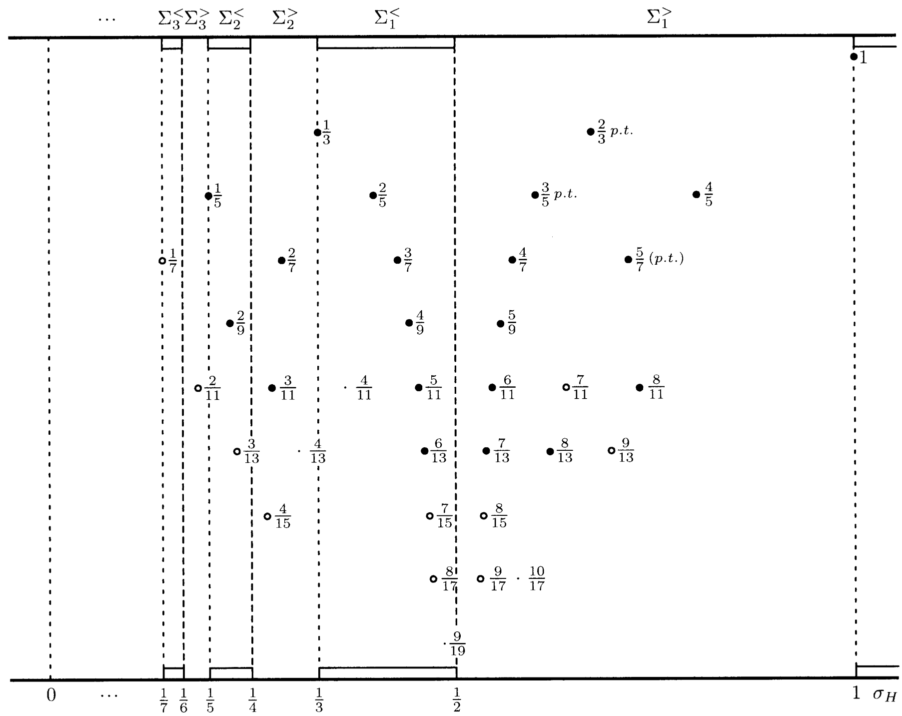


Figure 3. Observed Hall fractions σ_H in the range $0 < \sigma_H \leq 1$ and their experimental status in single-layer quantum Hall systems.

means that the pairs $(\Gamma_j, \mathbf{Q}^{(j)})$ are *indecomposable* cQHL's. Every indecomposable cQHL $(\Gamma_0, \mathbf{Q}_0)$ has a basis $\{\mathbf{q}_1, \dots \mathbf{q}_{N_0}\}$ with the property that $\langle \mathbf{Q}_0, q_\ell \rangle = -1$, for all $\ell = 1, \dots, N_0$. The set of all such bases is denoted by $\mathcal{B}(\Gamma_0, \mathbf{Q}_0)$. We then define an invariant $\ell_{\max.}$ (called "relative-angular-momentum invariant" [28]) by setting

$$\ell_{\max.}(\Gamma_0, \mathbf{Q}_0) := \min_{\{\mathbf{q}_i\}_{i=1}^{N_0} \in \mathcal{B}(\Gamma_0, \mathbf{Q}_0)} \Big(\max_{1 \leq i \leq N_0} \langle \mathbf{q}_i, \mathbf{q}_i \rangle \Big). \tag{2.17}$$

If $(\Gamma, \mathbf{Q})$ is a *decomposable*, primitive cQHL, i.e.,

$$(\Gamma, \mathbf{Q}) = \bigoplus_{j=1}^{k} \big(\Gamma_j, \mathbf{Q}^{(j)} \big), \tag{2.18}$$

as in (2.15), (2.16), we define

$$\ell_{\max.}(\Gamma, \mathbf{Q}) = \max_{1 \leq j \leq k} \ell_{\max.}\big(\Gamma_j, \mathbf{Q}^{(j)} \big). \tag{2.19}$$

Our *stability principle* for cQHL's says that an incompressible quantum Hall fluid corresponding to a primitive, chiral quantum Hall lattice $(\Gamma, \mathbf{Q})$ is the *more stable*, the *smaller* the value of the invariant $\ell_{\max.}(\Gamma, \mathbf{Q})$ and the *smaller* its dimension N. Available experimental data suggest that

$$\ell_{\max.}(\Gamma, \mathbf{Q}) \ \leq \ 7, (\text{or } 9), \tag{2.20}$$

for an *arbitrary* cQHL $(\Gamma, \mathbf{Q})$ describing a *physically realizable* IQHF. This is confirmed, qualitatively, by heuristic theoretical and numerical arguments [27]. Furthermore, there are fairly convincing, but heuristic theoretical arguments suggesting that, for a *real* IQHF with a nonzero density of impurities of finite strength, the dimension N of the corresponding cQHL is bounded above by a *finite* integer, N_*, depending on the filling factor ν, the density of electrons, and the density and strength of the "impurities", with $N_* \to \infty$, as the density of "impurities" tends to 0.

It is an elementary result in the theory of chiral quantum Hall lattices that the total number of cQHL's, $(\Gamma, \mathbf{Q})$, with $\ell_{\max.}(\Gamma, \mathbf{Q}) \leq \ell_*$ and $N = \dim \Gamma \leq N_*$, for arbitrary finite values of ℓ_*, N_*, is *finite* (though rapidly growing in ℓ_*, N_*).

A simple consequence of the Cauchy-Schwarz inequality tells us that the Hall conductivity σ_H of an IQHF corresponding to a cQHL $(\Gamma, \mathbf{Q})$ obeys the inequality

$$\sigma_H \ \equiv \ \sigma_H(\Gamma, \mathbf{Q}) \ = \ \langle \mathbf{Q}, \mathbf{Q} \rangle \ \geq \ \ell_{\max.}(\Gamma, \mathbf{Q})^{-1}. \tag{2.21}$$

This bound has interesting consequences (confirming a prejudice of Mark Kac [36]): if $\sigma_H \in \Sigma_p$, i.e.,

$$\frac{1}{2p+1} \ \leq \ \sigma_H \ (\Gamma, \mathbf{Q}) \ < \ \frac{1}{2p-1},$$

then

$$\ell_{\max.}(\Gamma, \mathbf{Q}) \ \geq \ 2p+1. \tag{2.22}$$

Our stability principle for cQHL's then says that the *most stable* IQHF's with $\sigma_H \in \Sigma_p$ are those described by cQHL's $(\Gamma, \mathbf{Q})$ satisfying

$$\ell_{\max.}(\Gamma, \mathbf{Q}) \ = \ 2p+1 \quad (N \text{ as small as possible}). \tag{2.23}$$

Combining the universal upper bound (2.20), i.e., $\ell_{\max.}(\Gamma, \mathbf{Q}) \leq 7$, with the bound (2.21), we conclude that there should not exist any physically realizable IQHF's with $\sigma_H < \frac{1}{7}$, and that, for σ_H in the window Σ_3, $\ell_{\max.}(\Gamma, \mathbf{Q})$ *must* take the smallest possible value compatible with (2.21), i.e., $\ell_{\max.}(\Gamma, \mathbf{Q}) = 7$. These conclusions are compatible with the data displayed in Figure 3.

The family of all primitive cQHL's $(\Gamma, \mathbf{Q})$, with $\sigma_H(\Gamma, \mathbf{Q}) \in \Sigma_p$ and $\ell_{\max.}(\Gamma, \mathbf{Q}) = 2p+1$ (the smallest possible value), is henceforth denoted by $\mathcal{H}_p$. In [22] we have proven an easy, yet remarkable theorem that says that there exist *bijections*, called "*shift maps*",

$$\mathcal{S}_p \ : \ \mathcal{H}_1 \to \mathcal{H}_{p+1}, \ \ p \ = \ 1, 2, 3, \ldots, \tag{2.24}$$

between the cQHL's in $\mathcal{H}_1$ and those in $\mathcal{H}_{p+1}$, with the properties that

$$\sigma_H\big(\mathcal{S}_p(\Gamma,\mathbf{Q})\big)^{-1} = \sigma_H(\Gamma,\mathbf{Q})^{-1} + 2p,$$

and

$$\ell_{\max.}\big(\mathcal{S}_p(\Gamma,\mathbf{Q})\big) = \ell_{\max.}(\Gamma,\mathbf{Q}) + 2p. \tag{2.25}$$

Furthermore, we have proven a somewhat deeper, but still rather easy *uniqueness theorem*[22]: let

$$\mathcal{H}_p^< := \big\{(\Gamma,\mathbf{Q}) \in \mathcal{H}_p \ : \ \sigma_H(\Gamma,\mathbf{Q}) \in \Sigma_p^<\big\}. \tag{2.26}$$

Then *all* the cQHL's $(\Gamma,\mathbf{Q})$ in $\mathcal{H}_p^<$ are known explicitly: the possible values in $\Sigma_p^<$ of the Hall conductivity σ_H corresponding to IQHF's described by cQHL's in $\mathcal{H}_p^<$ are given by the formula

$$\sigma_H = \frac{N}{2p\,N+1}, \tag{2.27}$$

and to each $N = 1,2,3,\dots$, with σ_H given by (2.27), there corresponds a *unique* cQHL, $(\Gamma_{N,p},\mathbf{Q})$, of dimension N, and there are no further cQHL's in $\mathcal{H}_p^<$!

Note that it follows from the bound (2.20) on $\ell_{\max.}$ that $\mathcal{H}_p^<$ contains *all* possible cQHL's with $\sigma_H \in \Sigma_p^<$ (as given by (2.27)), for $p = 3$.

The lattices $(\Gamma_{N,p},\mathbf{Q})$ with $\sigma_H(\Gamma_{N,p},\mathbf{Q}) = \langle\mathbf{Q},\mathbf{Q}\rangle = N(2pN+1)^{-1}$ can be described as follows: the lattice $\Gamma_{N,p}$ has a basis $\{\mathbf{q},\mathbf{e}_1,\dots,\mathbf{e}_{N-1}\}$ with the property that

$$\langle\mathbf{Q},q\rangle = -1, \quad \langle\mathbf{Q},\mathbf{e}_j\rangle = 0, \quad j = 1,\dots,N-1, \tag{2.28}$$

and with a Gram matrix K given by

$$K = \begin{pmatrix} 2p+1 & -1 & & & & \\ -1 & 2 & -1 & & & 0 \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ 0 & & & & & -1 \\ & & & & -1 & 2 \end{pmatrix}, \tag{2.29}$$

where $2p+1 = \langle\mathbf{q},\mathbf{q}\rangle$, and $K_{i+1,j+1} = \langle\mathbf{e}_i,\mathbf{e}_j\rangle$ are the matrix elements of the A_{N-1}-Cartan matrix. Thus, the Witt sublattice [8] of $\Gamma_{N,p}$ is the A_{N-1}-root lattice, and it is natural to call the series $(\Gamma_{N,p},\mathbf{Q}) \in \mathcal{H}_p^<$, $N = 1,2,3,\dots$, of cQHL's the *fundamental A-series* in the window Σ_p. The cQHL's $(\Gamma_{N,p},\mathbf{Q})$ described here are typical examples of a general class of so-called *maximally symmetric* cQHL's [28], [27], which can be classified. The shift map $\mathcal{S}_{p-1}$ acts on the A-series in $\mathcal{H}_1^<$ by replacing $K_{11} = 3$ by $K_{11} = 2p+1$ and leaving the other matrix elements in the Gram matrices unchanged.

If you compare these results with the data in the windows $\Sigma_p^<$ of Figure 3 and recall that an IQHF is the less stable, the larger the values of p and N of the corresponding cQHL, the agreement between theory and experiment is remarkable. Is there a problem with the data point at $\sigma_H = \frac{4}{11} \in \Sigma_1^<$? There are no cQHL's with $\sigma_H = \frac{4}{11}$ and $\ell_{\text{max.}} = 3$, but there actually are at least two distinct, low-dimensional cQHL's, with $\sigma_H = \frac{4}{11}$ and $\ell_{\text{max.}} = 5(!)$, one obtained by applying the shift map $\mathcal{S}_1$ to the lattice $\mathbb{Z} \oplus 3\mathbb{Z}$, hence of dimension 2, and another one of dimension 7 (among, perhaps, further lattices of high dimension). Because, for these lattices, $\ell_{\text{max.}}$ does *not* have the minimal value, 3, allowed in the window Σ_1, an IQHF with $\sigma_H = \frac{4}{11}$ is expected to be quite unstable against perturbations.

To the mathematician, the results just described may look disappointing, because they do not involve interesting lattices. The situation changes when we study the cQHL's belonging to the family $\mathcal{H}_p^> := \mathcal{H}_p \backslash \mathcal{H}_p^<$, corresponding to the range $\Sigma_p^>$ of values of σ_H. Because the shift map $\mathcal{S}_{p-1}$ is a bijection between $\mathcal{H}_1^>$ and $\mathcal{H}_p^>$, $p = 2, 3, 4, \dots$, the classification of the *most stable* cQHL's with $\sigma_H \in \Sigma_p^>$, that is of all the lattices in $\mathcal{H}_p^>$, reduces to the classification of lattices in $\mathcal{H}_1^>$. But this is not an easy job. Although the number of cQHL's in $\mathcal{H}_1^>$ of dimension $N < N_*$ is *finite*, it grows rapidly in N_*.

In order to make progress, one may attempt to translate physical properties of IQHF's (related e.g. to electron spin, or to the spectrum of quasi-particles in such systems) into mathematical properties of quantum Hall lattices (related to the structure of their Witt sublattices and of the so-called glue group; see [28], [22], [27]). This enables one to introduce subfamilies of quantum Hall lattices, likely to describe physically realizable IQHF's, whose classification is feasible.

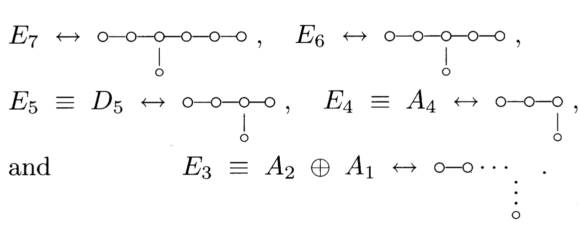
A prominent *finite* series of cQHL's in $\mathcal{H}_1^>$ is the one corresponding to the values

$$\sigma_H = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}. \tag{2.30}$$

It is called the *E-series*, for the following reasons. Let $\mathcal{O} \oplus \Gamma_W$ denote the Kneser shape [8] of an integral lattice Γ,

$$\mathcal{O} \oplus \Gamma_W \subseteq \Gamma \subseteq \Gamma^* \subseteq \mathcal{O}^* \oplus \Gamma_W^*,$$

where Γ_W is the Witt sublattice generated by vectors of squared lengths 1 and 2. To every σ_H in the E-series (2.30) there corresponds a cQHL $(\Gamma, \mathbf{Q})$ with the property that the $\mathcal{O}$-sublattice in its Kneser shape is a one-dimensional, odd lattice, denoted $\mathcal{O}_k$, $\Gamma_W \equiv \Gamma_{E_k}$ is an E_k-root lattice, with $k = 7, 6, 5, 4, 3$, and $\mathbf{Q} \in \mathcal{O}_k^*$ is orthogonal to Γ_W [28]. Here we define the lattices Γ_{E_k} as the root lattices of the Lie algebras corresponding to the following Dynkin diagrams:



There is also a cQHL with $\sigma_H = \frac{7}{13}$ and $\ell_{\text{max.}} = 3$. It has a two-dimensional $\mathcal{O}$-sublattice, and its Witt sublattice is the A_1-root lattice. This cQHL may be viewed as an irregular endpoint of the E-series. For there is *no* cQHL with $\sigma_H = \frac{8}{15}$ and $\ell_{\max.} = 3$ in dimension $N \leq 4$, or with discriminant $\triangle \leq 15$ and $N \leq 9$.

A lattice Γ is obtained from its Kneser shape, $\mathcal{O} \oplus \Gamma_W$, by gluing, namely by adding cosets of vectors in $\mathcal{O}^* \oplus \Gamma_W^*$, to $\mathcal{O} \oplus \Gamma_W$. The lattices Γ_k obtained from $\mathcal{O}_k \oplus \Gamma_{E_k}$, where Γ_{E_k} is the E_k-root lattice, $k = 7, 6, 5$, are unlikely to correspond to physically realizable IQHF's, as their dimensions (and the number of quasi-particles of the corresponding IQHF's) are large. However, they contain quantum Hall *sublattices*, with the *same* values for σ_H and $\ell_{\max.}$, which *are* realistic. For example, for $k = 7$, $\sigma_H = \frac{2}{3}$, the cQHL obtained from $\mathcal{O}_7 \oplus \Gamma_{E_7}$ by gluing contains a decomposable, two-dimensional QH sublattice, $3\mathbb{Z} \oplus 3\mathbb{Z}$, and an indecomposable, three-dimensional QH sublattice, whose Witt sublattice is the A_1-root lattice which, physically, could describe electron spin [28], or an internal symmetry that we call "isospin" symmetry — as well as less realistic sublattices of dimension 4, 5, 6, and 7. All these sublattices yield cQHL's with $\sigma_H = \frac{2}{3}$, $\ell_{\max.} = 3$. We thus predict that there should be *at least three* rather stable IQHF's with $\sigma_H = \frac{2}{3}$. They differ from each other in the rôle played by electron spin (which can be tuned by tilting the external magnetic field $\vec{B}_c$) or by "isospin". One therefore expects a magnetic-field driven phase transition between different IQHF's with $\sigma_H = \frac{2}{3}$. These predictions of our theory are in remarkable agreement with experimental data.

There is also a *D-series* of cQHL's, leading, e.g., to values of $\sigma_H = \frac{n_H}{d_H}$ with an *even* denominator d_H: $\sigma_H = \frac{1}{2}$ (arbitrary D_n), and $\sigma_H = \frac{4}{12-n}$, corresponding to $\Gamma_W = \Gamma_{D_n}$ with $n \leq 7$. Let $(\Gamma, \mathbf{Q})$ be a primitive cQHL. It has been shown in [28] that the sublattice of Γ orthogonal to $\mathbf{Q}$ cannot contain any self-dual lattice.

Besides the D- and the E-series, there is also an A_{N-1}-series of cQHL's in $\mathcal{H}_1^>$ that could describe single-layer IQHF's if N is an odd integer ≥ 5. They yield the values

$$\sigma_H = \frac{N}{N+4} \tag{2.31}$$

of the Hall conductivity $\left(\frac{5}{9}, \frac{7}{11}, \frac{9}{13}, \ldots\right)$.

Furthermore, we have classified *all* two-dimensional, three-dimensional, and four-dimensional cQHL's in $\mathcal{H}_1^>$; see [27]. (With an efficient computer program one could extend these results to $N = 5, 6$.) They correspond to the values $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{13}$ $(N = 3)$, and $\frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \frac{4}{5}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{5}{8}, \frac{5}{9}, \frac{6}{11}, \frac{8}{11}, \frac{10}{11}, \frac{11}{13}, \frac{13}{17}, \frac{14}{19}, \frac{6}{21}$, and $\frac{26}{31}$ $(N = 4)$.

Besides the lattices discussed above, there are plenty of *decomposable* cQHL's in $\mathcal{H}_1^>$ obtained as the direct sum of *two* cQHL's of the fundamental A-series of cQHL's in $\mathcal{H}_1^<$. They correspond to the sequence

$$\sigma_H = \frac{4NM + N + M}{(2N+1)(2M+1)}\,, \quad N, M = 1, 2, 3, \ldots, \tag{2.32}$$

of values of the Hall conductivity. Because there is a very stable single-layer IQHF with $\sigma_H = 1$, described by $(\Gamma = \mathbb{Z}, \mathbf{Q} = 1)$, one does not expect to see plateaux in the Hall conductivity around the points given in (2.32), for values of N and M larger than 2 or 3.

Finally, our theory provides candidates of IQHF's described by *pairs* $(\Gamma_e, \mathbf{Q}_e)$ and $(\Gamma_h, \mathbf{Q}_h)$ of cQHL's corresponding to values of $\sigma_H = \sigma_e - \sigma_h$ in the window $\Sigma_1^>$. These IQHF's would be charge-conjugate to those described by the fundamental A-series in $\mathcal{H}_1^<$. They are obtained by setting $\Gamma_e = \mathbb{Z}$, $\mathbf{Q}_e = 1$, $\Gamma_h = \Gamma_{N,1}$; see (2.27), (2.28). One finds that

$$\sigma_H = \sigma_e - \sigma_h = 1 - \frac{N}{2N+1}, \quad N = 1,2,3,\ldots \tag{2.33}$$

For $N \leq 6$, these values of σ_H coincide with the ones of the E-series. The existence (and uniqueness) of these pairs of cQHL's makes it plausible that $\sigma_H = \frac{6}{11}, \frac{7}{13}, \frac{8}{15}, \frac{9}{17}$ are values of the Hall conductivity of physically realizable IQHF's.

Those values of σ_H that correspond to *several* cQHL's in $\mathcal{H}_1^>$ (e.g. $\frac{2}{3}, \frac{3}{5}, \frac{4}{5}, \frac{5}{7}, \ldots$) tend to be values where, experimentally, phase transitions are observed.

We emphasize that, logically, our theory predicts the values of σ_H that *cannot* appear in IQHF's — indeed, it predicts plenty of gaps if bounds on $\ell_{\text{max.}}$ and N are imposed. (For example, it tells us that values of $\sigma_H = \frac{n_H}{d_H}$, with d_H very large, require large values of either $\ell_{\text{max.}}$ or N and hence should *not* be observed!) Furthermore, it tells us that *if* an *allowed* value of σ_H *is* observed in an IQHF, the structure of the IQHF can be described by a certain set of cQHL's. That's all our theory does if no *heuristic* principles are added to it.

Next, we propose to sketch how the physics of IQHF's leads us to study the mathematics of chiral quantum Hall lattices.

3 From incompressible quantum Hall fluids to chiral quantum Hall lattices via Chern-Simons theory

The starting point of our analysis is the idea to look for a theoretical description of the physics of an IQHF in the limiting regime of large-distance and long-time (low-frequency) scales. This limiting regime is called the *scaling limit* of the system, and experience shows that the theoretical description of physical systems simplifies in the scaling limit. An IQHF can be characterized by the following physical properties.

(P1) The temperature T of the system is close to 0 K. The longitudinal resistance, R_L, of an IQHF at $T = 0$ K vanishes, and the total electric charge is a good quantum number to label quantum-mechanical state vectors of the system [28], [19]. The charge of the groundstates of the system is normalized to be zero.

(P2) In the scaling limit, the total electric charge and current densities of an IQHF are the sum of $N = 1,2,3,\ldots$ separately conserved charge and current densities describing electron and/or hole transport in N separate "channels" distinguished by conserved quantum numbers. In our analysis, N will be treated as a free parameter. (Physically, N turns out to depend on the filling factor ν and other parameters characterizing the system.)

(P3) In units where $e = h = 1$, the electric charge of an electron/hole is $-1/1$. A local excitation of the system composed of electrons and holes and of total electric charge $q_{\text{el.}}$ satisfies *Fermi-Dirac statistics* if $q_{\text{el.}}$ is *odd* and *Bose-Einstein statistics* if $q_{\text{el.}}$ is *even.*

The quantum statistics of any local excitation of the system of electric charge $q_{\mathrm{el.}} \in 2\,\mathbb{Z}+1$ must be Fermi-Dirac statistics (i.e., the Pauli principle must hold), and if $q_{\mathrm{el.}} \in 2\mathbb{Z}$ it must be Bose-Einstein statistics.

(P4) The quantum-mechanical state vector describing an *arbitrary* physical state of an IQHF is *single valued* in the position of all those excitations that are multi-electrons/-holes.

The properties (P1)–(P4), believed to be true in every IQHF, are physical properties. Part of the art of theoretical physics is to translate physical properties, deduced from experiments, into precise mathematical hypotheses. This cannot be done in the form of theorems and requires intuition. But once this exciting part of the job is completed, one must attempt to use mathematical theorems to derive new predictions on the behavior of a physical system.

The assumption that the longitudinal resistance R_L of an IQHF vanishes is translated into the mathematical assumption that the energy spectrum of the quantum-mechanical Hamiltonian describing the dynamics of the system exhibits what is called a *mobility gap* δ above the groundstate energy which is *strictly positive*, uniformly in the size of the system. This is actually an assumption that one can try to derive from the underlying microscopic Schrödinger quantum mechanics of nonrelativistic electrons. This is a difficult, but *not* hopelessly difficult, problem of analysis; see [15] and references given there.

The quantum-mechanical electric charge and current densities of a physical system are operator-valued distributions

$$j(x) \;=\; \big(j^0(x),\, j^1(x), \ldots, j^d(x)\big), \tag{3.1}$$

where d is the dimension of physical space, and $x = (\vec{x}, t)$ is a space-time point. They satisfy the continuity equation (conservation of electric charge)

$$\frac{\partial}{\partial t}\, j^0(x) + \vec{\nabla} \cdot \vec{j}(x) \;=\; 0. \tag{3.2}$$

Let $J(x) = *j(x)$ be the d-form dual to j. Then (3.2) says that

$$dJ(x) \;=\; 0. \tag{3.3}$$

For a two-dimensional system confined to a disk $\Omega \subseteq \mathbb{R}^2$, the Poincaré lemma tells us that (3.3) implies that

$$J(x) \;=\; db(x), \tag{3.4}$$

where $b(x)$ is a 1-form; b is determined by J up to the gradient of a scalar distribution χ, i.e., b has the properties of an abelian gauge field. By property (P2)

$$J(x) \;=\; \sum_{i=1}^{N} Q_i\, J^i(x), \tag{3.5}$$

where Q_i is the unit of electric charge transported by the current J^i, and J^i satisfies the continuity equation

$$d\, J^i(x) \;=\; 0, \quad \text{for } i = 1, \ldots, N, \tag{3.6}$$

so that, by Poincaré's lemma,

$$J^i(x) \;=\; d\,b^i(x), \quad i \;=\; 1,\ldots,N. \tag{3.7}$$

The *key idea* is to describe the physics of an IQHF in the scaling limit in terms of an *effective field theory* of the gauge fields $\mathbf{b}(x) = \big(b^1(x),\ldots,b^N(x)\big)^T$. Because, by property (P1), an IQHF has a strictly positive mobility gap δ, that effective field theory can only be a *topological field theory.* The presence of a nonzero, external magnetic field transversal to the plane to which the electrons of an IQHF are confined implies that the quantum dynamics of the system violates the symmetries of parity (reflections in lines) and time-reversal. The only topological field theory of the gauge fields $\mathbf{b}(x)$ breaking these symmetries and respecting invariance under the gauge transformations

$$\mathbf{b}(x) \;\mapsto\; \mathbf{b}(x) + d\boldsymbol{\chi}(x) \tag{3.8}$$

is *abelian Chern-Simons theory,* with $G = \mathbb{R}^N$. This has been shown in [29], [26]. (The same conclusion can be reached by starting from the laws (i)–(iv), Section 2, preceding eq. (2.5), of electrodynamics in quantum Hall fluids [20], or by studying gauge anomaly cancellations [59], [26].) The action functional of abelian Chern-Simons theory is given by

$$S_\Lambda(\mathbf{b}) \;=\; \frac{1}{4\pi} \int\limits_\Lambda \mathbf{b}^T \wedge C d\mathbf{b} \;+\; \Gamma_{\partial\Lambda}(\mathbf{b}), \tag{3.9}$$

where $\Lambda = \Omega \times \mathbb{R}$ is the three-dimensional space-time of the system, $C = (C_{ij})_{i,j=1}^N$ is some metric on "field space" $\mathbb{R}^N$, and $\Gamma_{\partial\Lambda}(\mathbf{b})$ is the two-dimensional, anomalous chiral action only depending on the restriction of the gauge fields $\mathbf{b}$ to the boundary $\partial\Lambda$ of Λ; see [50]. Note that, individually, the two terms on the r.h.s. of (3.9) are not invariant under gauge transformations (3.8) not vanishing on $\partial\Lambda$. The boundary action $\Gamma_{\partial\Lambda}(\mathbf{b})$ is chosen such that their sum *is* gauge invariant (and is essentially determined by this requirement [50]). It is quadratic in $\mathbf{b}\,\big|_{\partial\Lambda}$.

Quantum Hall fluids are quantum-mechanical systems, and hence the Chern-Simons theory, with action functional S_Λ given in eq. (3.9), must be quantized. Because S_Λ is quadratic in $\mathbf{b}$, quantization may proceed via Feynman functional integrals. This task is not a big deal; see Section 1, and [25], [63], [23]. It turns out that the only *dynamical* degrees of freedom of the theory are localized on $\partial\Lambda$ and describe chiral $\widehat{u(1)}$-currents [43], [16]. Their dynamics is described by the term $\Gamma_{\partial\Lambda}(\mathbf{b})$, (after having taken into account the equations of motion of Chern-Simons theory). The number of clockwise moving currents is equal to the number of positive (negative) eigenvalues of the metric C; the number of counterclockwise moving currents is equal to the number of negative (positive) eigenvalues of C, (depending on the direction of $\vec{B}_c$). These are the experimentally observed edge currents first predicted by Halperin [32]. We shall focus our attention on the analysis of IQHF's with edge currents of only one chirality. Then C may be chosen to be positive-definite.

As sketched in Section 1, states in the quantum-mechanical Hilbert space of Chern-Simons theory can be viewed as solutions ϕ of the Knizhnik-Zamolodchikov equations [23] in $n = 0, 1, 2, \ldots$ variables. For our abelian Chern-Simons theory introduced in (3.9), these equations take the form

$$\frac{d\phi}{dt} = \left\{ \sum_{1 \le i < j \le n} \langle \mathbf{q}_i, \mathbf{q}_j \rangle \frac{\dot{z}_i - \dot{z}_j}{z_i - z_j} + \sum_{i=1}^{n} \langle \mathbf{q}_i, \mathbf{q}_{\partial\Omega} \rangle \, \dot{z}_i \, h'(z_i) \right\} \phi, \tag{3.10}$$

where

$$\mathbf{q}_i = (q_i^1, \ldots, q_i^N)^T \in \mathbb{R}^N, \quad i = 1, \ldots, n,$$

are n N-tuples of charges, mathematically: characters of $\mathbb{R}^N$, localized at the points $z_1, \ldots, z_n$, resp., $\mathbf{q}_{\partial\Omega}$ is an N-tuple of boundary charges,

$$\langle \mathbf{q}, \mathbf{q}' \rangle = \sum_{i,j=1}^{N} q^i C_{ij} q'^j, \tag{3.11}$$

$z_1(t), \ldots, z_n(t)$ are n paths in the domain Ω of the complex plane parameterized by a real parameter t, with $\dot{z}_i(t) = \frac{dz_i(t)}{dt}$, and h is a harmonic function on Ω, with $h' = \frac{dh}{dz}$; see [16].

The solutions of eq. (3.10) are functions on the universal covering space $\widetilde{M}_n$ of the space $\Omega^n \backslash \mathcal{D}$, where $\mathcal{D}$ is the diagonal $\{z_i = z_j, \text{ for some } i \neq j\}$. At $t = t_1$, with $z_i \equiv z_i(t_1)$, for $i = 1, \ldots, n$, the solution $\phi_{t_1} = \phi(z_1, \mathbf{q}_1, \ldots, z_n, \mathbf{q}_n)$ of (3.10) is given by

$$\begin{aligned} \phi(z_1, \mathbf{q}_1, \ldots, z_n, \mathbf{q}_n) &= \text{const.} \left[\prod_{1 \le i < j \le n} (z_i - z_j)^{\langle \mathbf{q}_i, \mathbf{q}_j \rangle} \right] \\ &\quad \times \exp \left(\sum_{i=1}^{n} \langle \mathbf{q}_i, \mathbf{q}_{\partial\Omega} \rangle \, h(z_i) \right), \end{aligned} \tag{3.12}$$

with $(z_1, \ldots, z_n)$ viewed as a point of $\widetilde{M}_n$, i.e., $(z_1, \ldots, z_n)$ stands for $\big(z_1(t_1), \ldots, z_n(t_1)\big)$, together with the homotopy class of the path $\big(z_1(t), \ldots, z_n(t)\big)_{t \in [t_0, t_1]}$; see Section 1.

To see that the characters q_j^i, $i = 1, \ldots, N$, are charges, we consider the charge operators

$$\int_{D_j} J^i = \oint_{\partial D_j} b^i \tag{3.13}$$

of the Chern-Simons theory, where D_j is a disk in Ω containing z_j, but not containing z_k, $k \neq j$. From the results in [23] one easily derives that

$$\Big(\int_{D_j} J^i \Big) \, \phi\left(z_1, \mathbf{q}_1, \ldots, z_n, \mathbf{q}_n\right) = q_j^i \, \phi\big(z_1, \mathbf{q}_1, \ldots, z_n, \mathbf{q}_n\big), \tag{3.14}$$

i.e., $\phi(z_1, \mathbf{q}_1, \ldots, z_n, \mathbf{q}_n)$ is an eigenvector of the ith charge operator $\int_{D_j} J^i$, with eigenvalue q^i_j, for $i = 1, \ldots, N$, $j = 1, \ldots, n$. By eq. (3.5) the operator detecting the total electric charge in the disk D_j is given by

$$\int\limits_{D_j} J \;=\; \sum_{i=1}^{N} Q_i \int\limits_{D_j} J^i \;\equiv\; \int\limits_{D_j} \langle \mathbf{Q}, \mathbf{J} \rangle, \tag{3.15}$$

and, by (3.14), ϕ is an eigenvector of $\int\limits_{D_j} J$ with eigenvalue

$$q_{\text{el.}}(D_j, \phi) \;=\; \sum_{i=1}^{N} Q_i\, q^i_j \;=\; \langle \mathbf{Q}, \mathbf{q}_j \rangle. \tag{3.16}$$

Suppose that $\mathbf{q}_i = \mathbf{q}_j \equiv \mathbf{q}$, for some $i \neq j$. Let us continue the solution ϕ along the path $\big(z_1(t), \ldots, z_n(t)\big)$ from $t = t_1$ to $t = t_2$, assuming that $\dot{z}_k(t) = 0$, for $k \neq i, j$, $t_1 \leq t \leq t_2$, and that $\big(z_i(t), z_j(t)\big)_{t_1 \leq t \leq t_2}$ exchanges z_i and z_j along counterclockwise oriented arcs not including any point z_k, for $k \neq i, j$. Then

$$\phi_{t_2} \;=\; \exp\big(i\pi \langle \mathbf{q}, \mathbf{q} \rangle\big)\, \phi_{t_1}, \tag{3.17}$$

i.e., the half-monodromy (called "Aharonov-Bohm phase factor" by the physicists) of the solution ϕ of (3.10) in the pair z_i, z_j is given by

$$\exp\big(i\,\pi \langle \mathbf{q}, \mathbf{q} \rangle\big). \tag{3.18}$$

Similarly, if $\dot{z}_i(t) = 0$, $t_1 \leq t \leq t_2$, $i \neq k$, and $\big(z_k(t)\big)_{t_1 \leq t \leq t_2}$ describes a counterclockwise oriented loop around the point z_ℓ not including any point z_i, $i \neq k, \ell$, then

$$\phi_{t_2} \;=\; \exp\big(i2\pi \langle \mathbf{q}_k, \mathbf{q}_\ell \rangle\big)\, \phi_{t_1}, \tag{3.19}$$

i.e., the monodromy of the solution ϕ of (3.10) in the pair z_k, z_ℓ is given by

$$\exp\big(i2\pi \langle \mathbf{q}_k, \mathbf{q}_\ell \rangle\big). \tag{3.20}$$

The groundstate of an incompressible quantum Hall fluid (IQHF) described by the Chern-Simons theory (3.9) is the vector $\phi \equiv \phi_0 \equiv 1$ ($n = 0$ in (3.12)); the charge densities J^i are normalized such that

$$\int\limits_{\Omega} J^i\, \phi_0 \;=\; 0.$$

The states $\phi(z_1, \mathbf{q}_1, \ldots, z_n, \mathbf{q}_n)$ given in (3.12) might correspond to *excited states* of the IQHF. To make this idea precise, we must find conditions on the characters, or charge vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ that guarantee that properties (P1)–(P4) of an IQHF are valid. Thus, suppose that

$$q_{\mathrm{el.}}(\mathbf{q}_j) = \sum_{i=1}^{N} Q_i\, q_j^i = \langle \mathbf{Q}, \mathbf{q}_j \rangle$$

is an *odd* integer. By property (P3), a physical excitation with charges $\mathbf{q}_j$ must then satisfy Fermi-Dirac statistics. Hence the half-monodromy (3.18) must satisfy

$$\exp\left(i\pi \langle \mathbf{q}_j, \mathbf{q}_j \rangle\right) = -1,$$

i.e.,

$$\langle \mathbf{q}_j, \mathbf{q}_j \rangle \in 2\,\mathbb{Z} + 1. \tag{3.21}$$

Similarly, if $\langle \mathbf{Q}, \mathbf{q}_j \rangle$ were *even*, the half-monodromy (3.18) would have to be $+1$, and hence

$$\langle \mathbf{q}_j, \mathbf{q}_j \rangle \in 2\,\mathbb{Z}. \tag{3.22}$$

Summarizing (3.21) and (3.22), we have that

$$\langle \mathbf{Q}, \mathbf{q} \rangle = \langle \mathbf{q}, \mathbf{q} \rangle \quad \mod 2, \tag{3.23}$$

whenever $\langle \mathbf{Q}, \mathbf{q} \rangle \in \mathbb{Z}$.

Next, suppose that $q_{\mathrm{el.}}(\mathbf{q}_j) \in \mathbb{Z}$, for some j (i.e., $\mathbf{q}_j$ corresponds to a multi-electron/-hole excitation of the fluid). By property (P4), the state vector $\phi(z_1, \mathbf{q}_1, \ldots, z_j, \mathbf{q}_j, \ldots, z_n, \mathbf{q}_n)$ must then be a *single-valued* function of z_j (for fixed z_i, $i \neq j$), provided $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are the charge vectors of (finite-energy) physical excitations of the IQHF. Thus, by (3.20),

$$\langle \mathbf{q}_j, \mathbf{q}_i \rangle \in \mathbb{Z}, \quad \text{for all } i \neq j. \tag{3.24}$$

Next, if $\mathbf{q}$ is the charge vector of a localized physical excitation of an IQHF then so is $-\mathbf{q}$, by a principle of charge conjugation. Furthermore, if $\mathbf{q}$ and $\mathbf{q}'$ are the charge vectors of two localized physical excitations of an IQHF then so is $\mathbf{q} + \mathbf{q}'$, because one may let their positions approach each other arbitrarily closely. Thus, the charge vectors of localized physical excitations of an IQHF form an additive group, denoted $\Gamma_{\mathrm{phys.}}$. By (3.23) and (3.24), the charge vectors $\mathbf{q}$ with $q_{\mathrm{el.}}(\mathbf{q}) = \langle \mathbf{Q}, \mathbf{q} \rangle \in \mathbb{Z}$ form an integral sublattice, Γ, in $\Gamma_{\mathrm{phys.}}$. Finally, by eq. (3.24) (which expresses property (P4)),

$$\Gamma_{\mathrm{phys.}} \subseteq \Gamma^*, \tag{3.25}$$

where Γ^* is the lattice dual to Γ. Because

$$q_{\mathrm{el.}}(\mathbf{q}) = \langle \mathbf{Q}, \mathbf{q} \rangle \in \mathbb{Z}, \quad \text{for all } \mathbf{q} \in \Gamma,$$

we conclude that $\mathbf{Q} \in \Gamma^*$. Furthermore, a single electron or hole is a physical excitation of an IQHF. Thus, there exists a vector $\mathbf{q} \in \Gamma$, with

$$\langle \mathbf{Q}, \mathbf{q} \rangle = 1,$$

i.e., $\mathbf{Q}$ is a *primitive* vector of Γ^*.

Suppose that $\Gamma_{\text{phys.}} \supsetneqq \Gamma$. Then there exists some local excitation of the IQHF with a charge vector $\mathbf{q} \in \Gamma_{\text{phys.}}$ such that $\mathbf{q} \mod \Gamma \neq 0$. The electric charge $q_{\text{el.}}(\mathbf{q}) = \big\langle \mathbf{Q}, \mathbf{q} \big\rangle$, of this excitation is then necessarily nonintegral (in units where $e = 1$), and its quantum statistics, as described by the half-monodromy $\exp\big(i\pi\langle \mathbf{q}, \mathbf{q}\rangle\big) \neq \pm 1$, is neither Fermi-Dirac nor Bose-Einstein statistics. It determines abelian, unitary representations of the braid groups B_n, $n = 2, 3, 4, \ldots$, and is therefore called abelian braid statistics. Thus, if $\Gamma_{\text{phys.}} \supsetneqq \Gamma$, there are local excitations in an IQHF with *fractional electric charge* and *braid statistics* ("Laughlin vortices").

Our analysis has enabled us to safely land on the notion of *chiral quantum Hall lattices.* It should be emphasized, once more, that the general analysis described here does *not* imply that Γ is a *Euclidian* lattice. The quadratic form $\langle \cdot, \cdot \rangle$ could be indefinite; see [22]. For simplicity, this general situation is not considered here and is presumably not relevant physically.

We are still missing one important point: that the Hall conductivity is given by

$$\sigma_H = \big\langle \mathbf{Q}, \mathbf{Q} \big\rangle. \tag{3.26}$$

To prove eq. (3.26), we study the response of an IQHF to a perturbation given by a small magnetic field $\vec{B}$ in the interior of the region Ω. Let B be the component perpendicular to Ω, and let $A = \sum_{\mu=0}^{2} A_\mu dx^\mu$ be an electromagnetic vector potential on Λ with

$$B = (dA)_{12}. \tag{3.27}$$

Now, recall that Q_i is the unit of electric charge transported by the current J^i. Thus, J^i couples to the electromagnetic vector potential A through a term

$$\frac{1}{2\pi} \int\limits_\Lambda J^i \wedge Q_i\, A = -\frac{1}{2\pi} \int\limits_\Lambda b^i \wedge Q_i\, dA$$

(up to a boundary term). The action functional of the IQHF in the scaling limit is therefore given by

$$S_\Lambda(\mathbf{b}) = \frac{1}{4\pi} \int\limits_\Lambda \mathbf{b}^T \wedge C\, d\mathbf{b} - \frac{1}{2\pi} \int\limits_\Lambda \mathbf{b}^T \wedge \mathbf{Q}\, dA,$$

up to a boundary term only depending on $\mathbf{b}\, |_{\partial\Lambda}$ and $A\, |_{\partial\Lambda}$. The equations of motion obtained by variation of S_Λ with respect to $\mathbf{b}$ are found to be

$$db^j(x) = \sum_{i=1}^{N} (C^{-1})^{ji} Q_i\, dA(x), \tag{3.28}$$

for x in the interior of Λ. Thus,

$$\begin{aligned} J_{12}(x) &= Q_j\, J^j_{12}(x) = Q_j (db^j)_{12}(x) \\ &= \Big(\sum_{i,j=1}^{N} Q_j (C^{-1})^{ji} Q_i \Big) (dA)_{12}(x) \\ &= \big\langle \mathbf{Q}, \mathbf{Q} \big\rangle (dA)_{12}(x). \end{aligned}$$

Integrating this equation over Ω, we find, using (3.27), that

$$q_{\text{el.}} = \int\limits_{\Omega} J_{12} = \langle \mathbf{Q}, \mathbf{Q} \rangle \int\limits_{\Omega} B \equiv \langle \mathbf{Q}, \mathbf{Q} \rangle\, \Phi.$$

Comparing this identity with eq. (2.5), we conclude that $\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle$, which proves eq. (3.26). Following [51], [1], [4], one can show that σ_H can also be expressed in terms of a first Chern number of a vector bundle of Chern-Simons groundstates on a two-dimensional torus of magnetic fluxes — this is physically somewhat contrived, though — or as a "generalized index", [20]. These matters will be discussed in more detail elsewhere.

We conclude this report with a list of important invariants of cQHL's $(\Gamma, \mathbf{Q})$ and their physical interpretations. For details and proofs, see [28], [22].

(I) Invariants of Γ

Invariant	**Physical quantity**
$\dim \Gamma$	number of independently conserved currents ("channels").
$\triangle = \mid \Gamma^*/\Gamma \mid$	number of fractionally charged Laughlin vortices (assuming that $\Gamma_{\text{phys.}} = \Gamma^*$);
$\triangle \langle \mathbf{Q}, \mathbf{Q} \rangle \mod 8$	———————
genus of Γ	monodromies, $\{\exp(i\, 2\pi \langle \mathbf{q}, \mathbf{q}' \rangle) \,:\, \mathbf{q}, \mathbf{q}' \in \Gamma^*\}$ of fractionally charged Laughlin vortices.
Witt sublattice, Γ_W	root lattice of simply laced Lie algebra of nonabelian symmetries of IQHF in scaling limit.

(II) Invariants of $(\Gamma, \mathbf{Q})$

Invariant	**Physical quantity**
$\sigma_H = \langle \mathbf{Q}, \mathbf{Q} \rangle$	Hall conductivity.
orbit of $\mathbf{Q}$ under orthogonal trsfs. of Γ	assignment of electric charges to quasi-particles.
"level" $\ell = $ g.c.d. $(\triangle, \triangle\, \sigma_H)$	———————
$\ell_{\max.}(\Gamma, \mathbf{Q})$ (see (2.17))	relative angular momentum of a pair of electrons.
$q^* = \min\limits_{\substack{\mathbf{q} \in \Gamma^* \\ \langle \mathbf{Q}, \mathbf{q} \rangle \neq 0}} \mid \langle \mathbf{Q}, \mathbf{q} \rangle \mid$	smallest fractional electric charge $\neq 0$.

These invariants and their physical counterparts permit us to elucidate fairly specific physical properties of IQHF's. But this goes beyond the present report.

4 Epilogue: Origins of the problems discussed in this lecture

In 1986, we became interested in two seemingly unrelated topics: three-dimensional gauge theories with a Chern-Simons term in their Lagrangian (or action), and the braid statistics of charged particles described by such theories, on one hand, and the fractional quantum Hall effect, on the other hand. It had already been suggested that these two topics are related to each other [61], [31], but it appeared that nobody understood the relationship in precise terms.

Between the fall of 1986 and 1990, we focused our attention primarily on the problems of understanding Chern-Simons gauge theory, the related two-dimensional conformal field theories, the general theory of braid statistics and of quantized symmetries in two- and three-dimensional quantum field theory, and some mathematical problems in knot theory and the theory of braided tensor categories related to low-dimensional quantum field theory. Our main results on these topics appeared in [23], [21], [24], [17], [30]; see also [13], [42].

In studying Chern-Simons-Higgs theories [25], Fröhlich and Marchetti understood that abelian, pure Chern-Simons theory was, in essence, just a way of reproducing the Gauss linking number. In 1987, during a sabbatical at I.H.É.S., Fröhlich was taught the basics of subfactor and knot theory by Jones. Jones expressed the intriguing idea that, in analogy to the Gauss linking number, more general knot invariants should be calculable from some "field theories" defined on links. Thanks to the presence of Felder and Gawędzki at I.H.É.S., Fröhlich also acquired some rudimentary knowledge in two-dimensional conformal field theory.

These strands of ideas naturally merged and led to some preliminary understanding of braid statistics in low-dimensional quantum field theory and its connection with the theory of knots and links [14]. Seminar notes of Jones and a preprint by Turaev [55] were very helpful in attempting to make those insights more precise. They soon led to the conjecture that, just as abelian pure Chern-Simons theory gives rise to the Gauss invariant of links, nonabelian pure Chern-Simons theory ought to give rise to more interesting link invariants. Apparently, Schwarz independently arrived at the same conjecture, around the same time (1987) [48]. Unfortunately, it appeared to be difficult to identify those invariants. It is well known that, in 1988, Witten independently came up with the same ideas, identified the link invariants emerging from nonabelian Chern-Simons theory, and went on to define new invariants for three-dimensional manifolds [63]. His work provided new motivation for us (Fröhlich and King) to return to the ideas leading to the original conjecture. We found a way of deriving the so-called Knizhnik-Zamolodchikov (KZ-)equations [38] from formal Chern-Simons functional integrals; see Section 1. We showed how to calculate some knot polynomials generalizing the Jones polynomial from solutions of the KZ-equations. The existence of appropriate solutions of the KZ-equations was proven by using convergent power series expansions in $\lambda = \pm(k+c_2)^{-1}$, where k is the level of some Kac-Moody algebra and c_2 is the dual Coxeter number of the underlying Lie algebra [23]. Our results gave substance to Jones' idea of constructing invariants of links from some "field theory" defined on links.

The KZ-equations are the equations for horizontal sections of certain vector bundles equipped with flat connections, called KZ-connections. The construction of KZ-connections is based on solutions of the so-called infinitesimal pure braid relations (a special case of which are the classical Yang-Baxter equations [3]). In fact, *every* solution of the infinitesimal pure braid relations gives rise to a KZ-connection. Horizontal sections of vector bundles can be constructed, locally, with the help of Chen's iterated integrals, more appropriately called Dyson series by the physicists. This method was used in [23].

Later on, the results and methods of [23] – see also Section 6.3 of [24] – were confirmed and put in a more general context of Vassiliev invariants [56] in [40].

In 1990, Morf taught us the basic facts about the (fractional) quantum Hall effect. A paper by Halperin [32] made it clear to us that there is a fundamental relationship between the quantum Hall effect and the theory of Kac-Moody algebras. We found that the quantum Hall effect is actually described by abelian pure Chern-Simons theories [20]. This insight, combined with the theory of the chiral anomaly in two-dimensional gauge theory, provided a completely general explanation of Halperin's findings (in a more general context than the one he had envisaged); see also [26]. Similar results were found, independently and somewhat earlier, by Wen [59] and were later confirmed by many other groups; see e.g. [49].

The work of Fröhlich and King on Chern-Simons theory now turned out to be very useful: it said that physical state vectors of incompressible quantum Hall fluids ($R_L = 0, \sigma_H$ on a plateau) could be constructed in terms of solutions of KZ-equations derived from certain abelian pure Chern-Simons theories. The known monodromy of solutions of the KZ-equations provided an essential clue to understanding the rôle played by the theory of integral quadratic forms on lattices in the theoretical analysis of incompressible quantum Hall fluids. Our analysis led us to the notion of chiral quantum Hall lattices. A partial classification of those chiral quantum Hall lattices that appear in the analysis of incompressible quantum Hall fluids was accomplished in joint work of Fröhlich and Thiran, with contributions by Kerler and Studer. Incidentally, such lattices also appear in algebraic topology (algebraic surfaces in algebraic four-manifolds). Our enterprise has taken quite a lot of time and effort. We are grateful to L. Michel for explaining to us many basic facts concerning integral lattices. Our results have appeared in [29], [26], [28], [22], [27].

Now that the classification of incompressible quantum Hall fluids in terms of chiral quantum Hall lattices has reached a satisfactory stage, it would be time to develop analytical proofs of existence of incompressible quantum Hall fluids. Interesting ideas on this problem have appeared in [64]. The strategy followed there leads to rather beautiful variational problems on spaces of sections of some line bundles — somewhat similar to the vortex problems in Higgs models [33] — which are described in [15].

Another line of research concerns the definition of Chern-Simons actions on noncommutative spaces, in the sense of Connes [7], and the analysis of the corresponding Chern-Simons theories [6]. This leads to a unifying point of view on topological field theory [63], [62]. The interplay between noncommutative geometry and quantum field theory appears to be a promising area for future work [18].

I believe we had "fun imagining it" — even though the job has sometimes been pretty hard.

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