majority of the spin sites. The expression for $\mathscr{A}^*$ can be shortened. The quantity in brackets is (where $K = \beta J$):

$$\begin{aligned}\cosh(B^* + K)\ \cosh(B^* - K) &= \cosh^2(B^*)\ \cosh^2(K) - \sinh^2(B^*)\ \sinh^2(K) \\ &= \cosh^2(B^*)\ \cosh^2(K)[1 - \tanh^2(B^*)\ \tanh^2(K)] \\ &= \cosh^2(B^*)\ \cosh^2(K)[1 - \tanh^2((B^* - \beta h)/r)] \\ &= \frac{\cosh^2(B^*)\ \cosh^2(K)}{\cosh^2[(B^* - \beta h)/r]} \qquad (4.462)\end{aligned}$$

$$\mathscr{A}^* = 2\frac{\cosh(B^*)\cosh(K)}{\cosh[(B^* - \beta h)/r]} \qquad (4.463)$$

where Eq. (4.444) is used to simplify the expression.

The ferromagnetic system has a sharp phase transition only at zero magnetic field ($h = 0$). In this case $B^* = 0$ above the transition temperature, and then $\mathscr{A}^* = 2\cosh(K)$. The expression (4.463) gives the formula in the ordered state by setting $h = 0$. In the case of nonzero magnetic field ($h \neq 0$), the ordering of the spins is gradual as the temperature is lowered. Even at high temperatures, the parameter B^* does not vanish but approaches $B^* \to \beta h$. In this case one should use (4.463) with all of its various factors.

4.5. TOMONAGA MODEL

The Tomonaga model (Tomonaga, 1950) describes a one-dimensional electron gas. The procedure is to examine the Hamiltonian of the one-dimensional electron gas and make some approximations on it. As a consequence of these approximations, the Hamiltonian becomes exactly solvable. The one-dimensional electron gas is not exactly solvable but only an approximate version of it.

The important physics is the recognition that the excitations of the electron gas are approximate bosons, although the elementary particles, electrons, are fermions. The excitations involve two-particle states, for example, moving an electron from one state to another. The wave function of the two fermion states has boson properties. The Tomonaga model assumes that the excitations are exactly bosons, which is the important approximation.

The model has been useful in several kinds of problems. First, there are organic solids such as TTF–TCNQ whose conductivity is thought to be largely one dimensional (see Heeger, 1977). The Tomonaga model has played a role in the interpretation of electrical conductivity in these materials (see Luther and Emery, 1974). Second, in impurity problems, or X-ray absorption problems, the response of the electron gas to the central impulse can be factored into spherical harmonics associated with different angular momentum states l. Each angular momentum channel l then becomes a one-dimensional electron gas to which one may apply the Tomonaga model. Recently, semiconductor nanotechnology permits the construction of semiconductor channels which act as one-dimensional conductors. The Tomonaga model is used in the theory of these systems. Single wall carbon nanotubes are another one-dimensional conductor.

4.5.1. Tomonaga Model

The original model of Tomonaga (1950) discusses the following Hamiltonian for the one-dimensional interacting electron gas:

$$H = v_F \sum_{ks} |k| a_{ks}^\dagger a_{ks} + \frac{1}{2L} \sum_k V_k \rho(k)\rho(-k) \tag{4.464}$$

$$\rho(k) = \sum_{ps} a_{p-k/2,s}^\dagger a_{p+k/2,s} \tag{4.465}$$

The system has length L, and v_F is the Fermi velocity of the particles, which are assumed to have a linear dispersion relation. The label $s = \pm 1$ denotes spin, and $\rho(k)$ is the electron density operator. The electron–electron interaction term V_k will be specified below. It is not $4\pi e^2/k^2$, which is dimensionally incorrect in one dimension, since V_k has units of Joule-meter. Dimensional analysis suggests the form $V_k \propto e^2(k_F/k)^n$, where n is any exponent. The summation over k states may be turned into integrals by the usual transformation as $L \to \infty$:

$$\sum_k f(k) = \frac{L}{2\pi} \int dk f(k) \tag{4.466}$$

The basic step in the Tomonaga model is to divide the density operator into two terms:

$$\rho_1(k) = \sum_{p>0,s} a_{p-k/2,s}^\dagger a_{p+k/2,s} \tag{4.467}$$

$$\rho_2(k) = \sum_{p<0,s} a_{p-k/2,s}^\dagger a_{p+k/2,s} \tag{4.468}$$

$$\rho(k) = \rho_1(k) + \rho_2(k) \tag{4.469}$$

The density operator $\rho(k)$ commutes with any other density operator $\rho(k')$. However, the two parts ρ_1 and ρ_2 do not commute with the same parts for other wave vectors. Examine the commutation relations:

$$\begin{aligned}[\rho_1(k), \rho_1(k')] &= \sum_{s,s'} \sum_{p,p'>0} [a_{p-k/2,s}^\dagger a_{p+k/2,s}, a_{p'-k'/2,s'}^\dagger a_{p'+k'/2,s'}] \\ &= \sum_{s,p>0} [a_{p-k/2,s}^\dagger a_{p+k'+k/2,s} \Theta(p + k/2 + k'/2) \\ &\quad - a_{p-k'-k/2,s}^\dagger a_{p+k/2,s} \Theta(p - k/2 - k'/2)]\end{aligned} \tag{4.470}$$

An important special case is $k' = -k$

$$[\rho_1(k), \rho_1(-k)] = \sum_{s,p>0} [n_{p-k/2,s} - n_{p+k/2,s}] = \sum_s \sum_{-k/2 \le p \le k/2} n_{p,s} \tag{4.471}$$

The right-hand side shows that the commutation relations depend on the operator n_{ps} over a range of p values. The operator n_{ps} is replaced by its average in the ground state of the free-particle system.

$$\sum_s \sum_{-k/2 \le p \le k/2} n_{ps} = 2 \sum_{-k/2 \le p \le k/2} \Theta(k_F - |p|) = \begin{cases} 2(kL/2\pi), & k < 2k_F \\ 2k_F L/\pi, & k > 2k_F \end{cases}$$

and the commutation relations (4.471) can be written for $k < 2k_F$ as

$$[\rho_1(k), \rho_1(-k)] = \left(\frac{kL}{\pi}\right) \tag{4.472}$$

$$[\rho_2(k), \rho_2(-k)] = -\left(\frac{kL}{\pi}\right) \tag{4.473}$$

$$[\rho_1(k), \rho_2(-k)] = 0 \tag{4.474}$$

The analogous results are included for the other commutators, which can be derived in the same fashion. The Tomonaga model assumes that these density operators obey the exact commutation relations of

$$\begin{aligned} [\rho_1(k), \rho_1(-k') &= \delta_{k,k'}\left(\frac{kL}{\pi}\right) \\ [\rho_2(k), \rho_2(-k')] &= -\delta_{k,k'}\left(\frac{kL}{\pi}\right) \\ [\rho_1(k), \rho_2(-k')] &= 0 \end{aligned} \tag{4.475}$$

These relations are the central approximation of the Tomonaga model. The commutation relations are not exact, since the commutators give operators, as in (4.470). However, these results are obtained when taking the expectation value of the exact commutation relations. For example, in (4.470)

$$\begin{aligned} \langle[\rho_1(k), \rho_1(k')]\rangle = \sum_{s,p>0} [&\langle a^\dagger_{p-k/2,s} a_{p+k'+k/2,s}\rangle \Theta(p + k/2 + k'/2) \\ &- \langle a^\dagger_{p-k'-k/2,s} a_{p+k/2,s}\rangle \Theta(p - k/2 - k'/2)] \end{aligned} \tag{4.476}$$

In the right-hand side, the averages are zero unless $k' = -k$, so that

$$\langle[\rho_1(k), \rho_1(-k')]\rangle = \delta_{k,k'} \sum_{s,p>0} [\langle n_{p-k/2}\rangle - \langle n_{p+k/2}\rangle] = 2\delta_{k,k'} \sum_{-k/2 \le p \le k/2} \langle n_{ps}\rangle \tag{4.477}$$

Although the commutation relations (4.475) are not exact, the expectation values of these commutators are given exactly. The approximation is not a very bad one.

It is convenient to express the density operators $\rho_j(\pm k)$ in terms of creation and destruction operators. This step is done so that the creation operators are dimensionless and

the commutation relations (4.475) are obeyed. The creation and destruction operators are for bosons. These definitions are given below, where the symbol k is always positive:

$$\begin{aligned}\rho_1(k) &= b_k\sqrt{\frac{kL}{\pi}} \\ \rho_1(-k) &= b_k^\dagger\sqrt{\frac{kL}{\pi}} \\ \rho_2(k) &= b_{-k}^\dagger\sqrt{\frac{kL}{\pi}} \\ \rho_2(-k) &= b_{-k}\sqrt{\frac{kL}{\pi}}\end{aligned} \tag{4.478}$$

$$[b_k, b_{k'}^\dagger] = \delta_{k,k'} \tag{4.479}$$

When k is positive $\rho_1(k) \propto b_k$, and when k is negative $\rho_1(k) \propto b_{-k}^\dagger$. The operators ρ_1 always commute with ρ_2. The choice (4.478) does satisfy the approximate commutation relations (4.475).

The second term in the Hamiltonian (4.465) may be written in terms of these boson operators:

$$\frac{1}{2L}\sum_k V_k\rho(k)\rho(-k) = \sum_k \bar{V}_k(b_k + b_{-k}^\dagger)(b_k^\dagger + b_{-k}) \tag{4.480}$$

$$\bar{V}_k = \frac{|k|V_k}{2\pi} \tag{4.481}$$

The electron–electron interaction term has been recast into an interaction between the boson excitations of the electron gas.

The first term in (4.465) is the particle kinetic energy. It requires some additional work in order to express it in terms of boson coordinates. It is not immediately obvious how to express $a_k^\dagger a_k$ in terms of the new boson operators. When faced with this predicament, it is useful to examine the commutation relations of this operator. The objective is to find a boson representation of the kinetic energy operator which reproduces the commutation relations. If this cannot be done exactly, at least try to find a good approximation. The commutator algebra completely specifies the excitation spectrum of the system, so that the excitations are adequately described by operators with accurate commutation relations.

Call the kinetic energy term H_0. Its commutator with $\rho_1(k)$ is

$$\begin{aligned}[\rho_1(k), H_0] &= v_F \sum_{s,p>0}\sum_{s'k'} |k'|[a_{p-k/2,s}^\dagger a_{p+k/2,s}, a_{k's'}^\dagger a_{k's'}] \\ &= v_F \sum_{s,p>0} a_{p-k/2,s}^\dagger a_{p+k/2,s}(|p+k/2| - |p-k/2|)\end{aligned} \tag{4.482}$$

$$|p+k/2| - |p-k/2| = \begin{cases} k & \text{if } p > k/2 \\ 2p & \text{if } p < k/2 \end{cases} \tag{4.483}$$

For small values of k, then $p > k/2$ over most of the p summation. In this case the above commutator is approximately given by

$$[\rho_1(k), H_0] = v_F k \sum_{s,p>0} a^\dagger_{p-k/2,s} a_{p+k/2,s} = v_F k \rho_1(k) \tag{4.484}$$

The above is a desirable form for the commutator, since the right-hand side is also proportional to $\rho_1(k)$. With the boson representations (4.475) and (4.478) for $\rho_1(k)$, the approximate commutation relation (4.484) is

$$[b_k, H_0] = k v_F b_k = \omega_k b_k \tag{4.485}$$

Of course, the same result would be given by the choice of $H_0 = \sum_k \omega_k b_k^\dagger b_k$. Next consider the commutator of H_0 with ρ_2. The same approximation in this case leads to

$$[\rho_2(k), H_0] = -\omega_k \rho_2(k) \tag{4.486}$$

Both of these approximate commutators are satisfied with the following choice for H_0:

$$H_0 = \sum_k \omega_k b_k^\dagger b_k \tag{4.487}$$

$$H = \sum_k \{\omega_k b_k^\dagger b_k + \bar{V}_k (b_k + b_{-k}^\dagger)(b_k^\dagger + b_{-k})\} \tag{4.488}$$

The one-dimensional electron gas (4.465) has been recast into the boson Hamiltonian (4.488). The latter is exactly solvable, as will soon be shown. The Tomonaga model (4.488) has been derived from (4.465) with several key approximations on commutation relations. The form (4.488) is a description of the boson excitations of the electron gas.

Equation (4.488) may be solved exactly by a variety of techniques. Probably the easiest method is to change to a coordinate representation for the boson operators:

$$Q_k = \frac{1}{\sqrt{2\omega_k}} (b_k + b_{-k}^\dagger) \tag{4.489}$$

$$P_k = i\sqrt{\frac{\omega_k}{2}} (b_k^\dagger - b_{-k}) \tag{4.490}$$

$$[Q_k, P_{k'}] = i\delta_{k,k'} \tag{4.491}$$

In this representation the Hamiltonian is written as

$$H_0 = \tfrac{1}{2} \sum_k (P_{-k} P_k + \omega_k^2 Q_k Q_{-k}) \tag{4.492}$$

$$H = \tfrac{1}{2} \sum_k (P_{-k} P_k + E_k^2 Q_k Q_{-k}) \tag{4.493}$$

$$E_k^2 = \omega_k^2 + 4\omega_k \bar{V}_k \tag{4.494}$$

The new eigenfrequencies are E_k. Now change back to a new set of boson normal mode operators, which are normalized to the new eigenfrequencies.

$$Q_k = \frac{1}{\sqrt{2E_k}}(\alpha_k + \alpha_{-k}^\dagger) \tag{4.495}$$

$$P_k = i\sqrt{\frac{E_k}{2}}(\alpha_k^\dagger - \alpha_{-k}) \tag{4.496}$$

$$[\alpha_k, \alpha_{k'}^\dagger] = i\delta_{k,k'} \tag{4.497}$$

$$H = \sum_k E_k(\alpha_k^\dagger \alpha_k + \tfrac{1}{2}) \tag{4.498}$$

These series of steps may be summarized by the observation that the boson operators are changed in the following way:

$$b_k + b_{-k}^\dagger = \sqrt{\frac{\omega_k}{E_k}}(\alpha_k + \alpha_{-k}^\dagger) \tag{4.499}$$

$$b_k^\dagger - b_{-k} = \sqrt{\frac{E_k}{\omega_k}}(\alpha_k^\dagger - \alpha_{-k}) \tag{4.500}$$

These transformations are useful for other problems.

The Hamiltonian of the one-dimensional electron gas (4.465) has been solved approximately. Only the excitation spectrum has been obtained. Some of these excitations are fluctuations in the density operator $\rho(k)$. Very similar results to the Tomonaga model are obtained by writing an equation of motion for the density operator and solving it approximately. This approach is used in Chapter 5.

So far the form of the interaction potential V_k has not been specified. In fact, physicists choose a variety of forms for this interaction to suit their problem. The units of V_k are the same as v_F: Joule-meter ($\hbar v_F$ is J-m). One possible choice is to take $V_k \propto e^2 = \text{constant} = V_0$. The energy spectrum is just altered by having the Fermi velocity increased:

$$E_k = \bar{v}_F k \tag{4.501}$$

$$\bar{v}_F = \sqrt{v_F\left(v_F + \frac{2}{\pi}V_0\right)} \tag{4.502}$$

The constant V_0 is assumed to be positive, since it describes interactions between electrons. The interactions increase the velocity of the acoustic plasmon.

Another possible choice is to take $V_k = 2/3(e^2k_F^2/k^2)$. This choice leads to long-wavelength modes with a constant frequency, which is the plasma frequency:

$$E_k = \sqrt{k^2v_F^2 + \omega_p^2} \tag{4.503}$$

$$\omega_p^2 = 4\omega_k\bar{V}_k = \frac{4\pi e^2 n_0}{m} \tag{4.504}$$

where $n_0 = k_F^3/3\pi^2$ is not the electron density in one dimension, but is a collection of constants. In the electron gas, there are two different types of excitations. One is the plasma modes at long wavelength, and the other is the electron–hole excitations at shorter wavelength. The latter are probably best described by the choice $V_k = V_0$.

4.5.2. Spin Waves

The Hamiltonian (4.464) of the one-dimensional electron gas has other collective excitations besides the density oscillations which were discussed above. These other excitations have the character of spin waves, or magnons. Overhauser (1965) has shown that the excitation spectrum is completely described by the sum of these two types of excitations: density oscillations and spin waves. This feature of one dimension does not apply to three dimensions. The density oscillations are the excitations which occur when there are external perturbations such as electric fields. The spin waves respond to magnetic perturbations and contribute to the spin susceptibility.

The spin waves are described by the operators

$$\sigma(k) = \sigma_1(k) + \sigma_2(k) \tag{4.505}$$

$$\sigma_1(k) = \sum_{p>0,s} s a^\dagger_{p-k/2,s} a_{p+k/2,s} \tag{4.506}$$

$$\sigma_2(k) = \sum_{p<0,s} s a^\dagger_{p-k/2,s} a_{p+k/2,s} \tag{4.507}$$

where the spin index is $s = \pm 1$ for $\uparrow$, $\downarrow$. The nature of the spin wave excitations is shown in Fig. 4.14. The spin-up and spin-down densities have opposite variations, so there is no net change in the particle density. There is a variation in $\rho_\uparrow - \rho_\downarrow$

$$\rho_s(k) = \sum_p a^\dagger_{p-k/2,s} a_{p+k/2,s} \tag{4.508}$$

$$\rho = \rho_\uparrow + \rho_\downarrow \tag{4.509}$$

$$\sigma = \rho_\uparrow - \rho_\downarrow \tag{4.510}$$

The spin operators are examined in the same fashion used for the density operators. The commutation relations are found among these operators and between them and the density operators. Some typical results are

$$[\sigma_1(k), \sigma_1(-k')] = \delta_{k,k'}\left(\frac{kL}{\pi}\right) \tag{4.511}$$

$$[\sigma_2(k), \sigma_2(-k')] = -\delta_{k,k'}\left(\frac{kL}{\pi}\right) \tag{4.512}$$

$$[\sigma_1(k), \sigma_2(k')] = 0 \tag{4.513}$$

$$[\sigma_i(k), \rho_j(-k')] = 0 \qquad (i, j = 1, 2) \tag{4.514}$$

The commutator $[\sigma_i(k), \rho_j(k)]$ contains one factor of s, and the term $s = 1$ cancels $s = -1$. This cancellation occurs when the two spin states are occupied with equal probability and the system is not magnetic. The spin operators commute with the density operators and so

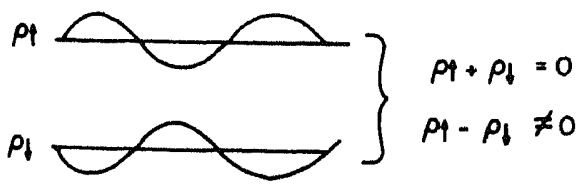

FIGURE 4.14 Spin-up and spin-down charge densities.

describe an independent set of excitations. These excitations can be represented by a new set of creation and destruction operators, which for $k > 0$ are

$$\begin{aligned}\sigma_1(k) &= c_k\sqrt{\frac{kL}{\pi}} \\ \sigma_1(-k) &= c_k^\dagger\sqrt{\frac{kL}{\pi}} \\ \sigma_2(k) &= c_{-k}^\dagger\sqrt{\frac{kL}{\pi}} \\ \sigma_2(-k) &= c_{-k}\sqrt{\frac{kL}{\pi}}\end{aligned} \tag{4.515}$$

$$[c_k, c_{k'}^\dagger] = \delta_{k,k'} \tag{4.516}$$

$$[c_k, b_{k'}^\dagger] = 0 \tag{4.517}$$

The next step is to examine the commutation relation of $\sigma_j(k)$ with the Hamiltonian (4.465), which will establish the energy spectrum of these spin wave operators. They commute with the second term in (4.465), from electron–electron interactions, since they commute with the density operators. The commutator with the kinetic energy term H_0 is:

$$[\sigma_1(k), H_0] = v_F \sum_{s,p>0} s a_{p-k/2,s}^\dagger a_{p+k/2,s}[|p + k/2| - |p - k/2|] \tag{4.518}$$

$$\approx v_F k \sigma_1(k) \tag{4.519}$$

The commutator is evaluated using the same approximation to get (4.484). Exactly the same result is obtained by representing the spin wave part of H_0 by $\sum_k \omega_k c_k^\dagger c_k$. The spin wave part of the Hamiltonian is

$$H_{sw} = \sum_k \omega_k c_k^\dagger c_k \tag{4.520}$$

$$H = \sum_k \{\omega_k b_k^\dagger b_k + \omega_k c_k^\dagger c_k + \bar{V}_k(b_k + b_{-k}^\dagger)(b_k^\dagger + b_{-k})\} \tag{4.521}$$

$$H = \sum_k \{E_k \alpha_k^\dagger \alpha_k + \omega_k c_k^\dagger c_k\} \tag{4.522}$$

The density operator parts in (4.488), (4.522), and (4.498) are combined with the spin wave parts to give the total Hamiltonian H for the excitation spectra of the one-dimensional electron gas. The original model of Tomonaga actually described a spinless electron gas. For spin one-half systems, the two possible spin orientations lead to another type of independent excitation which are called spin waves. The total Hamiltonian (4.522) has the density and spin wave excitations decoupled.

The original Hamiltonian (4.465) did not contain any terms which would cause interactions between spin waves; there were no terms of the type $\sigma(k)\sigma(-k)$. The spin wave excitation spectrum is unchanged by electron–electron interactions, at least in the Tomonaga model.

The spin wave part of the excitation spectrum can be used to derive the Pauli spin susceptibility. The starting point for this calculation is (3.458):

$$\chi(k, i\omega) = -\int_0^\beta d\tau e^{i\omega\tau}\langle T_\tau \sigma(k, \tau)\sigma(-k, 0)\rangle \tag{4.523}$$

In the Tomonaga model, the correlation function may be evaluated exactly by using the operator representation (4.515):

$$\sigma(-k, 0) = \sigma_1(-k) + \sigma_2(-k) = \left(\frac{kL}{\pi}\right)(c_k^\dagger + c_{-k}) \tag{4.524}$$

$$\sigma(k, \tau) = \left(\frac{kL}{\pi}\right)(c_{-k}^\dagger e^{\tau\omega_k} + c_k e^{-\tau\omega_k}) \tag{4.525}$$

The τ dependence is determined by H_{sw}. The further steps in the evaluation of the correlation function are identical to the derivation of the unperturbed phonon Green's function in (3.76):

$$\chi(k, i\omega) = \left(\frac{|k|L}{\pi}\right)\frac{2\omega_k}{(i\omega)^2 - \omega_k^2} \tag{4.526}$$

$$\chi_{\text{ret}}(k, \omega) = \left(\frac{|k|L}{\pi}\right)\frac{2\omega_k}{\omega^2 - \omega_k^2 + i2\omega\delta} \tag{4.527}$$

The retarded correlation function is found from the analytical continuation $i\omega \to \omega + i\delta$.

The susceptibility is found to be proportional to the length L of the electron gas. This dependence on L is correct, since the susceptibility is the total magnetization M divided by the magnetic field, and the total magnetization is indeed proportional to the size of the system. A more meaningful quantity would be the magnetization per unit volume, which is the above result divided by L. The susceptibility demonstrates a resonance phenomenon, so that it is singular whenever the external perturbations (k, ω) exactly match those of the excitation spectrum $\omega = \omega_k = kv_F$.

4.5.3. Luttinger Model

A model proposed by Luttinger (1963) is a slight variation on the Tomonaga model. It has the advantage of being exactly solvable, with fewer approximations, yet is identical to the Tomonaga model in some of its essential properties. The basic feature of the Luttinger model is that the system has two types of fermions. One has an energy spectrum given by $\varepsilon_k = kv_F$, while the other has an energy spectrum given by $\varepsilon_k = -kv_F$. They are shown by the solid and dashed lines in Fig. 4.15(a). There is an infinite number of each kind of particle, since the occupied energy states stretch to negative infinity.

In the Tomonaga model (4.465) it is assumed the energy spectrum is as shown in Fig. 4.15(b). The particles have a linear dispersion relation, but the same kind of particle is represented throughout the band of states.

The two kinds of fermions in the Luttinger model are denoted by the operators $a_{1,k,s}$ and $a_{2,k,s}$, where the subscript 1 or 2 designates the particle. The two bands are quite independent, so the two fermion operators anticommute:

$$\{a_{i,k,s}, a_{j,k',s'}^\dagger\} = \delta_{ij}\delta_{k,k'}\delta_{ss'} \tag{4.528}$$

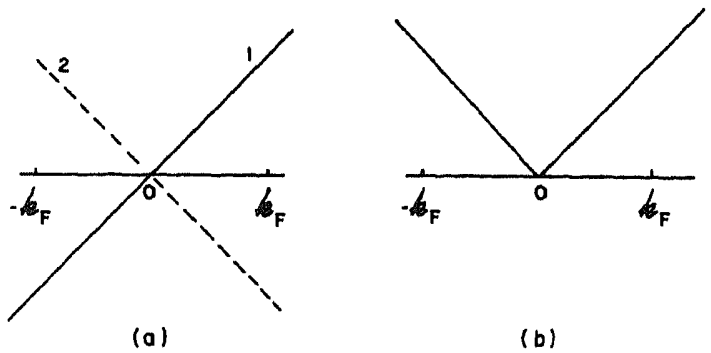

FIGURE 4.15 (a) The Luttinger model has two distinct particles, with separate energy bands. (b) The Tomonaga model has one particle, whose energy band is $v_F|k|$.

The operators $\rho_i(k)$ and $\sigma_j(k)$ are defined as in the Tomonaga model ($p > 0$):

$$\rho_i(p) = \sum_{ks} a^\dagger_{i,k+p,s} a_{i,k,s} \tag{4.529}$$

$$\rho_i(-p) = \sum_{ks} a^\dagger_{i,k,s} a_{i,k+p,s} = \rho_i(p)^\dagger \tag{4.530}$$

$$\sigma_i(p) = \sum_{ks} s a^\dagger_{i,k+p,s} a_{i,k,s} \tag{4.531}$$

$$\sigma_i(-p) = \sum_{ks} a^\dagger_{i,k,s} a_{i,k+p,s} = \sigma_i(p)^\dagger \tag{4.532}$$

The advantage of the Luttinger model is that it has the same kind of commutation relations as found for the Tomonaga model. However, they are valid for all p, whereas they were valid only for $p < 2k_F$ in the Tomonaga model:

$$[\rho_1(-p), \rho_1(p')] = \delta_{p,p'}\left(\frac{pL}{\pi}\right) \tag{4.533}$$

$$[\rho_2(p), \rho_2(-p')] = \delta_{p,p'}\left(\frac{pL}{\pi}\right) \tag{4.534}$$

$$[\rho_1(p), \rho_2(p')] = 0 \tag{4.535}$$

$$[\sigma_1(-p), \sigma_1(p')] = \delta_{p,p'}\left(\frac{pL}{\pi}\right) \tag{4.536}$$

$$[\sigma_2(-p), \sigma_2(p')] = \delta_{p,p'}\left(\frac{pL}{\pi}\right) \tag{4.537}$$

$$[\sigma_1(p), \sigma_2(p')] = 0 \tag{4.538}$$

$$[\sigma_i(p), \rho_j(p')] = 0 \tag{4.539}$$

These commutation relations depend, in an important way, on the assumption that there is an infinite number of negative-energy particles. For example, the first commutator is

$$[\rho_1(-p), \rho_1(p')] = 2\delta_{p,p'} \sum_k (n_{2,k} - n_{1,k+p}) \tag{4.540}$$

The factor of 2 comes from the summation over the two spin configurations $s = \pm 1$. For a finite number of particles, each summation over particle number would just give the number of 1-particles N_1,

$$N_1 = 2\sum_k n_{1,k} = 2\sum_k n_{1,k+p} \tag{4.541}$$

and the commutator would be zero. However, when there is an infinite number of particles in negative-energy states, a nonzero result is obtained. For a finite band, the difference

$$\sum_k (n_{1,k} - n_{1,k+p}) \tag{4.542}$$

equals $pL/2\pi$ at the top end of the band, but it equals the negative of this at the bottom end of the band, so that there is no net difference. For a semi-infinite band, there is no bottom contribution, so only the top difference is counted.

The kinetic energy term in the Luttinger model is

$$H_0 = v_F \sum_{ks} k(a^\dagger_{1,k,s} a_{1,k,s} - a^\dagger_{2,k,s} a_{2,k,s}) \tag{4.543}$$

H_0 has the exact commutation relations with the operators ($p > 0$)

$$[H_0, \rho_1(p)] = v_F p \rho_1(p) \qquad [H_0, \rho_2(p)] = -v_F p \rho_2(p) \tag{4.544}$$

$$[H_0, \sigma_1(p)] = v_F p \sigma_1(p) \qquad [H_0, \sigma_2(p)] = -v_F p \sigma_2(p) \tag{4.545}$$

The kinetic energy term is exactly represented by the operator

$$H_0 = \frac{\pi v_F}{L} \sum_{p>0} [\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p) + \sigma_1(p)\sigma_1(-p) + \sigma_2(-p)\sigma_2(p)] \tag{4.546}$$

In the Tomonaga model, the boson approximation applies only for excitation with small k. This restriction is removed in the Luttinger model. The transformation to boson operators is

$$\begin{aligned}
\rho_1(-p) &= b_{1p}\sqrt{\frac{pL}{\pi}}, & \rho_1(p) &= b^\dagger_{1p}\sqrt{\frac{pL}{\pi}} \\
\rho_2(-p) &= b^\dagger_{2,-p}\sqrt{\frac{pL}{\pi}}, & \rho_2(p) &= b_{2,-p}\sqrt{\frac{pL}{\pi}} \\
\sigma_1(-p) &= c_{1p}\sqrt{\frac{pL}{\pi}}, & \sigma_1(p) &= c^\dagger_{1p}\sqrt{\frac{pL}{\pi}} \\
\sigma_2(-p) &= c^\dagger_{2,-p}\sqrt{\frac{pL}{\pi}}, & \sigma_2(p) &= c_{2,-p}\sqrt{\frac{pL}{\pi}}
\end{aligned} \tag{4.547}$$

The Hamiltonian is now

$$H_0 = \sum_{p>0} p v_F [b^\dagger_{1p} b_{1p} + b^\dagger_{2,-p} b_{2,-p} + c^\dagger_{1p} c_{1p} + c^\dagger_{2,-p} c_{2,-p}] \tag{4.548}$$

The operator $\rho_1(p)$ for $p > 0$ takes a particle from state k and puts it into $p + k$. This operation will make an electron–hole pair when $k < k_F$ and $p + k > k_F$. The summation over all such electron–hole pairs is represented by the boson creation operator $b^\dagger_{1p}$. For particle 2, the Fermi "surface" is at the negative wave vector $-k_F$. Electron–hole pairs are made mostly at negative wave vectors. The operator $\rho_2(-p) = \sum_k a^\dagger_{2k} a_{2,k+p}$ for $p > 0$ creates these

bosons, since it takes an electron from the occupied state $k_F < k + p$ to the unoccupied state $k < -k_F$, where k is negative.

Various kinds of interaction terms may be added to the Luttinger model. Those which arise from electron–electron interactions are expressed as the product of four fermion operators, or two density operators. These Hamiltonians are exactly solvable, since they describes linear coupling between two harmonic oscillator systems.

The Luttinger model has the advantage of being exactly solvable. Of course, one could add other terms which might render it no longer exactly solvable. The disadvantage of the model is that it is unphysical, since it contains the infinite reservoir of negative-energy particles.

4.5.4. Single-Particle Properties

Some of the most interesting applications of the Tomonaga–Luttinger models are concerned with single-particle properties of the electron gas. An important quantity is the occupation number $n_{i,k,s} = \langle a^\dagger_{i,k,s} a_{i,k,s} \rangle$, in the interacting system. A more ambitious calculation would be the one-particle Green's function

$$G_{i,s}(k, t) = -i\langle T a_{i,k,s}(t) a^\dagger_{i,k,s}(0) \rangle \tag{4.549}$$

To obtain these quantities requires a representation of the single fermion operator $a_{i,k,s}$ in terms of boson operators. The discussion follows Mattis and Lieb (1965) and Luther and Peschel (1974).

The representation of the single-fermion operator in terms of bosons is found, as always, by examining the commutation relations. A representation of $a_{i,k,s}$ is satisfactory if it obeys all the proper commutation relations with the other operators. The first step is to Fourier-transform into a real-space representation:

$$\Psi_{is}(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} a_{i,k,s} \tag{4.550}$$

$$\Psi^\dagger_{is}(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} a^\dagger_{i,k,s} \tag{4.551}$$

The advantage of this representation becomes clear when considering the commutator of $\Psi_{is}(x)$ with the density operators. This discussion uses the Luttinger form of the Tomonaga model. Typical commutators are

$$[\Psi_{is}(x), \rho_j(p)] = \delta_{ij} e^{ipx} \Psi_{is}(x) \tag{4.552}$$

$$[\Psi_{is}(x), \sigma_j(p)] = \delta_{ij} s e^{ipx} \Psi_{is}(x) \tag{4.553}$$

which is derived in the following way:

$$\begin{aligned}
[\Psi_{is}(x), \rho_j(p)] &= \frac{1}{\sqrt{L}} \sum_{kk's'} e^{ikx} [a_{i,k,s}, a^\dagger_{j,k'+p,s'} a_{j,k',s'}] \\
&= \frac{\delta_{i,j}}{\sqrt{L}} \sum_{kk's'} e^{ikx} a_{j,k',s'} \delta_{ss'} \delta_{k=k'+p} \\
&= \frac{\delta_{i,j}}{\sqrt{L}} e^{ipx} \sum_{k'} e^{ik'x} a_{i,k',s} = \delta_{i,j} e^{ipx} \Psi_{is}(x)
\end{aligned} \tag{4.554}$$

The commutator $[\Psi_{is}(x), \rho_j(p)]$ has a simple form, since it is just proportional to $\Psi_{is}(x)$. The solution would be simpler if the commutator were a constant or even proportional to a density operator. It is not, so the solution of (4.552) is more complicated. One possible solution has the form

$$\Psi_{1s}(x) = F_1(x)\ \exp[J_1(x)] \tag{4.555}$$

$$J_1(x) = -\frac{\pi}{L}\sum_{p>0}\frac{1}{p}\{e^{-ipx}[\rho_1(p) + s\sigma_1(p)] - e^{ipx}[\rho_1(-p) + s\sigma(-p)]\} \tag{4.556}$$

The prefactor $F_1(x)$ can be a function of x but is a c number in the sense that it must commute with both $\rho_1(p)$ and $\sigma_1(p)$. Next show that this choice does satisfy Eq. (4.552)

$$\begin{aligned}[\Psi_{1s}, \rho_1] &= F_1(e^{J_1}\rho_1 - \rho_1 e^{J_1}) = F_1(e^{J_1}\rho_1 e^{-J_1} - 1)e^{J_1} \\ &= F_1[J_1, \rho_1]e^{J_1} = [J_1, \rho_1]\Psi_{1s}\end{aligned} \tag{4.557}$$

The last line is valid only when the commutator $[J_1, \rho_1]$ is a c number which commutes with the operator J_1. It does for the $J_1(x)$ in (4.556):

$$[J_1, \rho_1(p)] = \frac{\pi}{L}\sum_k \frac{1}{k} e^{ikx}[\rho_1(-k), \rho_1(p)] = e^{ipx} \tag{4.558}$$

The next observation is that the factor

$$\rho_1(p) + s\sigma_1(p) = \sum_{k,s'}(1 + ss')a^\dagger_{1,k+p,s'}a_{1,k,s'} = 2\sum_k a^\dagger_{1,k+p,s}a_{1,k} \tag{4.559}$$

since the factor $(1 + ss') = 0$ unless $s = s'$, and then it is 2. The equations can be condensed by introducing the notation of a spin-dependent density operator ($p > 0$):

$$\rho_{is}(p) = \sum_{\text{all } k} a^\dagger_{i,k+p,s}a_{i,k,s} \tag{4.560}$$

$$\rho_{is}(-p) = \sum_{\text{all } k} a^\dagger_{i,k,s}a_{i,k+p,s} \tag{4.561}$$

$$\rho_i(p) = \sum_s \rho_{is}(p) \tag{4.562}$$

$$\sigma_i(p) = \sum_s s\rho_{is}(p) \tag{4.563}$$

$$J_{1s}(x) = -\frac{2\pi}{L}\sum_{\text{all } p}\frac{e^{-ipx}}{p}\rho_{1s}(p) \tag{4.564}$$

$$\Psi_{1s}(x) = F_1(x)\ \exp[J_{1s}(x)] \tag{4.565}$$

The spin-dependent density operators can be represented by boson operators similar to (4.547) with an additional spin subscript.

The form of $\Psi_{1s}(x)$ in (4.556) is a solution to the commutator equation (4.552). Unfortunately this solution has some undesirable properties which will force a modification. The need for changes in $\Psi_{is}(x)$ may be understood by examining the form of the operator $\langle\Psi^\dagger_{1s}(x)\Psi_{1s}(x')\rangle$ for the noninteracting electron system, which is the Luttinger model with just

the Hamiltonian H_0 in (4.543). At zero temperature, the noninteracting system has the feature that the momentum distributions for particles 1 and 2 have the form

$$n_{1,k,s} = \theta(k_F - k) \tag{4.566}$$

$$n_{2,k,s} = \theta(k_F + k) \tag{4.567}$$

This fact can be used to evaluate the correlation function $\langle \Psi^\dagger_{1s}(x)\Psi_{1s}(x')\rangle$ by using the inverse of the transformation (4.550):

$$\langle \Psi^\dagger_{1s}(x)\Psi_{1s}(x')\rangle = \frac{1}{L}\sum_{kk'} e^{-i(kx-k'x')}\langle a^\dagger_{1ks}a_{1k's}\rangle \tag{4.568}$$

$$= \frac{1}{L}\sum_{kk'} e^{-i(kx-k'x')}\delta_{kk'}n_{1,k,s} \tag{4.569}$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ik(x-x')}\theta(k_F - k) \tag{4.570}$$

$$\langle \Psi^\dagger_{1s}(x)\Psi_{1s}(x')\rangle = -\frac{e^{-ik_F(x-x')}}{2\pi i(x - x' + i\eta)} \tag{4.571}$$

$$\langle \Psi^\dagger_{2s}(x)\Psi_{2s}(x')\rangle = \frac{e^{ik_F(x-x')}}{2\pi i(x - x' - i\eta)} \tag{4.572}$$

where the factors of $\pm i\eta$ are added to aid convergence at infinity.

The objective in choosing the representation (4.565) is to make the result for $\langle \Psi^\dagger_{1s}(x)\Psi_{1s}(x')\rangle$ be like (4.571) for the noninteracting Luttinger model. A method of doing this was suggested by Luther and Peschel (1974). It uses a limiting process, where the wave function $\Psi_{1s}(x)$ contains a parameter α, and the limit $\alpha \to 0$ is taken at the end of the calculation. Including the parameter α, the position space operators are represented as

$$\Psi_{1s}(x) = \frac{1}{\sqrt{2\pi\alpha}}\exp[ik_F x + J_{1s}(\alpha, x)] \tag{4.573}$$

$$\Psi_{2s}(x) = \frac{1}{\sqrt{2\pi\alpha}}\exp[-ik_F x - J_{2s}(\alpha, x)] \tag{4.574}$$

$$J_{is}(\alpha, x) = -\frac{2\pi}{L}\sum_{k>0}\frac{e^{-\alpha k/2}}{k}[e^{-ikx}\rho_{is}(k) - e^{ikx}\rho_{is}(-k)] \tag{4.575}$$

$$= -J_{is}(\alpha, x)^\dagger \tag{4.576}$$

This new form for $J_{is}(\alpha, x)$ may be expressed in terms of the boson operators:

$$J_{1s}(\alpha, x) = \sum_{p>0} e^{-\alpha p/2}\sqrt{\frac{2\pi}{pL}}(b_{1s,p}e^{ipx} - b^\dagger_{1s,p}e^{-ipx}) \tag{4.577}$$

$$J_{2s}(\alpha, x) = \sum_{p>0} e^{-\alpha p/2}\sqrt{\frac{2\pi}{pL}}(b^\dagger_{2s,-p}e^{ipx} - b_{2s,-p}e^{-ipx}) \tag{4.578}$$

The exponential factor $J_{1s}(\alpha, x)$ has the same form as (4.556) in the limit where $\alpha \to 0$, so the commutator (4.552) is obeyed in this limit. The prefactor $1/\sqrt{2\pi\alpha}$ is explained below. Now consider the evaluation of the quantity

$$\langle \Psi_{1s}^{\dagger}(x)\Psi_{1s}(x')\rangle = \frac{1}{2\pi\alpha} e^{-ik_F(x-x')} \langle e^{-J_{1s}(\alpha,x)} e^{J_{1s}(\alpha,x')} \rangle \tag{4.579}$$

$$= \frac{1}{2\pi\alpha} e^{-ik_F(x-x')} \Pi_{k>0} \left\langle \exp\left[e^{-\alpha k/2} \sqrt{\frac{2\pi}{kL}} (e^{-ikx} b_{1s,k}^{\dagger} - e^{ikx} b_{1s,k}) \right] \right.$$

$$\left. \times \exp\left[e^{-\alpha k/2} \sqrt{\frac{2\pi}{kL}} (e^{ikx'} b_{1s,k} - e^{-ikx'} b_{1s,k}^{\dagger}) \right] \right\rangle \tag{4.580}$$

The right-hand side of this expression is an average of exponential functions of boson operators. These expressions are evaluated in Sec. 4.3.2. Each exponent is separated by using the Feynman theorem $\exp(A+B) = \exp(A)\exp(B)\exp(-[A,B]/2)$. Then the factors are commuted until all the destruction operators are on the right:

$$\langle \Psi_{1s}^{\dagger}(x)\Psi_{1s}(x')\rangle = \frac{1}{2\pi\alpha} e^{-ik_F(x-x')-\phi_0(x-x')} \Pi_{k>0} \left\langle \exp\left[e^{-\alpha k/2} \sqrt{\frac{2\pi}{kL}} b_{1s,k}^{\dagger} (e^{-ikx} - e^{-ikx'}) \right] \right.$$

$$\left. \times \exp\left[e^{-\alpha k/2} \sqrt{\frac{2\pi}{kL}} b_{1s,k} (e^{ikx'} - e^{ikx}) \right] \right\rangle$$

$$\phi_0(x) = \frac{2\pi}{L} \sum_{k>0} \frac{e^{-\alpha k}}{k} (1 - e^{ikx}) = \int_0^{\infty} \frac{dk}{k} e^{-\alpha k} (1 - e^{ikx}) \tag{4.582}$$

At zero temperature, the quantity in the final brackets gives unity, which gives the following prediction for the noninteracting electron gas:

$$\langle \Psi_{1s}^{\dagger}(x)\Psi_{1s}(x')\rangle = \frac{1}{2\pi\alpha} e^{-ik_F(x-x')-\phi_0(x-x')} \tag{4.583}$$

The expression for $\phi_0(x)$ has the form of an infrared divergence as discussed in Sec. 9.3. Expand the exponential $\exp(ikx)$ and integrate term by term. The factor $\exp(-\alpha k)$ ensures the convergence of these integrals, which is the primary role played by α:

$$\phi_0(x) = -\sum_{l=0}^{\infty} \frac{(ix)^l}{l!} \int_0^{\infty} dk k^{l-1} e^{-\alpha k} = -\sum_{l=0}^{\infty} \frac{(ix)^l}{l\alpha^l}$$

$$= \ln\left(1 - \frac{ix}{\alpha}\right) \tag{4.584}$$

$$e^{-\phi_0(x)} = \frac{1}{1 - ix/\alpha} \tag{4.585}$$

The series for $\phi_0(x)$ is recognized as a logarithm, which gives the final result

$$\langle \Psi_{1s}^{\dagger}(x)\Psi_{1s}(x')\rangle = \frac{1}{2\pi\alpha} e^{-ik_F(x-x')} \frac{1}{1 - i(x-x')/\alpha} \tag{4.586}$$

$$= \frac{1}{2\pi} e^{-ik_F(x-x')} \frac{1}{\alpha - i(x-x')} \tag{4.587}$$

The limit $\alpha \to 0$ does indeed recover the noninteracting value (4.571). The parameter α becomes the convergence factor η.

It is easy to check that the factor $\langle \Psi_{2s}^{\dagger}(x)\Psi_{2s}(x')\rangle$ is also given correctly. The representation (4.575) for $\Psi_{1s}(x)$ reproduces the commutation relation with the density operators and also gives the correct ground state momentum distribution for the noninteracting system. All these results, of course, apply in the limit where $\alpha \to 0$. The commutator of $\Psi_{1s}(x)$ with H_0 is also given correctly, since the latter is expressed in terms of the density operators, which have the correct commutators. The representation (4.575) is suitable for the single-fermion operators.

This representation can be used to calculate many interesting properties of the Luttinger model. For example, the electron Green's function is

$$\begin{aligned} G_{1s}(x-x', t) &= -i\langle T\Psi_{1s}(x, t)\Psi_{1s}^{\dagger}(x', 0)\rangle \\ &= -i\theta(t)\langle e^{iH_0 t}\Psi_{1s}(x)e^{-iH_0 t}\Psi_{1s}^{\dagger}(x')\rangle \\ &\quad + i\theta(t)\langle \Psi_{1s}^{\dagger}(x')e^{iH_0 t}\Psi_{1s}(x)e^{-iH_0 t}\rangle \end{aligned} \tag{4.588}$$

The correlation functions can be evaluated at zero temperature, using the same steps which led to (4.581). The time dependence of $\Psi_{1s}(x, t)$ for the noninteracting Hamiltonian is

$$\Psi_{1s}(x, t) = \frac{1}{\sqrt{2\pi\alpha}} \exp[ik_F x + J_{1s}(\alpha, x, t)] \tag{4.589}$$

$$\begin{aligned} J_{1s}(\alpha, x, t) &= \sum_{k>0} e^{-\alpha k/2}\sqrt{\frac{2\pi}{kL}}(e^{i(kx-\omega_k t)}b_{1s,k} - e^{-i(kx-\omega_k t)}b_{1s,k}^{\dagger}) \\ J_{1s}(\alpha, x, t) &= J_{1s}(\alpha, x - v_F t) \end{aligned} \tag{4.590}$$

The time dependence of $\Psi_{1s}(x, t) = \Psi_{1s}(x - v_F t)$ merely changes the factor x in $J_{1s}(\alpha, x)$ to $x - v_F t$. This rather trivial change makes it possible to use the previous result for $\langle \Psi_{1s}^{\dagger}(x)\Psi_{1s}(x')\rangle$ to evaluate $\langle \Psi_{1s}^{\dagger}(x')\Psi_{1s}(x, t)\rangle$:

$$\begin{aligned} \langle \Psi_{1s}^{\dagger}(x')\Psi_{1s}(x, t)\rangle &= \frac{1}{2\pi\alpha} \exp[ik_F(x-x') - \phi_0(x' - x + v_F t)] \\ &= \frac{e^{ik_F(x-x')}}{2\pi i(x - x' - v_F t - i\alpha)} \end{aligned} \tag{4.591}$$

$$\begin{aligned} \langle \Psi_{1s}(x, t)\Psi_{1s}^{\dagger}(x')\rangle &= \frac{1}{2\pi\alpha} \exp[ik_F(x-x') - \phi_0^{*}(x' - x + v_F t)] \\ &= -\frac{e^{ik_F(x-x')}}{2\pi i(x - x' - v_F t + i\alpha)} \end{aligned} \tag{4.592}$$

The factor $\langle \Psi_{1s}(x, t)\Psi_{1s}^{\dagger}(x')\rangle$ has just the Hermitian conjugate of ϕ_0. The Green's function for the noninteracting system is easily obtained ($\alpha \to 0$):

$$G_{1s}(x, t) = \frac{e^{ik_F x}}{2\pi}\left[\frac{\theta(t)}{x - v_F t + i\alpha} + \frac{\theta(-t)}{x - v_F t - i\alpha}\right] \tag{4.593}$$

This equation can be Fourier-transformed to obtain the Green's function in the wave vector representation:

$$G_{1s}(k,t) = \int_{-\infty}^{\infty} dx e^{-ikx} G_{1s}(x,t) \tag{4.594}$$

$$= -ie^{-iv_F t(k-k_F)}[\Theta(t)\Theta(k-k_F) - \Theta(-t)\Theta(k_F-k) \tag{4.595}$$

This result is the same $G(k, t)$ which is obtain in the fermion representation:

$$G_{1s}(k,t) = -i\langle Ta_{i,k,s}(t)a^{\dagger}_{1,k,s}(0)\rangle \tag{4.596}$$

where the energy has been normalized to the Fermi energy: $\varepsilon_k = v_F(k - k_F)$. The correct result for $G_{1s}(k, t)$ again illustrates that the boson representation (4.575) for the single-particle operators will faithfully reproduce the results obtained directly from the fermion representation. The virtue of the Boson representation is that more difficult problems can be solved. In particular, interaction terms can be added to the Hamiltonian. Exact expressions can be found for Green's functions, or other correlation functions, although they are usually difficult to evaluate analytically.

4.5.5. Interacting System of Spinless Fermions

An exact solution can be obtained for various correlation functions, even for the interacting electron gas in one dimension. First solve for the occupation number. This solution relies upon the representation of the single-particle operators which was developed in the prior subsection. The Hamiltonian in this part is taken to be the Luttinger model for spinless fermions (Mattis and Lieb, 1965):

$$H = \sum_{p>0}[\omega_p(b^{\dagger}_{1,p}b_{1,p} + b^{\dagger}_{2,-p}b_{2,-p}) + V_p(b^{\dagger}_{1,p}b^{\dagger}_{2,-p} + b_{2,-p}b_{1,p})] \tag{4.597}$$

The interaction term comes from particle–particle interactions between the two types of fermions. Other interaction terms could be considered.

The first step in the solution is to learn the method of diagonalizing this Hamiltonian. There are several ways to do this, and all give the same result. A canonical transformation is used to obtain a new set of boson operators α_p, β_p, which are defined as

$$b_{1,p} = \beta_p \cosh(\lambda_p) - \alpha^{\dagger}_p \sinh(\lambda_p) \tag{4.598}$$

$$b^{\dagger}_{1,p} = \beta^{\dagger}_p \cosh(\lambda_p) - \alpha_p \sinh(\lambda_p) \tag{4.599}$$

$$b_{2,-p} = \alpha_p \cosh(\lambda_p) - \beta^{\dagger}_p \sinh(\lambda_p) \tag{4.600}$$

$$b^{\dagger}_{2,-p} = \alpha^{\dagger}_p \cosh(\lambda_p) - \beta_p \sinh(\lambda_p) \tag{4.601}$$

$$[b_{1,p}, b^{\dagger}_{1,p}] = [\beta_p, \beta^{\dagger}_p] \cosh^2(\lambda_p) + [\alpha^{\dagger}_p, \alpha_p] \sinh^2(\lambda_p) \tag{4.602}$$

$$= \cosh^2(\lambda_p) - \sinh^2(\lambda_p) = 1 \tag{4.603}$$

$$[\alpha_p, \beta^{\dagger}_p] = 0, \qquad [\alpha_p, \alpha^{\dagger}_p] = 1 \tag{4.604}$$

The various commutation relations are still obeyed in this new representation. The parameter λ_p is chosen so that the Hamiltonian (4.597) is diagonalized. It is first written out in terms of the transformed operators:

$$H = \sum_{p>0}[(\beta_p^\dagger\beta_p + \alpha_p^\dagger\alpha_p)\{[\cosh^2(\lambda_p) + \sinh^2(\lambda_p)]\omega_p - 2V_p \sinh(\lambda_p)\cosh(\lambda_p)\}$$
$$+ (\beta_p^\dagger\alpha_p^\dagger + \alpha_p\beta_p)\{[\cosh^2(\lambda_p) + \sinh^2(\lambda_p)]V_p - 2\omega_p \sinh(\lambda_p)\cosh(\lambda_p)\}]$$

Since these are boson operators, the ordering of terms such as $\alpha\beta = \beta\alpha$ does not matter. The zero-point motion terms are ignored. Two combinations of hyperbolic functions seem to occur:

$$\cosh^2(\lambda_p) + \sinh^2(\lambda_p) = \cosh(2\lambda_p) \tag{4.605}$$
$$2\sinh(\lambda_p)\cosh(\lambda_p) = \sinh(2\lambda_p) \tag{4.606}$$

The Hamiltonian is diagonalized by setting to zero the coefficient of the term $(\beta_p^\dagger\alpha_p^\dagger + \alpha_p\beta_p)$. This step gives $\tanh(2\lambda_p) = V_p/\omega_p$, so that the diagonalized Hamiltonian is

$$H = \sum_{P>0} E_p(\beta_p^\dagger\beta_p + \alpha_p^\dagger\alpha_p) \tag{4.607}$$
$$E_p = \sqrt{\omega_p^2 - V_p^2} \tag{4.608}$$
$$\cosh(2\lambda_p) = \frac{\omega_p}{E_p} \tag{4.609}$$

The transformation to the new operators is used to evaluate the properties of the interacting system. The α_p and β_p operators refer to the actual boson normal modes in the interacting system. The ground state of the system is the vacuum of α_p and β_p particles; i.e., $\alpha_p|0\rangle = 0$, $\beta_p|0\rangle = 0$. These are the same set of normal modes in the Tomonaga model (4.498).

Consider the evaluation of the fermion occupation number, which is given in (4.581) as the ground state expectation value of the operator combination:

$$\langle\Psi_1^\dagger(x)\Psi_1(x')\rangle = \frac{1}{2\pi\alpha}\exp[-ik_F(x-x') - \phi_0(x-x')] \tag{4.610}$$
$$e^{-\phi_0(x)} = \langle e^{-J_1(\alpha,x)}e^{J_1(\alpha,0)}\rangle \tag{4.611}$$

where $J_1(\alpha, x)$ is given in (4.577). The ground state of the system must be the particle vacuum of the bosons with excitation energy E_p in (4.608), since these are the normal modes. The $J_1(\alpha, x)$ operator must be expressed in the α_p and β_p representation. The transformation (4.600) produces a redefined operator form

$$J_1(\alpha, x) = \sum_{p>0} e^{-\alpha p/2}\sqrt{\frac{2\pi}{pL}}\{e^{ipx}[\beta_p \cosh(\lambda_p) - \alpha_p^\dagger \sinh(\lambda_p)]$$
$$- e^{-ipx}[\beta_p^\dagger \cosh(\lambda_p) - \alpha_p \sinh(\lambda_p)]\} \tag{4.612}$$

It contains operators of both types α_p and β_p. These operators are independent, since they each describe an independent Boson system. Each of these boson systems are averaged independently. The ground state average gives

$$\phi_0(x) = \phi_a(x) + \phi_b(x) \tag{4.613}$$

$$e^{-\phi_a(x)} = \langle e^{-J_a(x)} e^{J_a(x)} \rangle \tag{4.614}$$

$$e^{-\phi_b(x)} = \langle e^{-J_b(x)} e^{J_b(x)} \rangle \tag{4.615}$$

$$J_a(x) = \sum_{p>0} e^{-\alpha p/2} \sqrt{\frac{2\pi}{pL}} \sinh(\lambda_p)(e^{-ipx}\alpha_p - e^{ipx}\alpha_p^\dagger) \tag{4.616}$$

$$J_b(x) = \sum_{p>0} e^{-\alpha p/2} \sqrt{\frac{2\pi}{pL}} \cosh(\lambda_p)(e^{ipx}\beta_p - e^{-ipx}\beta_p^\dagger) \tag{4.617}$$

The separate averages for $\phi_a(x)$ and $\phi_b(x)$ are similar to those found earlier in (4.582). The average for $\phi_b(x)$ is identical to the earlier average for $\phi_0(x)$, except for the extra kernel $\cosh(\lambda_p)$. The average for $\phi_a(x)$ also contains a unique kernel $\sinh(\lambda_p)$ and has $x \to -x$. By analogy with (4.582), at zero temperature

$$\phi_b(x) = \frac{2\pi}{L} \sum_{p>0} \frac{e^{-\alpha p}}{p} \cosh^2(\lambda_p)(1 - e^{ipx}) \tag{4.618}$$

$$\phi_a(x) = \frac{2\pi}{L} \sum_{p>0} \frac{e^{-\alpha p}}{p} \sinh^2(\lambda_p)(1 - e^{-ipx}) \tag{4.619}$$

The result for $\phi_b(x)$ is manipulated by replacing $\cosh^2(\lambda_p)$ by its equivalent $1 + \sinh^2(\lambda_p)$. The term with "1" is identical to $\phi_0(x)$

$$\phi_b(x) = \phi_0(x) + \frac{2\pi}{L} \sum_{p>0} \frac{e^{-\alpha p}}{p} \sinh^2(\lambda_p)(1 - e^{ipx}) \tag{4.620}$$

so that

$$\begin{aligned} \phi_a + \phi_b &= \phi_0 + \phi_s \\ \phi_s(x) &= \frac{2\pi}{L} \sum_{p>0} \frac{e^{-\alpha p}}{p} \sinh^2(\lambda_p)[(1 - e^{-ipx}) + (1 - e^{ipx})] \end{aligned} \tag{4.621}$$

$$\langle \Psi_1^\dagger(x)\Psi_1(x') \rangle = -\frac{e^{-ik_F(x-x')}}{2\pi i(x - x' + i\alpha)} e^{-\phi_s(x-x')} \tag{4.622}$$

The effect of the interactions on the electron gas is contained in the exponential factor $\exp(-\phi_s)$. The other terms in (4.622) are the same as for the noninteracting electron gas.

By using the relation

$$\sinh^2(\lambda_p) = \frac{1}{2}[\cosh(2\lambda_p) - 1] = \frac{1}{2}\left[\frac{\omega_p}{E_p} - 1\right] \tag{4.623}$$

$$\phi_s(x) = \int_0^\infty \frac{dp}{p} e^{-\alpha p} \left(\frac{\omega_p}{E_p} - 1\right)[1 - \cos(px)] \tag{4.624}$$

which uses the prior result (4.609).

Any evaluation of the factor $\phi_s(x)$ must assume some specific form of the potential V_p between electrons. One possible model is to take $V_p = pV_0$, $V_0 =$ constant. This form of the potential is obtained from a delta function interaction in real space. This model assumes that the particles interact only when they directly collide. In this case ($\omega_p = pvF$)

$$E_p = p\sqrt{v_F^2 - V_0^2} \tag{4.625}$$

$$g \equiv \sinh^2(\lambda_p) = \frac{1}{2}\left[\frac{v_F}{\sqrt{v_F^2 - V_0^2}} - 1\right] \tag{4.626}$$

$$\phi_s(x) = 2g\int_0^\infty \frac{dp}{p} e^{-\alpha p}[1 - \cos(px)] \tag{4.627}$$

The factor $\sinh^2(\lambda_p)$ is a constant, which is called g. The integral for the exponential factor $\phi_s(x)$ is now simple to evaluate, since it has the same form as earlier for $\phi_0(x)$ in (4.582):

$$\phi_s(x) = g[\phi_0(x) + \phi_0^*(x)] = g\ln\left(1 + \frac{x^2}{\alpha^2}\right) \tag{4.628}$$

The delta function model makes the following prediction for the momentum distribution of the 1-particles:

$$\langle \Psi_1^\dagger(x)\Psi_1(x')\rangle = -\frac{e^{-ik_F(x-x')}}{2\pi i(x - x' + i\alpha)}\frac{1}{[1 + (x - x')^2/\alpha^2]^g} \tag{4.629}$$

$$\begin{aligned} n_{1,k} &= \int_{-\infty}^{\infty} dx e^{ik(x-x')}\langle \Psi_1^\dagger(x)\Psi_1(x')\rangle \\ &= -\int_{-\infty}^{\infty} \frac{dx}{2\pi i}\frac{e^{ix(k-k_F)}}{x + i\alpha}\frac{\alpha^{2g}}{(x^2 + \alpha^2)^g} \end{aligned} \tag{4.630}$$

Setting $g = 0$ recovers the noninteracting case $n_{1,k} = \Theta(k_F - k)$, which is obtained by closing the contour of integration in the UHP (upper half-plane) when $k_F < k$ and in the LHP (lower half-plane) when $k_F > k$. The pole at $x = -i\alpha$ is circled only in the latter case.

Mattis and Lieb (1965) showed that a more interesting result is found for the case where the coupling constant g is nonzero. Then $n_{1,k} =$ constant, independent of k, so the Fermi distribution is totally destroyed. This happens even in the limit where g is infinitesimally small. As $g \to 0$, then $n_{1,k} = \frac{1}{2}$.

This result is obtained by changing the integration variable to $y = x/\alpha$:

$$n_{1,k} = -\int_{-\infty}^{\infty} \frac{dy}{2\pi i}\frac{e^{iy\alpha(k-k_F)}}{y + i}\frac{1}{(y^2 + 1)^g} \tag{4.631}$$

The only α dependence is in the exponential factor. This exponential factor is needed for $g = 0$, since it tells us whether to close the integration contour in the upper or lower half-plane. However, for a nonzero value of g the integral converges even without the exponential factor. Therefore set $\alpha = 0$ before doing the integral and consider

$$n_{1,k} = -\int_{-\infty}^{\infty} \frac{dy}{2\pi i}\frac{1}{y + i}\frac{1}{(y^2 + 1)^g} \tag{4.632}$$

The right-hand side is no longer a function of $k - k_F$, and is a constant. The integral for $y > 0$ is added to that for $y < 0$ by changing the variable $y \to -y$ in the latter to give the real integral:

$$n_{1,k} = -\int_0^\infty \frac{dy}{2\pi i} \frac{1}{(y^2+1)^g} \left(\frac{1}{y+i} + \frac{1}{-y+i} \right) \tag{4.633}$$

$$= \int_0^\infty \frac{dy}{\pi} \frac{1}{(1+y^2)^{1+g}} = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}+g)}{\Gamma(1+g)} \tag{4.634}$$

The integral is in a standard form, which is given in tables [G&R, 3.194(3) after changing $y^2 = x$] in terms of gamma functions. In the limit where $g \to 0$ then $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$ so that

$$\lim_{g \to 0} n_{1,k} = \tfrac{1}{2} \tag{4.635}$$

The distribution function is a nonanalytic function of the coupling constant g. The usual noninteracting distribution function is found in the case of $g = 0$. The introduction of an arbitrarily weak delta function potential destroys the Fermi distribution, and each wave vector state is occupied with an equal probability. For the case where $g \to 0$, this probability approaches $\frac{1}{2}$. The $g = 0$ result is not obtained in the $g \to 0$ limit.

This result would be difficult to prove by perturbation theory and shows the value of an exact solution. These results pertain only to the one-dimensional electron gas. Behavior of this type is called *non-Fermi liquid behavior.* Fermi liquid theory is discussed in Chapter 11. Another name to describe such systems is *Luttinger liquids.*

4.6. POLARITONS

4.6.1. Semiclassical Discussion

The word *polariton* was coined by Hopfield (1958) to describe the normal modes in solids which propagate as electromagnetic waves. The word is a combination of *polarization* and *photon*, because these modes are combinations of free photons and the polarization modes of the solid. A new word was needed, because a new view was then emerging about the optical properties of solids. Hopfield popularized this new physics, although similar ideas had been discussed earlier by Fano (1956, 1960) and by Born and Huang (1954).

In the old view of electromagnetic wave propagation in solids, the light shone upon the surface of a sample and went into it. The polarization modes of the solid, e.g., TO phonons, could absorb some of this light.

The new view is that the light and the polarization modes in the solid are coupled into a new set of normal modes. These new modes are called polaritons. When light is shone upon the surface, polaritons are created which propagate inward. The mathematics is trivial; since both the photons and the polarization modes are usually described by harmonic oscillator equations, the new modes are obtained by solving coupled harmonic oscillator equations. The physical effect is semiclassical and need not involve quantum mechanics. The photon Green's function $\mathscr{D}_{\mu\nu}(\mathbf{q}, \omega)$ in a system with dielectric function $\varepsilon_{\mu\nu}(\mathbf{q}, \omega)$ was derived in (2.185). A