Date. No.

2021.1.17

本学期内容回顾(按课程次数):

一. 简介

二. $\delta^{(2)}(x)$, RG

三. 发散, Path Integral

四. 无穷维积分 (Gaussian integral)

五. 具体计算 moment 展开 / cumulant 展开

connected diagram

$$\langle u e^{-O}\rangle_0 \leftarrow \mathcal{L}_0$$

$$= \sum_l \frac{\langle u O^l\rangle_c^0}{l!}(-1)^l$$

六、ϕ^4 theory

七、ϕ^4 theory

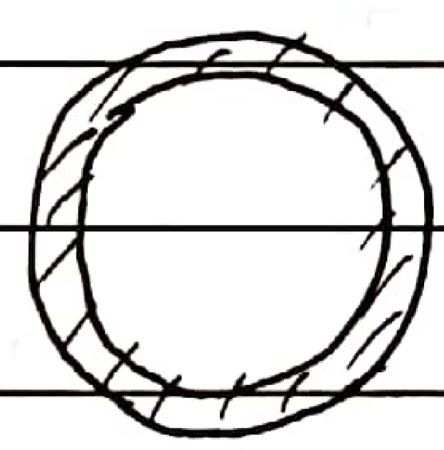
八、ϕ^4 theory

九、ϕ^4 theory

十、ϕ^4 theory

(六–十) R, RG 等价性

Callan-Symmzik eq



十一、BKT

十二、BKT

十三、BKT

十四、BKT

十五、~~Boson~~ BKT

(十一–十五) 简单 Model

Wilson

K-W transformation (duality)

十六、Bosonization

十七、Bosonization

十八、Bosonization 的应用

补充:

1. BKT $\leftrightarrow$ Bosonization 有联系
   - Bosonization $\longrightarrow$ (t, x) Sine-Gordon model
   - BKT $\longrightarrow$ (x, y) model

2. 没有讲 QED, Maxwell / gauge, Goldstone / Higgs mechanism

Date. No.

$$\psi = \frac{\eta}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi}$$

ϕ → Bose field

$\sqrt{4\pi}$ ↑ 满足反对易

$\sqrt{2\pi\alpha}$ ↑ cutoff，让 $\langle\psi^\dagger(x)\psi(y)\rangle$ 相同

两个应用
- 1. XXZ model
- 2. ~~同~~分数 Hall state (X. G. Wen, 1992)
  - $\psi \to$ anyon : $\sqrt{4\pi} \to \beta$
- 3. Random Ising model (Shankar) ⇒ Bosonization（可解）* 不讲，涉及到 replica trick（最早是由 Anderson 提出的）

二. fermion如何用boson表示

问题：$\psi \sim e^{i\sqrt{4\pi}\phi}$
- 数学上没问题（上节课已解决）
- 物理上如何实现 $F \to B$?

有一个图像 ______o o o o o o o o

fermion 不能交换（被禁戒）
所以只有集体激发 ⟶ 看起来像 boson

(1) 具体计算上还需要考虑 fermion 算符 c $\xrightarrow{\text{用}}$ boson 算符 b 来表示.

* 最早是在 Luttinger Liquid 中给出这种算符代数关系.

(2) 还要解释其物理意义.

(3) $[\phi(x), \phi(y)] = \frac{i}{4}\varepsilon(x-y)$

$[\partial\phi(x), \phi(y)] \propto \delta(x-y)$

$[\partial\phi(x), \partial\phi(y)] \propto \delta'(x-y) \iff [\rho(x), \rho(y)] \propto \delta'(x-y)$

以 Luttinger Liquid 为例.

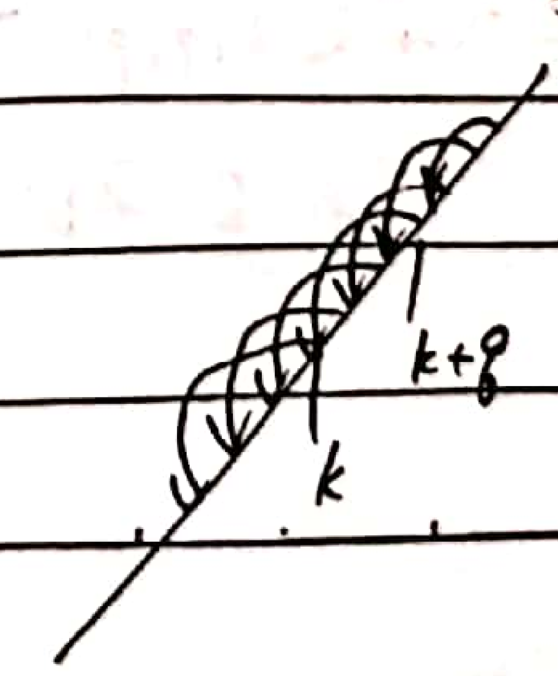
Ref: (1) 最好的教材：Mahan

(2) Luttinger (1963)；纠正错误 Mattis, Lieb (1965)

理解： $:\rho(x): = \frac{N}{L} + \frac{1}{L}\sum_{q\neq 0} e^{-iqx}\rho_q$

$\rho_q = \sum_k C_k^\dagger C_{k+q}$ 有明确的物理意义

其中
$$\begin{aligned}\rho_q &= \int \rho(x) e^{iqx}dx \\ &= \int C^\dagger(x)C(x)e^{iqx}dx \\ &= \int C_k^\dagger C_{k+q} e^{-i(k+q)\cdot x + ikx + iqx}dx \\ &= \sum_k C_k^\dagger C_{k+q}\end{aligned}$$

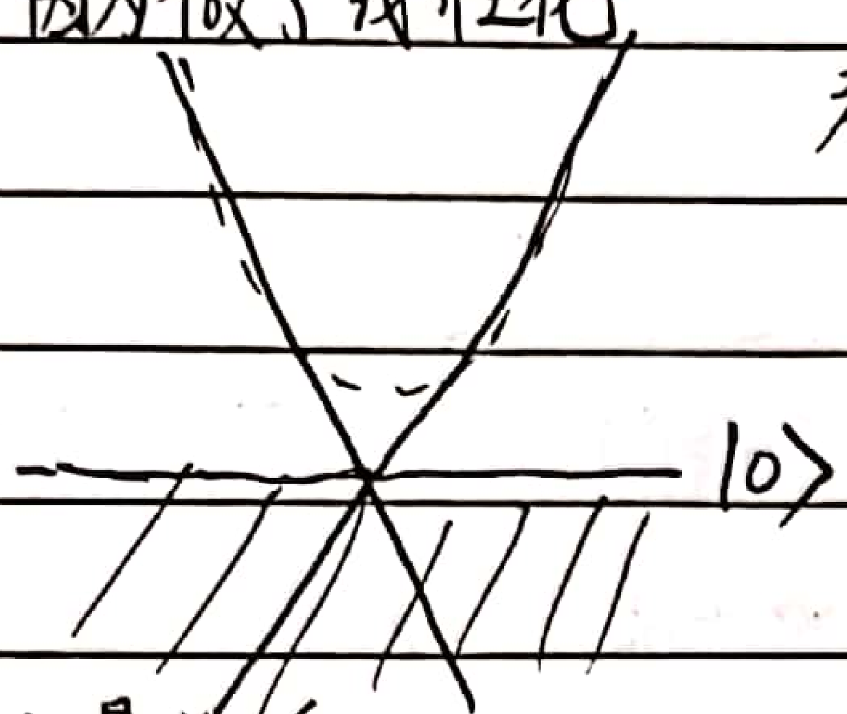


目标：1.证明 $\rho_q \propto b_q$ （ρ_q代数实际上是Kac-Moody algebra）

2.计算：$\rho(x)$：

麻烦：$q=0$时 $\Rightarrow \rho_0=\sum_k C_k^\dagger C_k=\sum_k n_k$

因为做了线性化



考虑的谱 能量 $\to -\infty$，一定会发散

解决办法：做正规化，扣除掉 $|0\rangle$ 以下的态

故：$\boxed{:\rho_0: = \rho_0 - \langle 0|\rho_0|0\rangle}$

计算对易关系

$[\rho_q, \rho_{q'}]$

错误计算：

ρ与θ是共轭量 故 $[\rho(x),\theta(x')]\propto\delta(x-x')$

$[\rho(x),\rho(x')]=0$

$\to [\rho_q,\rho_{q'}]e^{iqx+iq'x'}=0$

正确 ~~正确~~计算

$[\rho_q,\rho_{q'}]$

$=\sum_{kk'}[C_k^\dagger C_{k+q}, C_{k'}^\dagger C_{k'+q'}]$

$=\sum_{kk'}(C_k^\dagger C_{k+q}C_{k'}^\dagger C_{k'+q'} - C_{k'}^\dagger C_{k'+q'}C_k^\dagger C_{k+q})$

$=\sum_{kk'} C_k^\dagger(\cancel{-C_{k'}^\dagger C_{k+q}} + \delta_{k'=k+q})C_{k'+q'} - C_{k'}^\dagger(\delta_{k=k'+q'} \cancel{- C_k^\dagger C_{k'+q'}})C_{k+q}$

$\cancel{=\sum_{kk'}(C_k^\dagger C_{k'+q'}\delta_{k'=k+q} - C_{k'}^\dagger C_{k+q}\delta_{k=k+q})}$

~~错误计算~~

$\cancel{=\sum_k C_k^\dagger C_k - \underbrace{\sum_k C_{k+q}^\dagger C_{k+q}}_{平移}}$

$=\sum_{kk'}(C_k^\dagger C_{k'+q'}\delta_{k'=k+q} - C_{k'}^\dagger C_{k+q}\delta_{k=k'+q'})$

错误计算：

$=\sum_k C_k^\dagger C_{k+q+q'} - \sum_{k'} C_{k'}^\dagger C_{k'+q+q'}=0$

这种计算在2D、3D都成立，但只有1D不成立

$$= \sum_k \left(C_k^\dagger C_{k+q+q'} - C_{k-q'}^\dagger C_{k+q} \right)$$

我们关心的是 $q+q'=0 \Rightarrow q'=-q$

$$= \sum_k \left(C_k^\dagger C_k - C_{k+q}^\dagger C_{k+q} \right)$$

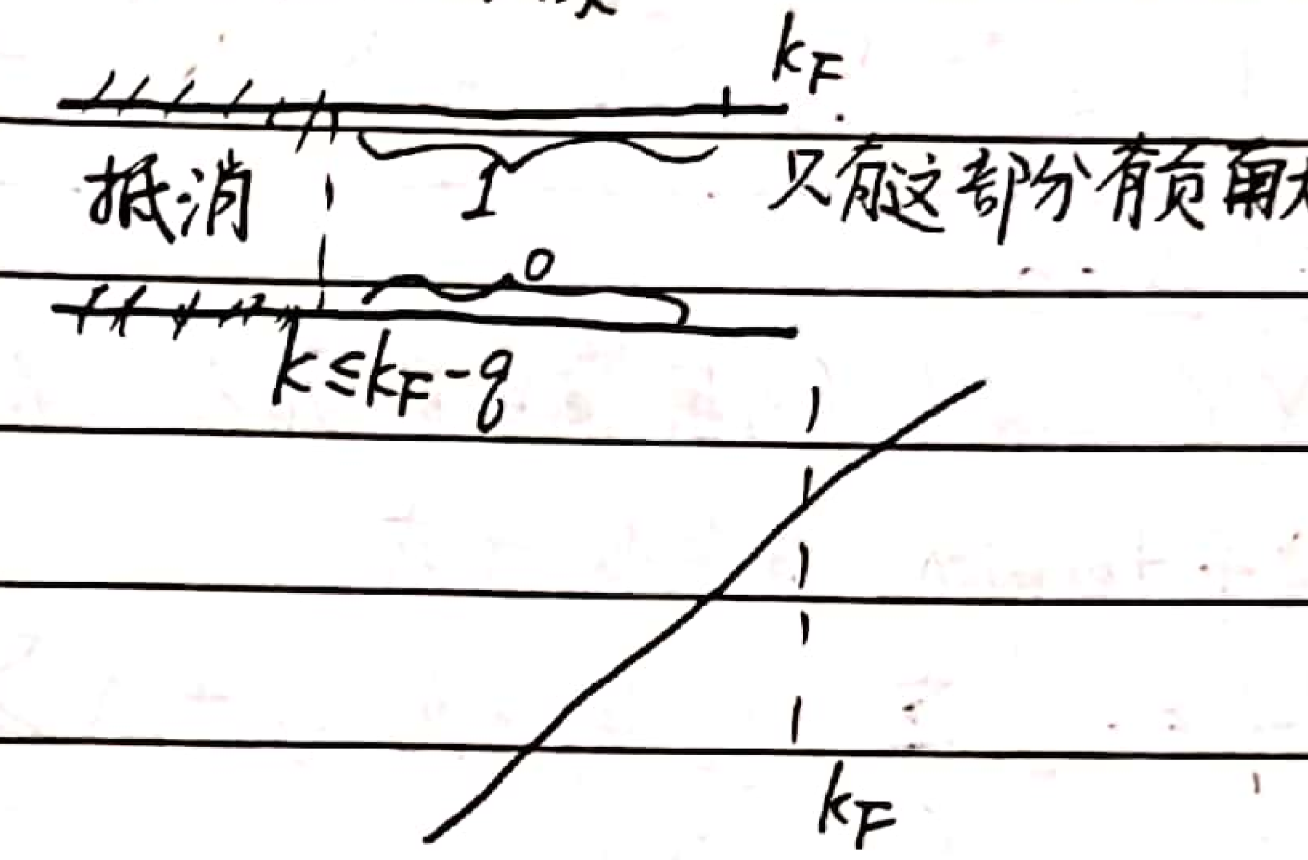
原因：在基态（GS）不能平移，$k \to k+q$ （X）.

$$\langle 0 | [\rho_q, \rho_{q'}] | 0 \rangle \big|_{q'=-q}$$

$$= \langle 0 | [\rho_q, \rho_{-q}] | 0 \rangle$$

$$= \sum_k \langle 0 | \underset{①}{C_k^\dagger C_k} - \underset{②}{C_{k+q}^\dagger C_{k+q}} | 0 \rangle \propto q$$

物理图像



两项的非零区间 $\left\{ \begin{array}{l} \text{IF } |k| > k_F \\ \text{① 的非零区间：} k \le k_F \\ \text{② 的非零区间：} k+q \le k_F \end{array} \right.$

回到从共轭量计算的方法：

$$[\phi(x), \phi(y)] = \frac{i}{4} \mathrm{sgn}(x-y)$$

$$[\partial\phi, \partial\phi] \propto \delta'(x-y)$$

$$[\rho(x), \rho(y)] \propto \delta'(x-y)$$

$$[\rho_q, \rho_{-q}] e^{iq(x-y)} \propto \boxed{\delta'(x-y)}$$

$$\int dq\, [\rho_q, \rho_{-q}] e^{iq(x-y)} \propto \int q e^{iq(x-y)} dq$$

（正反皆可推出）

$$\delta(x-y) = \int dq\, e^{iq(x-y)}$$

$$\delta'(x-y) \propto \int dq\, q\, e^{iq(x-y)}$$

对于有相互作用情况：

$$\int dx\, (\partial_x \phi)^2 \sim \int \rho^2(x)\, dx \sim \int \rho_q \rho_{-q}\, dq$$

令 $[\rho_q, \rho_{-q}] = \frac{Lq}{(2\pi)}$

$$\rho_{-q} \triangleq \rho_q^\dagger$$

$$[\rho_q, \rho_q^\dagger] = \frac{Lq}{(2\pi)} \Leftrightarrow [b_q, b_q^\dagger] = 1$$

Define : $b_q = \sqrt{\frac{2\pi}{L|q|}}\, \rho_q$

故 $\int dx\, (\partial_x \phi)^2 \sim \int \rho^2(x) dx \sim \int \rho_q \rho_{-q} dq \propto \int q^2\, b_q^\dagger b_q\, dq$

物理上 $b_q = \sum_k \sqrt{\frac{2\pi}{L|q|}}\, C_k^\dagger C_{k+q}$

即 $\boxed{\text{particle - hole excitation} \xleftrightarrow{\text{等价于}} \text{boson}}$

~~* 类似于超导中 cooper pair~~

Date. No.

XXZ model.

ref. Shankar. §18.4 P346 - P352

目标：推出 Sine-Gordon model

后续：Shankar §18.4.2 SG model's RG (Wilson)

(但这部分已经在BKT相变部分讲完了)

$$\boxed{\text{XXZ model} \quad H = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z}$$

IF $\Delta = 0$

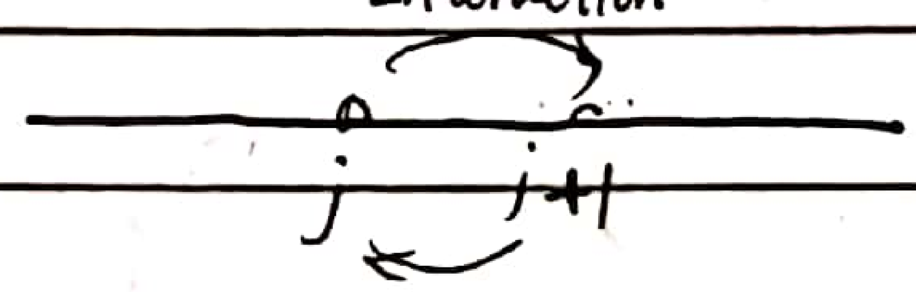
得XX model (可用简单的 Jordan-Wigner 变换解决)

对于fermion，有类似形式

$$H = -\frac{1}{2}\sum_j (\psi_{j+1}^\dagger \psi_j + h.c.) + \Delta \sum_j (\psi_j^\dagger \psi_j - \tfrac{1}{2})(\psi_{j+1}^\dagger \psi_{j+1} - \tfrac{1}{2}) + h \sum_j \psi_j^\dagger \psi_j$$

(Bette-Ansatz 解)

Interaction

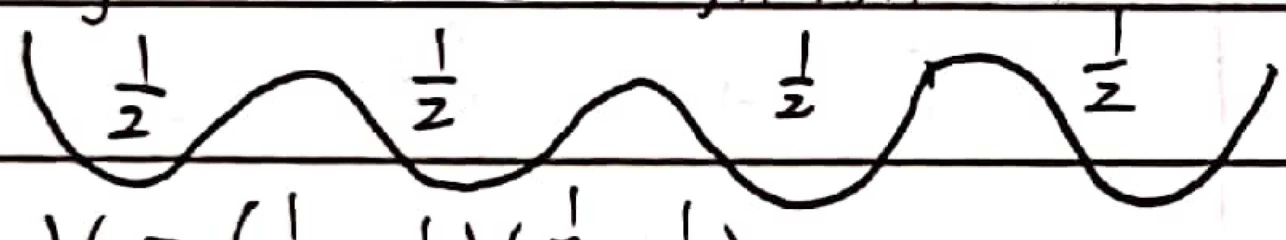


可以有化学势

讨论：

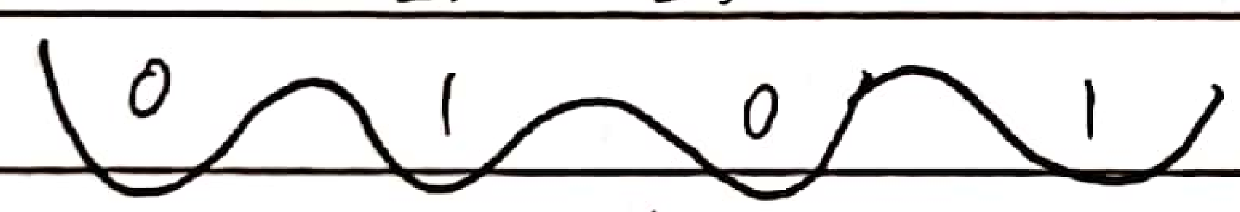
(1) 若 Δ 较大

$$V = \Delta \sum_j (\psi_j^\dagger \psi_j - \tfrac{1}{2})(\psi_{j+1}^\dagger \psi_{j+1} - \tfrac{1}{2})$$



可能产生 CDW (charge density wave)

$$V = (\tfrac{1}{2} - \tfrac{1}{2})(\tfrac{1}{2} - \tfrac{1}{2}) = 0$$

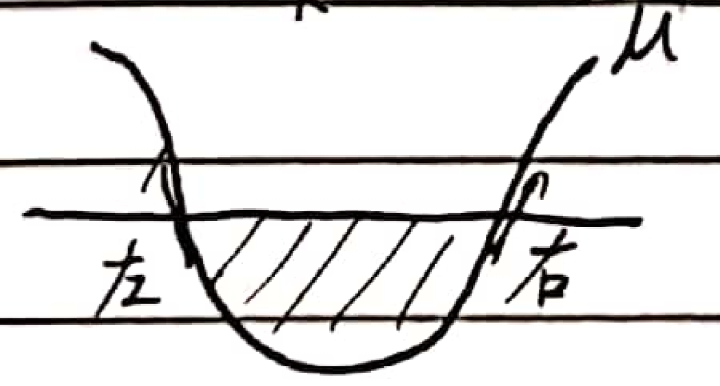


$$V = -\tfrac{1}{2} \cdot \tfrac{1}{2} = -\tfrac{1}{4}$$

(2) 若 Δ 较小

$$\text{FT: } H = -\sum_k 2t \cos k \, C_k^\dagger C_k + h C_k^\dagger C_k$$

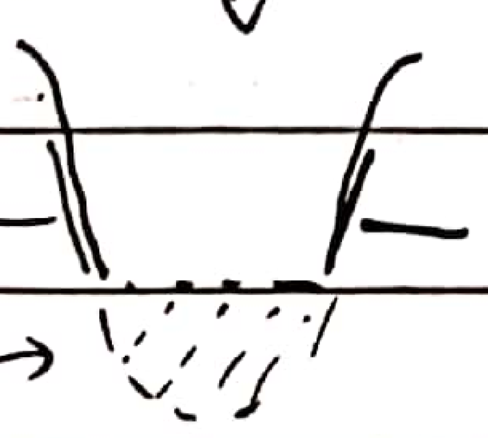
$$= -\sum_k (\cos k - \underbrace{h}_{\mu}) C_k^\dagger C_k$$



$$\psi_j = \int_{-\pi}^{\pi} \psi(k)\, e^{ikj} \left(\frac{dk}{2\pi}\right)$$

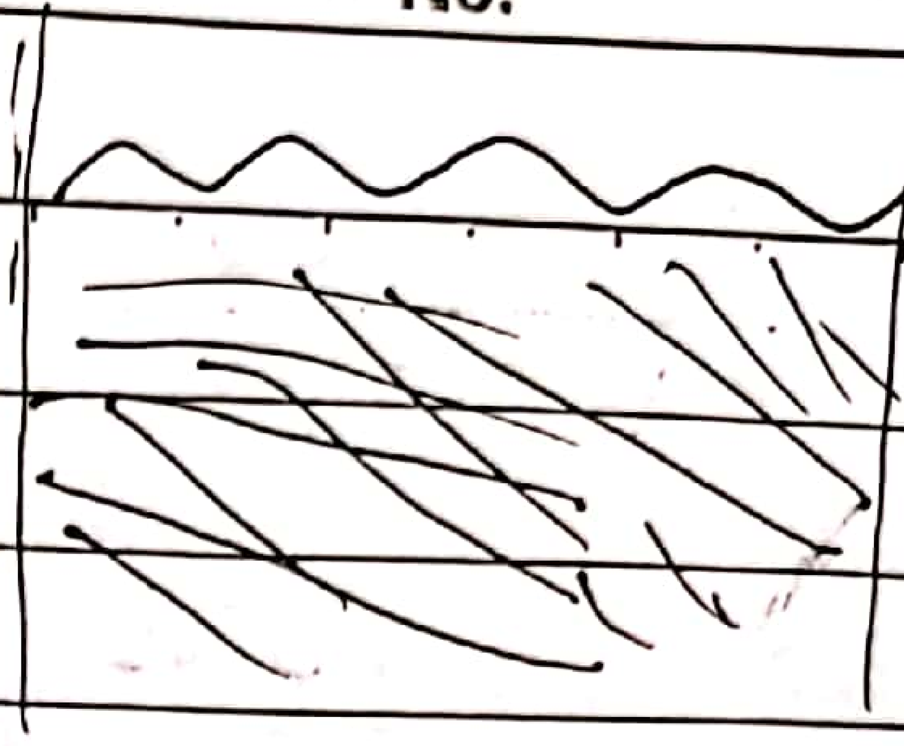
$$\Downarrow$$

引入截断 Λ 将下半部分砍掉



$$= \int_{-\Lambda}^{\Lambda} \underbrace{\psi(k_F + k)\, e^{i(k_F+k)\cdot j}}_{\text{Right}} dk + \int_{-\Lambda}^{\Lambda} \psi(-k_F + k)\, e^{i(-k_F+k)\cdot j} dk$$

$$\psi_j = \sqrt{a}\left[\underbrace{e^{ik_F \cdot j}}_{快}\underbrace{\psi_+(x=a_j)}_{慢} + \underbrace{e^{-ik_F\cdot j}}_{快}\underbrace{\psi_-(x=a_j)}_{慢}\right]$$

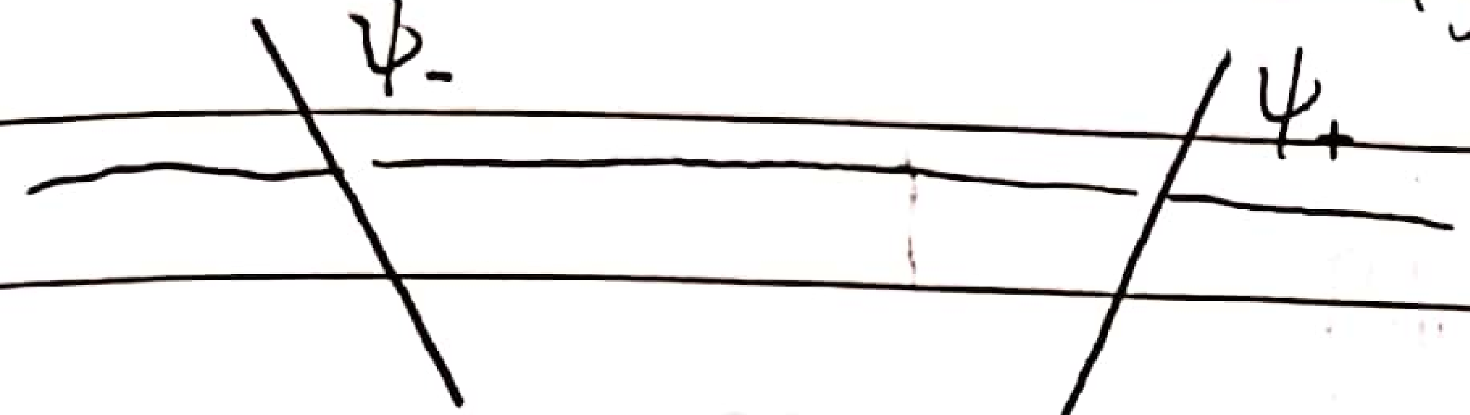


$$e^{ik_F\cdot x} = e^{ik_F^0 x} e^{i\delta k_F x} \quad (\delta k_F / k_F^0 \sim 0)$$

当$k_F = \frac{\pi}{2}$时

$$\psi_j = \sqrt{a}\left[e^{i\frac{\pi}{2}j}\psi_+(a_i) + e^{-i\frac{\pi}{2}j}\psi_-(a_j)\right]$$

~~其中a是晶格常数，来自于离散→连续~~ $\begin{cases}\int f(x)dx = \sum_i f_i a \\ \int \psi^\dagger(x)\psi(x)dx = \sum_i \psi_i^\dagger \psi_i a\end{cases}$



(1) IF $\Delta = 0$. ($\psi^\dagger \frac{p^2}{2m}\psi \to \frac{k^2}{2m}$, $\psi^\dagger \mathcal{L}\psi = vk\,\psi_k^\dagger\psi_k$)

$$H_0 = -iv\psi_+^\dagger\partial_x\psi_+ + iv\psi_-^\dagger\partial_x\psi_-$$

$$\psi^\dagger\psi \sim \rho \sim \partial\phi$$

$$\psi^\dagger\partial_x\psi \sim (\partial_x\phi)^2$$

(2) IF $\Delta \neq 0$ (interaction)

$$\sum_j \Delta(\psi_j^\dagger\psi_j - \tfrac{1}{2})(\psi_{j+1}^\dagger\psi_{j+1} - \tfrac{1}{2})$$

类似于之前的

$$\psi^\dagger(x)\psi(x) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon \gg \alpha}} \psi^\dagger(x+\varepsilon)\psi(x) - \underbrace{\langle\psi^\dagger(x+\varepsilon)\psi(x)\rangle}$$ 扣去背底

$$\rho = \frac{\hat{N}}{L} + \frac{1}{L}\sum_{q\neq 0} e^{-iqx}\rho_q$$

$$= \cancel{\Delta\sum_j \psi_j^\dagger\psi_j} \quad \Delta\sum_j \underbrace{:\psi_j^\dagger\psi_j:}\underbrace{:\psi_{j+1}^\dagger\psi_{j+1}:}$$

$$\psi_j = \sqrt{a}\left[e^{i\frac{\pi}{2}j}\psi_+(a_j) + e^{-i\frac{\pi}{2}j}\psi_-(a_j)\right]$$

$$\psi_j^\dagger\psi_j = a\left[e^{-i\frac{\pi}{2}j}\psi_+^\dagger(a_j) + e^{i\frac{\pi}{2}j}\psi_-^\dagger(a_j)\right]\left[e^{i\frac{\pi}{2}j}\psi_+(a_j) + e^{-i\frac{\pi}{2}j}\psi_-(a_j)\right]$$

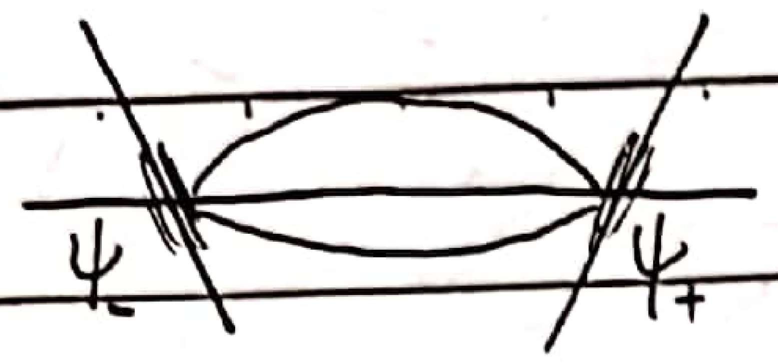
$\boxed{e^{-i\pi j} = (-1)^j}$

$$\propto \underbrace{\psi_+^\dagger(a_j)\psi_+(a_j) + \psi_-^\dagger(a_j)\psi_-(a_j)}_{\text{Normal 设为 } X} + \underbrace{(-1)^j\psi_+^\dagger(a_j)\psi_-(a_j) + (-1)^j\psi_-^\dagger(a_j)\psi_+(a_j)}_{\text{OSC (振荡部分) 设为 } Y(-1)^j}$$

$$\to (X + (-1)^j Y)(X + (-1)^{j+1}Y) = X^2 - Y^2$$

Date. No.

$X \propto (\partial\phi) \propto \rho$



~~原式~~ h.c. + $\psi_+^\dagger \psi_-$

$\sim e^{-i\sqrt{4\pi}\phi_+} e^{-i\sqrt{4\pi}\phi_-} + h.c.$

$\sim e^{-i\sqrt{4\pi}\phi} + h.c.$

$\sim \cos(\sqrt{4\pi}\phi)$

原式 $= a\Delta \sum_j \underbrace{\overbrace{\left(\frac{1}{\sqrt{\pi}}\partial_x\phi\right)^2}^{X}}_{\rho} - \overbrace{[\psi_+^\dagger \psi_- + h.c.]^2}^{Y}$

$= \Delta \int dx \left[\frac{1}{\pi}(\partial_x\phi)^2 - \frac{1}{\pi^2\alpha^2}\sin^2(\sqrt{4\pi}\phi)\right]$

$\boxed{\text{公式}\left(-\frac{1}{\pi\alpha}\cos(\sqrt{4\pi}\phi)\right)^2 = -\frac{1}{\pi}(\partial_x\phi)^2 + \frac{1}{2\pi^2\alpha^2}\cos(\sqrt{16\pi}\phi)}$

$= \Delta \int dx \left[\frac{2}{\pi}(\partial_x\phi)^2 + \frac{1}{2\pi^2\alpha^2}\cos(\sqrt{16\pi}\phi)\right]$

(SG model 形式)

最终：XXZ model

$\Rightarrow H = \int dx \left[\frac{1}{2}\pi^2 + \left(1+\frac{4\Delta}{\pi}\right)(\partial_x\phi)^2 + \frac{\Delta}{2\pi^2\alpha^2}\cos(\sqrt{16\pi}\phi)\right]$

$\begin{cases} Z = \int D\phi\, e^{-S} \\ S = \int dt dx\, H \end{cases}$

XY model ($dx\,dy$) $\longleftrightarrow$ XXZ model ($dx\,dt$)

取 $K = \left[1+\frac{4\Delta}{\pi}\right]^{-1/2}$

$HK = \int dx \frac{1}{2}\left[K\pi^2 + \frac{1}{K}(\partial_x\phi)^2\right] + \frac{y}{2\pi^2\alpha^2}\cos(\sqrt{16\pi}\phi)$, $y = k\Delta$

令 $\begin{cases} \phi' = \frac{1}{\sqrt{K}}\phi \\ \pi' = \sqrt{K}\phi \end{cases}$

$H_c = \int dx \frac{1}{2}\left[\pi'^2 + (\partial_x\phi')^2\right] + \frac{y}{2\pi^2\alpha^2}\cos(\sqrt{16\pi K}\phi')$

$S = \int dt dx\, H_c$

$\rightarrow \int dx dy\, H_{xxz}$

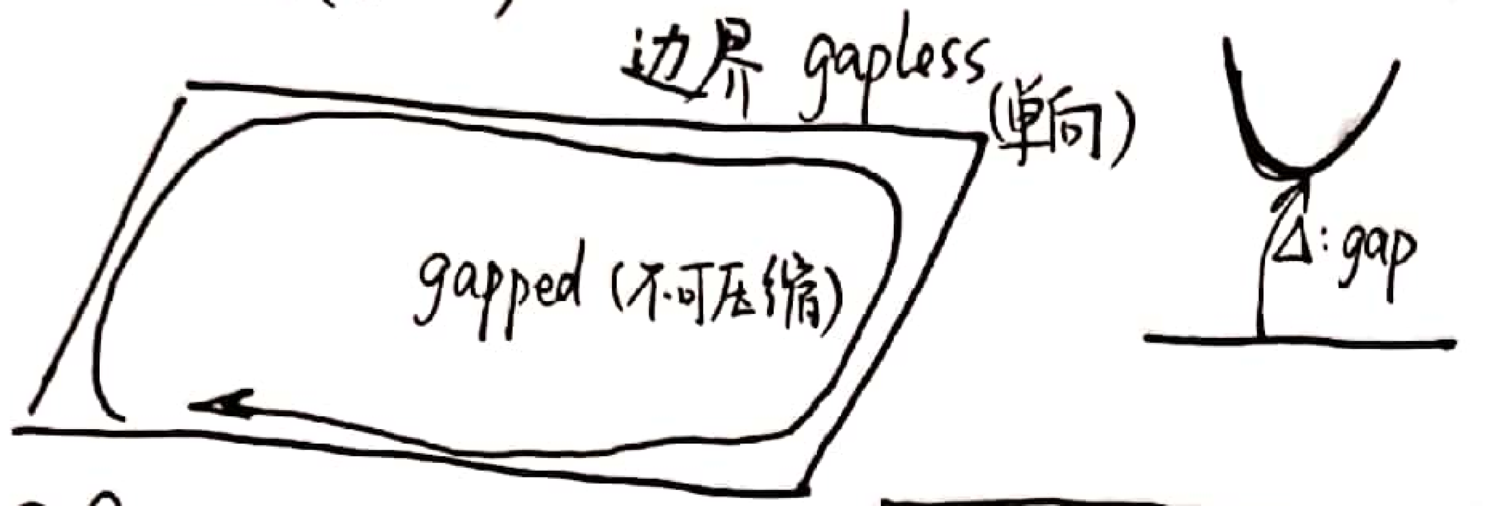
三. 任

目的：任意子的 Bosonization

$$\psi(x)\psi(y) = (-1)^{1/\nu}\psi(y)\psi(x)$$

* 这种形式的 anyon 很难计算（严格对角化等）

~~W~~ XG Wen (1992)



$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0}$$

$$H = \pi \int dx \frac{v}{\nu}\rho^2(x)$$

真实相互作用 $E = \int \rho(x) V(x-y)\rho(y)\, dy$

↓

短程相互作用 $E = \int V_0 \rho^2(x)\, dx = V_0 \int (\rho_0 + \delta\rho)^2 dx = V_0 \int \rho_0^2 dx + V_0 \int \delta\rho^2\, dx$

$(\rho(x) = \rho_0 + \delta\rho,\ \int \delta\rho\, dx = 0)$

联立

① $H = \pi \frac{v}{\nu}\int \rho^2(x)\, dx \xrightarrow{FT} 2\pi \frac{v}{\nu}\sum_{k>0} \rho_{-k}\rho_k$

其中 $\rho_k = \int dx \frac{1}{\sqrt{L}}\rho(x) e^{ikx}$

② $-\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0$

$$\dot{\rho}_k = \int dx \frac{1}{\sqrt{L}}\left(\frac{\partial \rho}{\partial t}\right) e^{ikx} = ivk\rho_k$$

将 ρ_k 与 π_k 看作正则量
(q) (p)

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

$$\Leftrightarrow \boxed{[\rho_k, \pi_{k'}] = i\delta_{kk'}}$$

$$\dot{\rho}_k = \frac{\partial H}{\partial \pi_k}$$

$\rho_k \Rightarrow q$ 要求 $\pi_k = i2\pi\rho_{-k}/(\nu k)$

$$\left.\begin{aligned}\dot{\rho}_{-k} &= -ivk\rho_{-k}\\ &= -\frac{\partial H}{\partial \rho_k}\end{aligned}\right\}$$

$[\rho_k, \rho_{k'}] = \frac{\nu}{2\pi}k\,\delta_{k+k'=0}$

OR

$[\rho_k, \rho_{k'}^\dagger] = \frac{\nu}{2\pi}k\,\delta_{k=k'}$

即 Kac-Moody algebra

$$[\rho(x),\rho(y)] = \frac{i\nu}{2\pi}\delta'(x-y)\ \boxed{\neq 0}\ !$$

原因：有基态 $\langle 0|[\ ,\ ]|0\rangle$ 破缺平移对称性.

$$[\rho(x),\phi(y)] = -i\nu\,\delta(x-y)$$

$$\rho = \frac{1}{2\pi}\partial\phi$$

$$\psi \sim e^{i\frac{1}{\nu}\phi(x)} \Longrightarrow \psi(x)\psi(y) = (-1)^{\nu}\psi(y)\psi(x)$$

四.

Ref: Univeral Theory of Nonlinear Luttinger Liquids
Glazman (Yale), Science, 2009.

- 要考虑动力学性质时，一定要考虑激发态.

$$\frac{k^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 \Rightarrow$$ quantum hydrodynamic theory

补充：

将XY model 推广到 3D ： particle-vortex duality ((2+1)d). David Tong, Song (越南人)
(XY model couple 规范势).

non-Abelian bosonization : Witten, 1983

今年未讲：

NLσM
Topo term