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An Exactly Soluble Model of a Many-Fermion System*

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An exactly soluble model of a one-dimensional many-fermion system is discussed. The model has a fairly realistic interaction between pairs of fermions. An exact calculation of the momentum distribution in the ground state is given. It is shown that there is no discontinuity in the momentum distribution in this model at the Fermi surface, but that the momentum distribution has infinite slope there. Comparison with the results of perturbation theory for the same model is also presented, and it is shown that, for this case at least, the perturbation and exact answers behave qualitatively alike. Finally, the response of the system to external fields is also discussed.

I. INTRODUCTION

WE shall be concerned in this paper with a model of a many-fermion system which is exactly soluble. The model is quite unrealistic for two reasons: it is one-dimensional and the fermions are massless. On the other hand, it has the realistic feature that there is a true pair interaction between the particles. It is very closely related to the well-known Thirring Model[1] in field theory, though slightly more general. Our main interest in the model is in connection with the question of whether or not a sharp Fermi Surface (F.S.) exists in the exact ground state.

This question has only been investigated previously[2] by a special sort of many-body perturbation theory, when it has been shown for the usual realistic three-dimensional many-fermion system that each term of the series does give rise to a sharp F.S. This, of course, proves nothing about the entire series unless one can also prove something about its convergence, which has not been possible so far. The main point of this investigation therefore is to see if in this soluble model the exact solution and the perturbation solution (via propagators) behave in an essentially different fashion.

We now consider the exact formulation of the model. Consider first the case of no interaction between the particles. These are taken to be spinless, massless, fermions moving in a one-dimensional space. The analogue of the relativistic Dirac Hamiltonian is $v_0\sigma_3 p$ (σ_3 is the usual Pauli spin matrix; units such that $\hbar = 1$ are chosen). v_0 is the velocity of the particles, which would be c in the relativistic case. Then the Hamiltonian is

$$H_0 = v_0 \int_0^L \psi^+(x)\sigma_3 p\psi(x)\, dx. \tag{1}$$

Here ψ is the two component spinor

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{2}$$

and we are assuming that the particles are confined to a length L along the x axis. The quantity p is of course the ordinary momentum operator $1/i\, \partial/\partial x$.

Written out, (1) becomes

$$H_0 = v_0 \int_0^L [\psi_1^+ p\psi_1 - \psi_2^+ p\psi_2]\, dx. \tag{3}$$

If we go into momentum space via

$$\psi_j(x) = \sum_k a_{jk} e^{ikx}/L^{\frac{1}{2}} \tag{4}$$

(where the allowed values of k are

$$k = (2\pi/L)n, \qquad n = 0, \pm 1, \pm 2, \cdots, \pm\infty \tag{5}$$

since we shall impose periodic boundary conditions on our sample), we obtain

$$H_0 = v_0 \sum_k (a_{1k}^+ a_{1k} - a_{2k}^+ a_{2k})k. \tag{6}$$

The creation and destruction operators a, a^+ satisfy the commutation relationship

$$a_{jk}^+ a_{j'k'} + a_{jk'} a_{jk}^+ = \delta_{jj}\delta_{kk'}. \tag{7}$$

Since the allowed values of $a_{jk}^+ a_{jk}$ are zero and unity, the lowest state of H_0 is $-\infty$ since we can choose all the $j = 1$, $k < 0$ and the $j = 2$, $k > 0$ states occupied. This is the usual problem occurring in Dirac theory and requires a redefinition of the creation and destruction operators so that we deal only with "particles" and "holes". Define

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[1] W. Thirring, Ann. Phys. **3**, 91 (1958). See also V. Glaser, Nuovo Cimento **9**, 990 (1958); T. Pradhan, Nucl. Phys. **9**, 124 (1961); K. Johnson, Nuovo Cimento **21**, 773 (1961).
[2] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960); J. M. Luttinger, *ibid.* **119**, 1153 (1960); **121**, 942 (1961).

$$\begin{aligned} a_{1k} &= b_k \quad k>0 \\ &= c_k^+ \quad k<0, \\ a_{2k} &= b_k \quad k<0 \\ &= c_k^+ \quad k>0. \end{aligned} \tag{8}$$

We may also write this as

$$b_k = \theta_k^+ a_{1k} + \theta_k^- a_{2k}, \qquad c_k = \theta^- a_{1k}^+ + \theta_k^+ a_{2k}^+, \tag{9}$$

where

$$\begin{aligned} \theta_k^+ &= \begin{matrix} 1 & k>0 \\ 0 & k<0, \end{matrix} \\ \theta_k^- &= \begin{matrix} 1 & k<0 \\ 0 & k>0. \end{matrix} \end{aligned} \tag{10}$$

From (9) we see at once that b_k, c_k also have the commutation rules of fermions, i.e.,

$$b_k^+ b_{k'} + b_{k'} b_k^+ = \delta_{kk'}, \quad c_k^+ c_{k'} + c_{k'} c_k^+ = \delta_{kk'}, \tag{11}$$

and all the rest anticommute.

Inserting (8) in (6) we obtain

$$H_0 = v_0 \sum_k (b_k^+ b_k + c_k^+ c_k)\,|k| + v_0\Big(\sum_{k<0} k - \sum_{k>0} k\Big). \tag{12}$$

The last term is infinite, but a constant, and as usual we simply redefine H_0 without it, i.e., we take

$$H_0 = v_0 \sum_k (b_k^+ b_k + c_k^+ c_k)\,|k|. \tag{13}$$

We shall call the operators b_k and c_k the destruction operators for particles and holes respectively. The vacuum state ϕ_0 is clearly defined by

$$b_k \phi_0 = 0, \qquad c_k \phi_0 = 0. \tag{14}$$

The interaction Hamiltonian H' is taken to be (this special choice is what makes the model soluble)

$$H' = 2\lambda v_0 \iint_0^L \psi_1^+(x)\psi_1(x) V(x-y) \times \psi_2^+(y)\psi_2(y)\, dx\, dy. \tag{15}$$

$V(x-y)$ is an arbitrary two-body potential at this point. If we write this in momentum space [assuming also that $V(x-y)$ satisfies periodic boundary conditions], we obtain

$$H' = \frac{2\lambda v_0}{L} \sum_{k_1\cdots k_4} \delta_{k_1-k_2+k_3-k_4,0} \times v(k_3-k_4) a_{1k_1}^+ a_{1k_2} a_{2k_3}^+ a_{2k_4}, \tag{16}$$

where

$$\begin{aligned} V(x) &= \frac{1}{L}\sum_k v(k) e^{-ikx}, \\ v(k) &= \int_0^L dx e^{ikx} V(x). \end{aligned} \tag{17}$$

The term in (16) corresponding to $k_3 = k_4$, say H'', is given by

$$H'' = \frac{2\lambda}{L} v_0 v(0) \Big(\sum_k a_{1k}^+ a_{1k}\Big)\Big(\sum_k a_{2k}^+ a_{2k}\Big). \tag{18}$$

This term clearly gives rise to divergent effects, since for the unperturbed vacuum the number of "1" and "2" particles are infinite. To avoid this difficulty, we shoose $v(0)$ to be zero, which is the same as taking the average value of the potential ($\bar{V}$) equal to zero. We also express this by saying that in (15) we replace $V(x-y)$ by $V(x-y) - \bar{V}$.

The total Hamiltonian of the problem is now given by

$$H = H_0 + H'. \tag{19}$$

II. EXACT SOLUTION OF THE MODEL

We shall show that (19) can be diagonalized by a very simple canonical transformation. Consider

$$\tilde{H} = e^{i\lambda S} H e^{-i\lambda S}, \tag{20}$$

where

$$S \equiv \iint_0^L dx\, dy \psi_1^+(x)\psi_1(x) E(x-y) \psi_2^+(y)\psi_2(y). \tag{21}$$

Here $E(x)$ is defined by

$$dE(x)/dx = V(x) - \bar{V}. \tag{22}$$

Writing

$$V(x) - \bar{V} = \frac{1}{L}\sum_k{}' v(k) e^{-ikx}, \tag{23}$$

we obtain

$$E(x) = \frac{i}{L}\sum_k{}' \frac{v(k)}{k} e^{-ikx}. \tag{24}$$

Let us define

$$N_i(x) = \psi_i^+(x)\psi_i(x). \tag{25}$$

Then from the commutation rules it follows at once that

$$(N_i(x), N_{i'}(x')) = 0, \tag{26}$$

so that

$$e^{i\lambda S} H' e^{-i\lambda S} = H'. \tag{27}$$

(In the non-second quantized version of the theory H' and S are just functions of position.)

Therefore

$$\tilde{H} = H + i\lambda(S, H_0) + [(i\lambda)^2/2'](S, (S, H_0)) + \cdots . \quad (28)$$

Using the commutation rules for the ψ_i, we obtain at once

$$(S, H_0) = v_0 \iint_0^L dx\, dy \left\{ \frac{\partial N_1(x)}{\partial x} E(x - y) N_2(y) - N_1(x) E(x - y) \frac{\partial N_2(y)}{\partial y} \right\}. \quad (29)$$

Integrating by parts and using the periodic boundary conditions to drop the surface terms, we obtain

$$\begin{aligned}(S, H_0) &= -\frac{2v_0}{i} \iint_0^L dx\, dy N_1(x) E'(x - y) N_2(y) \\ &= -\frac{2v_0}{i} \iint_0^L dx\, dy N_1(x)(V(x - y) - \bar{V}) N_2(y) \\ &= -\frac{1}{i\lambda} H'. \end{aligned} \quad (30)$$

Since this commutes with S, there are no higher terms in the series (28), and we obtain

$$\tilde{H} = \tilde{H} - H' = H_0. \quad (31)$$

(Again these results are seen very easily by going over to the non-second quantized representation.)

Now $\tilde{H}$ is trivial to diagonalize, just being the noninteracting Hamiltonian. Therefore, all the energy levels of H are the same as those of H_0. (This is very unrealistic indeed.) On the other hand, the wavefunctions of H are very different from the free-particle ones. If ψ_n^0 is a wavefunction of H_0 corresponding to energy E_n^0, then the corresponding wavefunction for H (say, ψ_n) is

$$\psi_n = e^{-i\lambda S} \psi_n^0. \quad (32)$$

Therefore, although the energy levels do not change as a result of the interaction, other properties depending on more details of the wavefunction may be profoundly altered.

We next want to formulate the many-body problem for our system. We at once have the following problem: since particle-hole pairs can be produced by the interaction, the number of particles in an eigenstate of H is not fixed. However, we clearly must have that the number of particles minus the number of holes (call this n) is fixed in an eigenstate. Writing

$$n = \sum_k (b_k^+ b_k - c_d^+ c_k), \quad (33)$$

we can easily verify by direct calculation that n is a constant of the motion.

The noninteracting case for the N-particle problem is clearly the case of n having the eigenvalue N. Similarly, we define the N-particle problem for the interacting case as the system for which n has the value N. There will always be a certain number of holse present, but the smaller the interaction, the smaller this number will be.

The exact ground state of the N-particle system may be obtained as follows. Certainly the lowest state (ψ_N^0) of $\tilde{H}$ for which $n = N$ is obtained by having no holes present. Then the first N particle states will be occupied. That is

$$\begin{aligned} b_k^+ \psi_N^0 &= 0, \quad |k| < k_F, \\ b_k \psi_N^0 &= 0, \quad |k| > k_F, \\ c_k \psi_N^0 &= 0, \end{aligned} \quad (34)$$

where the Fermi momentum k_F is determined by

$$N = \sum_{|k|<k_F} 1 = \frac{L}{2\pi} \int_{-k_F}^{k_F} dk = \frac{L}{\pi} k_F. \quad (35)$$

We may also write

$$\psi_N^0 = b_{k_N}^+ \cdots b_{k_1}^+ \phi_0, \quad (36)$$

where ϕ_0 is the unperturbed vacuum and $k_1 \cdots k_N$ are the N allowed momenta between $-k_F$ and k_F.

Therefore the exact ground-state wavefunction (ψ_N) is given by

$$\psi_N = e^{-i\lambda S} \psi_N^0. \quad (37)$$

In order to study the sharpness of the F.S., we must investigate[2] the mean number of particles with momentum k, say $\bar{n}_k$. We have, of course,

$$n_k = b_k^+ b_k,$$

so

$$\bar{n}_k = (\psi_N, b_k^+ b_k \psi_N) = (\psi_N^0, e^{i\lambda S} b_k^+ b_k e^{-i\lambda S} \psi_N^0). \quad (38)$$

If we wanted to know the average number of holes $\bar{N}_h$ present we may use

$$\bar{N}_h = \sum_k \bar{n}_k - N. \quad (39)$$

Clearly $\bar{n}_k$ is an even function of k, so we shall restrict ourselves to $k > 0$. Then, by (9),

$$b_k^+ b_k = a_{1k}^+ a_{1k} = \frac{1}{L} \iint_0^L d\xi\, d\eta e^{ik(\xi-\eta)} \psi_1^+(\xi)\psi_1(\eta), \tag{40}$$

as one sees, by direct integration,

$$\bar{n}_k = \frac{1}{L} \iint_0^L d\xi\, d\eta e^{ik(\xi-\eta)} \times (\psi_N^0 | e^{i\lambda S} \psi_1^+(\xi)\psi_1(\eta) e^{-i\lambda S} | \psi_N^0). \tag{41}$$

Now we have the following operator identity

$$\exp\left[i\lambda \int_0^L g(x) N_1(x)\, dx \right] \psi_1(\eta) \times \exp\left[-i\lambda \int_0^L g(x) N_1(x)\, dx \right] = e^{-i\lambda g(\eta)} \psi_1(\eta), \tag{42}$$

if $g(x)$ commutes with $\psi_1(\eta)$. This is most easily proved by differentiating with respect to λ and making use of the fact that

$$(\psi_1(\eta), N_1(x)) = \delta(x-\eta)\psi_1(\eta). \tag{43}$$

Using (42), (41) becomes

$$\bar{n}_k = \frac{1}{L} \iint_0^L d\xi\, d\eta e^{ik(\xi-\eta)} \times \left(\psi_N^0 \left| \psi_1^+(\xi)\psi_1(\eta) \exp\left\{ i\lambda \int_0^L dy N_2(y) \times [E(\xi-y) - E(\eta-y)] \right\} \right| \psi_N^0 \right). \tag{44}$$

Expressed in terms of a_{ik}, (34) becomes

$$\begin{aligned} a_{1k}^+ \psi_N^0 &= 0, \quad k < k_F, \\ a_{1k} \psi_N^0 &= 0, \quad k > k_F, \end{aligned} \tag{45}$$

and

$$\begin{aligned} a_{1k}^+ \psi_N^0 &= 0, \quad k > -k_F, \\ a_{2k} \psi_N^0 &= 0, \quad k < -k_F, \end{aligned} \tag{46}$$

Writing $\psi_N^0 = \Psi_1 \Psi_2$ where Ψ_1, depends on the variables of the field "1" and is given by (45), and Ψ_2 depends on the variables of the field "2" and is given by (46), we have

$$\bar{n}_k = \frac{1}{L} \iint_0^L d\xi\, d\eta e^{ik(\xi-\eta)} (\Psi_1 | \psi_1^+(\xi)\psi_1(\eta) | \Psi_1) \times \left(\Psi_2 \left| \exp\left\{ i\lambda \int_0^L dy N_2(y) \times [E(\xi-y) - E(\eta-y)] \right\} \right| \Psi_2 \right). \tag{47}$$

From (45) we have at once

$$(\Psi_1 | \psi_1^+(\xi)\psi_1(\eta) | \Psi_1) = \frac{1}{L} \sum_{k'<k_F} e^{-ik'(\xi-\eta)}. \tag{48}$$

The second factor in (47) is also not difficult to reduce to simpler form. We have, in fact,

$$\left(\Psi_2 \left| \exp\left\{ i\lambda \int_0^L dy N_2(y) \times [E(\xi-y) - E(\eta-y)] \right\} \right| \Psi_2 \right) = \mathrm{Det}\,(g). \tag{49}$$

Det (g) is the determinant of the matrix $g_{\alpha\alpha'}$, where

$$g_{\alpha\alpha'} = \frac{1}{L} \int_0^L dy e^{-i(k_\alpha - k_{\alpha'})y} \times \exp\{ i\lambda [E(\xi-y) - E(\eta-y)] \}, \tag{50}$$

the k_α being the occupied states of the "2" particles in (46), i.e., the k_α are the set of discrete allowed k values greater than $-k_F$. The proof of (50) is given in the Appendix. The remarkable thing is that this (infinite) determinant can in fact be evaluated and the answer reduced to quadratures.

Writing Det $(g) = G(\xi, \eta)$, (47) becomes

$$\begin{aligned} \bar{n}_k &= \frac{1}{L^2} \iint_0^L d\xi\, d\eta \sum_{k'<k_F} e^{+i(k-k')(\xi-\eta)} G(\xi,\eta) \\ &\equiv \frac{2\pi}{L} \sum_{k'<k_F} F(k-k'), \end{aligned} \tag{51}$$

where

$$F(\kappa) = \frac{1}{2\pi L} \iint_0^L d\xi\, d\eta e^{i\kappa(\xi-\eta)} G(\xi,\eta). \tag{52}$$

III. EXPLICIT EVALUATION OF MOMENTUM DISTRIBUTION

We now must consider the determinant $G(\xi, \eta)$ in more detail. Since $k_\alpha = (2\pi/L)n$,

$$k_\alpha - k_{\alpha'} = (2\pi/L)(n-n');$$

$$n, n' = -n_F, -n_F+1, \cdots, \infty, \tag{53}$$

we may write

$$G = \begin{vmatrix} g_0 & g_{-1} & g_{-2} & \cdots \\ g_1 & g_0 & g_{-1} & \cdots \\ g_2 & g_1 & g_0 & \cdots \\ g_3 & g_2 & g_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \tag{54}$$

where

$$g_m = \frac{1}{L}\int_0^L dye^{-2\pi imy/L} \times \exp\{i\lambda[E(\xi - y) - E(\eta - y)]\}. \quad (55)$$

[(54) incidently, is independent of k_F.]

This type of determinant has been studied extensively, and is known as a *Toeplitz* determinant.[3] For very large order, an asymptotic formula can be given for them, which in our case (infinite-determinant) becomes exact. The result is the following: for a finite Toeplitz determinant

$$D_M = \begin{vmatrix} g_0 & g_{-1} & \cdots & g_{-M} \\ g_1 & g_0 & \cdots & g_{-M+1} \\ \vdots & \vdots & & \vdots \\ g_M & g_{M-1} & \cdots & g_0 \end{vmatrix},$$

we have[4]

$$\lim_{M\to\infty} \frac{D_M}{D^{M+1}} = \exp\left(\sum_{l=1}^{\infty} K_l K_{-l} l\right), \quad (56)$$

where

$$D = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} d\theta \log f(\theta)\right], \quad (57)$$

$$K_l = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-il\theta} \log f(\theta), \quad (58)$$

$$f(\theta) = \sum_{m=-\infty}^{\infty} g_m e^{im\theta}. \quad (59)$$

In the proof, $\log f(\theta)$ is defined by

$$\log f(\theta) = \log\{1 - [1 - f(\theta)]\} = -\sum_{n=1}^{\infty} \frac{[1 - f(\theta)]^n}{n}, \quad (60)$$

and it is assumed that this series converges.

In our case, this leads to particularly simple results. Changing variables in (55) from y to θ where

$$\theta = 2\pi y/L, \qquad 0 \le \theta \le 2\pi,$$

we obtain

$$g_m = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-im\theta} \times \exp\left\{i\lambda\left[E\left(\xi - \frac{L\theta}{2\pi}\right) - E\left(\eta - \frac{L\theta}{2\pi}\right)\right]\right\}. \quad (61)$$

Therefore from (59) we see at once that

$$f(\theta) = \exp\left\{i\lambda\left[E\left(\xi - \frac{L\theta}{2\pi}\right) - E\left(\eta - \frac{L\theta}{2\pi}\right)\right]\right\}. \quad (62)$$

Thus for sufficiently small λ, (60) is clearly satisfied since as one easily sees from (23) or (24), $E(x)$ is a bounded function of x. We shall for simplicity assume that λ is sufficiently small, and therefore we may write

$$\log f(\theta) = i\lambda\left[E\left(\xi - \frac{L\theta}{2\pi}\right) - E\left(\eta - \frac{L\theta}{2\pi}\right)\right]. \quad (63)$$

Now

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \log f(\theta) = \frac{i\lambda}{L}\int_0^L dy[E(\xi - y) - E(\eta - y)] = 0, \quad (64)$$

since, by (24), the average of $E(x)$ is zero. Therefore

$$D = 1. \quad (65)$$

Further,

$$K_l = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-il\theta} \log f(\theta) = \frac{i\lambda}{L}\int_0^L dy[E(\xi - y) - E(\eta - y)]e^{-2\pi ily/L} = \frac{\lambda}{L}\left[(e^{-ik\eta} - e^{-ik\xi})\frac{v(k)}{k}\right]_{k=2\pi l/L}. \quad (66)$$

Then

$$\sum_{l=1}^{\infty} K_l K_{-l} l = -\frac{\lambda^2}{2\pi}\frac{1}{L}\sum_{k>0}^{\infty} \frac{|v(k)|^2 \, |e^{-i\eta k} - e^{-i\xi k}|^2}{k} = -\frac{\lambda^2}{\pi}\frac{1}{L}\sum_{k>0}^{\infty} |v(k)|^2 \frac{1 - \cos k(\xi - \eta)}{k} \equiv -Q(\xi - \eta). \quad (67)$$

So finally we have

$$G(\xi, \eta) = e^{-Q(\xi-\eta)}. \quad (68)$$

Using the periodicity of Q in ξ and η, we see that (52) may be written

$$F(\kappa) = \frac{1}{2\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} d\xi e^{+L\kappa\xi} e^{-Q(\xi)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi e^{i\kappa\xi} e^{-Q(\xi)}. \quad (69)$$

Finally, replacing the sum by an integral in (67) we obtain

$$Q(\xi) = \frac{\lambda^2}{2\pi^2}\int_0^{\infty} dk \frac{1 - \cos k\xi}{k} |v(k)|^2. \quad (70)$$

[3] See, for example, V. Grenander and G. Szegö, *Toeplitz Forms and their Applications*, (University of California Press, Berkeley and Los Angeles, 1958), especially P. 176 ff. See also M. Kac, *Probability and Related Topics in Physical Sciences*, (Interscience Publishers, London and New York, 1959), p. 60 ff.

[4] The formula given in Grenander and Szegö, (reference 3) contains K_L^* instead of K_{-L} as given in (56). I am indebted to Professor M. Kac for pointing out to me that if $f(\theta)$ is complex, rather than real as Grenander and Szegö assume, this simple change is all that is necessary.

We cannot go further in the evaluation of $Q(\xi)$ without some further information on the potential. However, the nature of the discontinuity at the F.S. can be investigated.

We may write

$$\bar{n}_k = \frac{2\pi}{L}\sum_{k'<k_F} F(k-k') = \int_{k-k_F}^{\infty} d\kappa F(\kappa) \tag{71}$$
$$= \int_0^\infty d\kappa F(\kappa) + \int_{k-k_F}^{0} d\kappa F(\kappa).$$

The first term of (71) is a constant. To study the behavior of $\bar{n}_k$ near the F.S. ($k \cong k_F$) we therefore need $F(\kappa)$ only for very small κ. This in turn, from (69), requires the behavior of $Q(\xi)$ for large ξ. Since $Q(\xi)$ is an even function of ξ, we consider it for large positive ξ. We have

$$\frac{\partial Q(\xi)}{\partial \xi} = \frac{\lambda^2}{2\pi^2}\int_0^\infty dk \sin k\xi\, |v(k)|^2$$
$$= \frac{\lambda^2}{2\pi^2}\left[|v(0)|^2\frac{1}{\xi} + O\left(\frac{1}{\xi^2}\right)\right], \tag{72}$$

by successive integrations by parts. Integrating, we get

$$Q(\xi) = (\lambda^2/2\pi^2)[|v(0)|^2 \log\xi + C + O(1/\xi^2)], \tag{73}$$

where C is a constant which is in principle calculable from the potential. This may be written in the following way:

$$Q(\xi) = \alpha \log(\xi/a)^2 + O(1/\xi^2), \tag{74}$$

where

$$\alpha \equiv (\lambda^2/4\pi^2)\, |v(0)|^2, \tag{75}$$

and a is a constant with the dimensions of a length, which depends only on the shape of the potential (it is a measure of its range). Inserting (74) into (69) we obtain, for $|\kappa a| \ll 1$,

$$F(\kappa) = \frac{1}{\pi}\int_0^\infty d\xi \frac{\cos\kappa\xi}{(\xi/a)^{2\alpha}}$$
$$= \frac{\Gamma(1-2\alpha)\sin\pi\alpha}{\pi}\frac{a^{2\alpha}}{|\kappa|^{1-2\alpha}}. \tag{76}$$

Thus we obtain, for $|k-k_F|\, a \ll 1$,

$$\int_{k-k_F}^{0} d\kappa F(\kappa) = -\frac{\Gamma(1-2\alpha)\sin\pi\alpha}{2\pi\alpha}$$
$$\times\, |(k-k_F)a|^{2\alpha}\, \sigma(k-k_F), \tag{77}$$

where

$$\sigma(x) = 1, \quad x>0$$
$$= -1, \quad x<0.$$

Therefore we see that there is, for $\alpha \neq 0$, no discontinuity at the F.S. (because the factor $|(k-k_F)a|^{2\alpha}$ vanishes there) though the slope is infinite at this point. On the other hand, if $\alpha = 0$, (77) behaves like $-\frac{1}{2}\sigma(k-k_F)$, which just gives the usual discontinuity at the F.S. Thus, in this model, the smallest amount of interaction *always* destroys the discontinuity of $\bar{n}_k$ at the F.S.

The behavior of $\bar{n}_k$ for large k[i.e., $(k-k_F)a \gg 1$] is also not difficult to obtain. From (71) we need $F(\kappa)$ for large κ, which is the same as knowing $Q(\xi)$ for small ξ. From (70) this may be obtained by expanding

$$Q(\xi) = \frac{\lambda^2}{2\pi^2}\frac{1}{2}\int_0^\infty dk k|v(k)|^2\xi^2 + \cdots, \tag{78}$$

as long as the integral converges, which we shall assume. Writing this as

$$Q(\xi) \cong \tfrac{1}{2}\xi^2/b^2,$$
$$b^2 \equiv \frac{\lambda^2}{2\pi^2}\int_0^\infty dk\cdot k\cdot|v(k)|^2, \tag{79}$$

we obtain

$$F(\kappa) = b/(2\pi)^{\frac{1}{2}} e^{-b^2\kappa^2/2}. \tag{80}$$

Therefore, for large k, we have

$$\bar{n}_k \cong \frac{b}{(2\pi)^{\frac{1}{2}}}\int_k^\infty d\kappa e^{-\kappa^2 b^2/2} \cong \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{kb}e^{-k^2b^2/2}. \tag{81}$$

Therefore the momementum distribution decreases exponentially for large k.

For k close to the origin we may write

$$\bar{n}_k = \int_{k-k_F}^{\infty} F(\kappa)\, d\kappa$$
$$= \int_{-\infty}^{\infty} F(\kappa)\, d\kappa + \int_{-(k\ \ -k)}^{-\infty} F(\kappa)\, d\kappa. \tag{82}$$

From (69),

$$\int_{-\infty}^{\infty} d\kappa F(\kappa) = \int_{-\infty}^{\infty} d\xi\, \delta(\xi) e^{-Q(\xi)} = e^{-Q(0)} = 1. \tag{83}$$

Further, $F(\kappa)$ is an even function of κ. Thus

$$\bar{n}_k = 1 - \int_{k_F-k}^{\infty} F(\kappa)\, d\kappa = 1 - \bar{n}_{2k_F-k}, \tag{84}$$

$$\bar{n}_0 = 1 - \bar{n}_{2k_F}. \tag{85}$$

Therefore $\bar{n}_0 < 1$. If the interaction is such that $k = 2k_F$ is already in the asymptotic region for large k, then $\bar{n}_{2k_F}$ is exponentially small, and $\bar{n}_0$ is very close to unity.

Finally, we should like to conclude this section with a remark about the case where $V(x) = \delta(x)$, the Dirac δ function. In this case $v(k)$ is a constant, so that (70) diverges logarithmically.

If one regards the δ function as the limit of a smooth function [a very convenient choice, with which one can calculate explicitly, is $v(k) = e^{-|k|a/2}$, letting a approach zero in the final answer], it is easy to see that the result is simply $\bar{n}_k = \frac{1}{2}$. The anomalous behavior of the δ-function case is not surprising as it looks at first. Since the particle mass is zero and λ (as may easily be verified) is dimensionless, the only length which can come into the problem is the mean distance between particles or, equivalently k_F^{-1}. However, from (54), k_F does not enter into $F(\kappa)$, so that $\bar{n}_k$ is a function of $k - k_F$ alone, which must be dimensionless. One such example is the unperturbed distribution, which depends only on whether $|k| > k_F$ or not. Another is a constant, which is what we actually obtain for the δ-function potential. The physical origin of this distribution which extends to infinite k, is that the high fourier components of the δ function produce infinitely many pairs, so that infinitely many particles are present.

IV. COMPARISON WITH PERTURBATION THEORY

According to the general formulas[2] the momentum distribution in the ground state is given by

$$\bar{n}_k = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} d\zeta \frac{e^{\zeta 0^+}}{\zeta - \epsilon_k - G_k(\zeta)}, \qquad (86)$$

where $G_k(\zeta)$ is the proper self-energy part of the particle propagator. In this formalism one should calculate the correct propagator at finite temperature (including "anomalous" diagrams) and also use the correct chemical potential μ. It was found there that if the F.S. does not distort (spherical case) this is the same as using ordinary Goldstone perturbation theory (no anamolous diagrams) and the the unperturbed chemical potential. We shall assume that this is also the case here, there being nothing comparable to F.S. distortion in one dimension. Then we replace μ by v_0k_F and take for $G_k(\zeta)$ the lowest nonvanishing contribution. This is second order. A straightforward calculation yields, for $k > 0$,

$$G(\zeta) = \frac{}{2}\left(\frac{\lambda}{\pi}\right)^2 v_0\left(\int_{|k'|}^{\infty} + \int_{-\infty}^{-|k'|}\right) d\kappa \times \frac{|v[\frac{1}{2}(\kappa + |k'|)]|^2 \, |\frac{1}{2}(\kappa + |k'|)|}{z + \kappa}, \qquad (87)$$

where

$$k' = k - k_F, \qquad z = (\zeta - k_F v_0) \mid v_0.$$

This function is analytic in the cut z plane, the cuts extending from $-\infty$ to $-|k'|$ and from $|k'|$ to ∞.

If this is inserted in (86) (with μ replaced by v_0k_F), the resulting integral is quite complicated to discuss, even in the neighborhood of $k = k_F$, for an arbitrary potential, and we shall limit ourselves to a special case.

Writing $z = x - i0^+$, we have

$$G_k(\zeta) = v_0[K_{k'}(x) + iJ_{k'}(x)]. \qquad (88)$$

It is easy to see that by suitably deforming the contour in (86) we may write

$$\bar{n}_k = \frac{1}{2\pi i}\int_{-\infty}^{0} dx \times \left[\frac{1}{x - k' - K_{k'}(x) - iJ_{k'}(x)} - \text{c.c.}\right] \quad k' > 0 \qquad (89)$$

$$= 1 - \frac{1}{2\pi i}\int_0^{\infty} dx \times \left\{\frac{1}{x - k' - K_{k'}(x) - iJ_{k'}(x)} - \text{c.c.}\right\} \quad k' < 0. \qquad (90)$$

Now choosing

$$|v(\kappa)|^2 = 1 \quad |\kappa| < \tfrac{1}{2}q$$
$$= 0 \quad |\kappa| > \tfrac{1}{2}q, \qquad (91)$$

one easily sees

$$K_{k'}(x) = \alpha\,|k'|\left[-2 + \left(1 - \frac{x}{|k'|}\right) \times \log\left|\frac{q^2 - (x - |k'|)^2}{x^2 - k'^2}\right|\right],$$
$$J_{k'}(x) = 0 \quad \text{unless} \quad -q + |k'| < x < -|k'|, \quad \text{or} \quad |k'| < x < q + |k'|$$
$$= \pi\alpha\,|x - |k'|| \quad \text{otherwise}. \qquad (92)$$

We want to investigate $\bar{n}_k$ for small k'. It is not difficult, using (92), to show that, for small α and $|k'|$, $\bar{n}_k$ takes the form

$$\bar{n}_k = \tfrac{1}{2}\{1 - \sigma(k')/(1 - 2\alpha \log |k'a|)\}, \qquad (93)$$

where $a = 1/q$.

This expression is, just as the exact expression, continuous at $k = k$, and has infinite slope there. In fact if we write

$$|k'a|^{2\alpha} = (e^{-2\alpha \log |k'a|})^{-1}$$
$$= (1 - 2\alpha \log |k'a| + \cdots)^{-1},$$

forcing an expansion of the exact result (77) for small α, we see that in this sense (93) agrees exactly with the exact answer to the order involved.

Thus, unlike the realistic three-dimensional case, perturbation theory predicts no discontinuity at the F.S. Since the exact answer behaves in the same way, perturbation theory (for the proper self-energy part) in this problem at least is a reliable guide to the behavior of $\bar{n}_k$.

V. RESPONSE TO EXTERNAL FIELDS

If one considers particles to have a charge e, we can induce currents to flow by applying an external field. It follows at once from the commutation relationships that

$$\dot{\rho} + \partial j/\partial x = 0, \tag{94}$$

where

$$\rho(x) = e\psi^+(x)\psi(x) = e[N_1(x) + N_2(x)], \tag{95}$$

$$j(x) = ev_0\psi^+(x)\sigma_3 \quad \psi(x) = ev_0[N_1(x) - N_2(x)], \tag{96}$$

$$\dot{\rho} \equiv i[H, \rho]. \tag{97}$$

This is clearly the equation of continuity of charge, and we can identify ρ and j with the charge and current densities, respectively.[5]

Suppose we couple to our system an external electric field described by a potential $\varphi(x, t)$. The interaction is described by a Hamiltonian H_{ext} given by

$$H_{\text{ext}} = \int_0^L \rho(x)\varphi(x, t)\, dx, \tag{98}$$

$$H_T = H + H_{\text{ext}}. \tag{99}$$

If we again make the canonical transformation (20),

$$\tilde{H}_T = e^{i\lambda S} H_T e^{-i\lambda S}, \tag{100}$$

we find, since S commutes with H_{ext},

$$\tilde{H}_T = H_0 + H_{\text{ext}}. \tag{101}$$

Therefore, for a static field, all the energy levels are identical with the noninteracting case. In particular, this means that the Kohn effect[6] (which predicts a logarithmic singularity in $|q - 2k_F|$ for the change in energy of the system in the presence of an external field of wavenumber q) is completely unaltered by the interaction, this, in spite of the fact that the behavior of $\bar{n}_k$ in the neighborhood of $k = k_F$ is profoundly altered.

If we calculate the linear response, (i.e., the current that flows to terms linear in the external field) by means of (say) the Kubo formula,[7] then one sees immediately that the result is the same as in the unperturbed case. Again this result is due to the fact that both the charge and current densities depend only on N_1 and N_2, which commute with S. This is also true if the external field couples to the current or when there are impurities present which act on the individual particles.

Finally, we may consider "positron annihilation" in this model.[8] Usually this is thought of as an effect which gives a direct experimental measurement of $\bar{n}_k$. In the one-dimensional case one cannot measure an angular correlation between the photons which come out. However, one can ask questions about the probability of one of them having a momentum between q and $q + dq$. We do not want to enter into a long discussion of the various possibilities here. We mention, however, that if one couples massless "photons" described by a scalar field ϕ having velocities $u_0(<v_0)$, via an effective interaction for pair annihilation,

$$H''' = g \int_0^L dx\rho(x)\phi^2(x); \tag{102}$$

then again only the unperturbed momentum distribution plays a role. However, if one takes more complicated couplings (depending for example on other bilinear expressions than ρ or j) one can get a large effect from the interaction.

Thus we see that although the momentum distribution is very much altered by the interaction in this model, it is by no means true that effects due to "particles at the Fermi Surface" are correspondingly altered. In other words, the naive association of the existence of a discontinuity in the momentum distribution, and the quasiparticle-like behavior of a weakly excited system of interacting fermions is shown to be unjustified for this model.

APPENDIX

We want to evaluate expressions of the following type:

$$I = (\Psi, \Lambda\Psi),$$
$$\Lambda \equiv \exp\left[i \int_0^L Q(y)\psi^+(y)\psi(y)\, dy \right], \tag{A1}$$

where $Q(y)$ is an ordinary function, and where Ψ

[5] In reality these definitions should be modified by the subtraction of infinite constants corresponding to the redefinition of the vacuum state as that with no holes and no elections. We imagine this done in what follows.

[6] W. Kohn, Phys. Rev. Letters, 2, 393 (1959).

[7] R. Kubo, Can. J. Phys. 34, 1274 (1956).

[8] See, for example, R. Ferrell, Rev. Mod. Phys. 28, 308 (1956).

represents a wavefunction in which the single-particle states $n = 1, 2, \cdots, M$ are occupied. If we write

$$\psi(y) = \sum_n a_n \varphi_n(y), \tag{A2}$$

where the $\varphi_n(y)$ are a complete orthonormal set of single-particle states, then clearly

$$\Psi = a_1^+ \cdots a_M^+ \Psi_0. \tag{A3}$$

Ψ_0 is the unperturbed vacuum.

We may write (A1) as

$$I = (\Psi_0, a_M \cdots a_1 \Lambda a_1^+ \cdots a_M^+ \Psi_0). \tag{A4}$$

Writing

$$a_1 = \int_0^\infty dz_1 \varphi_1^*(z_1) \psi(z_1), \tag{A5}$$

we get, making use of (42),

$$a_1 \Lambda = \int dz_1 \varphi_1^*(z_1) \psi(z_1) \Lambda$$
$$= \Lambda \int dz_1 \varphi_1^*(z_1) e^{iQ(z_1)} \psi(z_1). \tag{A6}$$

Therefore, (A4) becomes

$$I = \left(\Psi_0, \Lambda \int d^M z \left(\prod_{n=1}^M \varphi_n^*(z_n) e^{iQ(z_n)} \right) \times \psi(z_M) \cdots \psi(z_1) a_1^+ \cdots a_M^+ \mid \Psi_0 \right). \tag{A7}$$

Since Ψ_0 is the unperturbed vacuum,

$$\Lambda \Psi_0 = \Psi_0, \tag{A8}$$

so that (A7) becomes

$$I = \int d^M z \left(\prod_{n=1}^M \varphi_n^*(z_n) e^{iQ(z_n)} \right) \times (\Psi_0, \psi(z_M) \cdots \psi(z_1) a_1^+ \cdots a_M^+ \Psi_0) \tag{A9}$$

$$= \int d^M z \, d^M z' \left(\prod_{n=1}^M \varphi_n^*(z_n) \varphi_n(z_n') e^{iQ(z_n')} \right) \times (\Psi_0 \, | \psi(z_M) \cdots \psi(z_1) \psi^+(z_1') \cdots \psi^+(z_M') | \, \Psi_0). \tag{A10}$$

The expectation value in (A10) is a familiar one in the many-body problem. It can be obtained by taking the sum of the products of the corresponding expectation value for all possible ψ, ψ^+ pairs. The sign of each term is given by a plus if the permutation necessary to bring them to the required position is even, a minus if it is odd. Clearly then

$$I = \int d^M z \, d^M z' \sum_P (-)^P P \left(\prod_1^M \varphi_n^*(z_n) \varphi_n(z_n') e^{iQ(z_n)} \right) \times (\Psi_0 \, | \psi(z_1) \psi^+(z_1') | \, \Psi_0) \cdots (\Psi_0 \, | \psi(z_M) \psi^+(z_M') | \, \Psi_0). \tag{A11}$$

The sum on P is over all possible permutations of the variables. Now

$$(\Psi_0 \, | \psi(z_1) \psi^+(z_1') | \, \Psi_0)$$
$$= (\Psi_0 \, | \psi(z_1) \psi^+(z_1') + \psi^+(z_1') \psi(z_1) | \, \Psi_0)$$
$$= \delta(z_1 - z_1')(\Psi_0, \Psi_0) = \delta(z_1 - z_1'), \tag{A12}$$

$$I = \int d^M z \sum_P (-)^P \varphi_1^*(z_1) e^{iQ(z_1)} \cdots \varphi_M^*(z_M) e^{iQ(z_m)} \times \varphi_{i_1}(z_1) \cdots \varphi_{i}\,(z_M), \tag{A13}$$

where

$$P(1, 2, \cdots, M) = (i_1, i_2, \cdots, i_M),$$

or

$$I = \sum_P (-)^P (\varphi_1, e^{iQ(z)} \varphi_{i_1}) \cdots (\varphi_M, e^{iQ(z)} \varphi_{i}\;). \tag{A14}$$

This, however, is just the definition of the determinant of the matrix g_{nn}, where

$$g_{nn'} = (\varphi_n, e^{iQ(z)} \varphi_{n'}). \tag{A15}$$

Therefore,

$$I = \mathrm{Det}\,(g).$$

If we take for the φ_n plane wave states, we get just the result used in the text.

Incidently, if one does this in configuration space and uses determinental wavefunctions, this becomes a well-known theorem about the integral over products of determinants.

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