Interacting fermions in one dimension

Within the context of many-body physics, a theory is termed **free** if the Hamiltonian is bilinear in creation and annihilation operators, i.e. $\hat{H} \sim \sum_{\mu\nu} a_\mu^\dagger H_{\mu\nu} a_\nu$, where H may be a finite- or infinite-dimensional matrix.[16] Such models are "solvable" in the sense that the solution of the problem simply amounts to a diagonalization of the matrix $H_{\mu\nu}$ (subject to the preservation of the commutation relations of the operators a and $a^\dagger$). However, only a few models of interest belong to this category. In general, interaction contributions typically quartic in the field operators are present and complete analytical solutions are out of reach.

Yet there are a few precious examples of genuinely interacting systems that are amenable to (nearly) exact solution. In this section we will address an important representative of this class, namely the one-dimensional interacting electron gas. Not only is the analysis of this system physically interesting but, in addition, it provides an opportunity to practice working with the second quantized operator formalism on a deeper level.

Qualitative discussion

Consider the nearly free electron Hamiltonian (2.18) and (2.19) reduced to a one-dimensional environment. Absorbing the chemical potential E_F into the definition of the Hamiltonian, and neglecting spin degrees of freedom (e.g. one might consider a fully spin polarized band),

$$\hat{H} = \sum_k a_k^\dagger \left(\frac{k^2}{2m} - E_\mathrm{F} \right) a_k + \frac{1}{2L} \sum_{kk',q\neq 0} V(q) a_{k-q}^\dagger a_{k'+q}^\dagger a_{k'} a_k. \tag{2.33}$$

INFO At first sight, the treatment of a one-dimensional electron system may seem an academic exercise. However, effective one-dimensional interacting fermion systems are realized in a surprisingly rich spectrum of materials. We have already met with **carbon nanotubes** above. A nanotube is surrounded by clouds of mobile electrons (see earlier discussion in section 2.2). With the latter, confinement of the circumferential direction divides the system into a series of one-dimensional bands, each classified by a sub-band index and a wavenumber k. At low temperatures, the Fermi surface typically intersects a single sub-band, allowing attention to be drawn to a strictly one-dimensional system. A similar mechanism renders certain **organic molecules** (such as the Bechgaard salt $(\mathrm{TMTSF})_2\mathrm{PF}_6$, where TMTSF stands for the tetramethyl-tetraselenafulvalene) one-dimensional conductors.

A third, solid state, realization is presented by artificial low-dimensional structures fabricated from semiconducting devices. Redistribution of electron charge at the interface of a GaAs/AlGaAs heterostructure results in the formation of a two-dimensional electron gas. By applying external gates, it is possible to fabricate quasi-one-dimensional **semiconductor quantum wires** in which electron motion in the transverse direction is impeded by a large potential gradient (Fig. 2.8 (a)). At sufficiently low Fermi energies, only the lowest eigenstate of the transverse Schrödinger equation (the lowest "quantum mode") is populated and one is left with a strictly one-dimensional electron system. There are other realizations, such as edge modes in **quantum Hall systems**, "**stripe phases**" in high-tempterature superconductors, or certain **inorganic crystals**, but we shall not discuss these here explicitly.

[16] More generally, a free Hamiltonian may also contain contributions $\sim a_\mu a_\nu$ and $a_\mu^\dagger a_\nu^\dagger$.

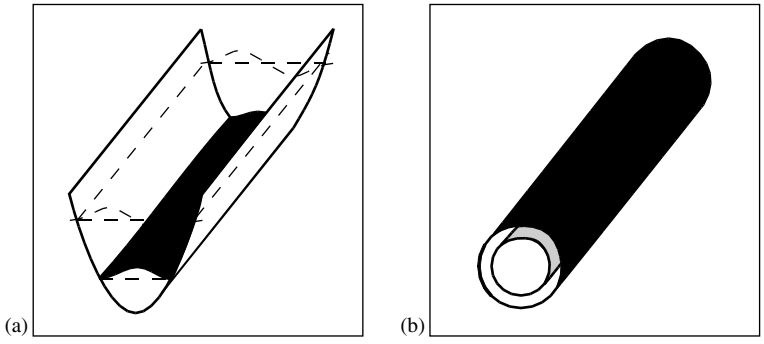


Figure 2.8 Different realizations of one-dimensional electron systems. (a) Steep potential well (realizable in, e.g., gated two-dimensional electron systems). (b) (Approximately) cylindrical quantum system (carbon nanotubes, quasi-one-dimensional molecules, etc.). In both cases, the single-particle spectrum is subject to mechanisms of size quantization. This leads to the formation of "minibands" (indicated by shaded areas in the figure), structureless in the transverse direction and extended in the longitudinal direction.

The one-dimensional fermion system exhibits a number of features not shared by higher-dimensional systems. The origin of these peculiarities can be easily understood from a simple qualitative picture. Consider an array of interacting fermions confined to a line. To optimize their energy the electrons can merely "push" each other around, thereby creating density fluctuations. By contrast, in higher-dimensional systems, electrons are free to avoid contact by moving around each other. A slightly different formulation of the same picture can be given in momentum space. The Fermi "sphere" of the one-dimensional system is defined through the interval $[-k_{\rm F}, k_{\rm F}]$ of filled momentum states. The Fermi "surface" consists of two isolated points, $\{k_{\rm F}, -k_{\rm F}\}$ (see the figure below). By contrast, higher-dimensional systems typically exhibit continuous and simply connected Fermi surfaces. It takes little imagination to anticipate that an extended Fermi sphere provides more phase space to two-particle interaction processes than the two isolated Fermi energy sectors of the one-dimensional system. The one-dimensional electron system represents a rare exception of an interacting system that can be solved under no more than a few, physically weak, simplifying assumptions. This makes it a precious test system on which non-perturbative quantum manifestations of many-body interactions can be explored.

Quantitative analysis

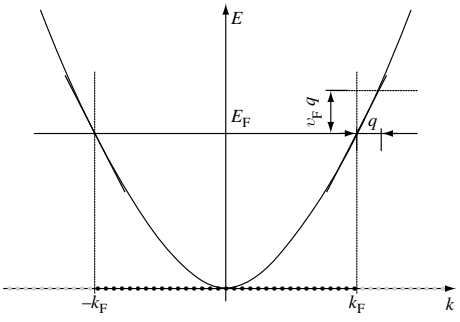
We now proceed to develop a quantitative picture of the charge density excitations of the one-dimensional electron system. Anticipating that, at low temperatures, the relevant dynamics will take place in the vicinity of the two Fermi points $\{k_{\rm F}, -k_{\rm F}\}$, the Hamiltonian (2.33) can be reduced further to an effective model describing the propagation of left and right moving excitations. To this end, we first introduce the notation that the subscripts R/L indicate that an operator $a^\dagger_{(+/-)k_{\rm F}+q}$ creates an electron that moves to the right/left with velocity $\simeq v_{\rm F} \equiv k_{\rm F}/m$.

We next observe (see the figure below) that, in the immediate vicinity of the Fermi points, the dispersion relation is approximately linear, implying that the non-interacting part of

the Hamiltonian can be represented as (exercise)

$$\hat{H}_0 \simeq \sum_{s=\mathrm{R,L}} \sum_q a^\dagger_{sq} \sigma_s v_\mathrm{F} q \, a_{sq}, \tag{2.34}$$

where $\sigma_s = (+/-)$ for $s = \mathrm{R/L}$ and the summation over q is restricted by some momentum cut–off $|q| < \Gamma$ beyond which the linearization of the dispersion is invalid. (Throughout this section, all momentum summations will be subject to this constraint.) Turning to the interacting part of the Hamiltonian, let us first define the operator



$$\hat{\rho}_{sq} = \sum_k a^\dagger_{sk+q} a_{sk}. \tag{2.35}$$

Crucially, the definition of these operators is not just motivated by notational convenience. It is straightforward to verify (exercise) that $\hat{\rho}_s(q)$ is obtained from the Fourier transform of the local density operator $\hat{\rho}(x)$. In other words, $\hat{\rho}_{sq}$ measures density fluctuations of characteristic wavelength q^{-1} supported by electron excitations with characteristic momentum $\pm k_\mathrm{F}$ (see Fig. 2.9 (a)). From our heuristic argument above, suggesting charge density modulations to be the basic excitations of the system, we expect the operators $\hat{\rho}_{sq}$ to represent the central degrees of freedom of the theory.

Represented in terms of the density operators, the interaction contribution to the Hamiltonian may be recast as

$$\hat{V}_\mathrm{ee} = \frac{1}{2L} \sum_{kk'q} V_\mathrm{ee}(q) \, a^\dagger_{k-q} a^\dagger_{k'+q} a_{k'} a_k \equiv \frac{1}{2L} \sum_{qs} \left[g_4 \hat{\rho}_{sq} \hat{\rho}_{s-q} + g_2 \hat{\rho}_{sq} \hat{\rho}_{\bar{s}-q} \right], \tag{2.36}$$

where $\bar{s} = \mathrm{L/R}$ denotes the complement of $s = \mathrm{R/L}$, and the constants g_2 and g_4 measure the strength of the interaction in the vicinity of the Fermi points, i.e. where $q \simeq 0$ and $q \simeq 2k_\mathrm{F}$. (With the notation $g_{2,4}$ we follow a common convention in the literature.)

EXERCISE Explore the relation between the coupling constants g_2, g_4 and the Fourier transform of V_ee. Show that to the summation $\sum_{kk'q}$, not only terms with $(k, k', q) \simeq (\pm k_\mathrm{F}, \pm k_\mathrm{F}, 0)$, but also terms with $(k, k', q) \simeq (\pm k_\mathrm{F}, \mp k_\mathrm{F}, 2k_\mathrm{F})$ contribute. When adequately ordered (do it!), these contributions can be arranged into the form of the right-hand side of Eq. (2.36). (For a detailed discussion see, e.g., T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2004) or G. Mahan, *Many Particle Physics* (Plenum Press, 1981)). At any rate, the only point that matters for our present discussion is that the interaction *can* be represented through density operators with positive constants $g_{2,4}$ determined by the interaction strength.

INFO Working with second quantized theories, one frequently needs to compute commutators of operators $\hat{A}(a, a^\dagger)$ polynomial in the elementary boson/fermion operators of the theory (e.g. $\hat{A} = aa^\dagger$, $\hat{A} = aaa^\dagger a^\dagger$, etc. where we have omitted the quantum number subscripts generally carried by a and $a^\dagger$). Such types of operation are made easier by a number of operations of elementary

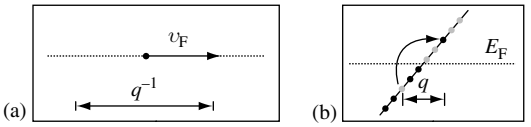


Figure 2.9 Two different interpretations of the excitations created by the density operators $\hat{\rho}_{sq}$. (a) Real space; $\hat{\rho}_{sq}$ creates density modulations of characteristic wavelength q^{-1} and characteristic velocity v_F. (b) Momentum space; application of $\hat{\rho}_{sq}$ to the ground state excites electrons from states k to $k+q$. This creates particle–hole excitations of energy $\epsilon_{k+q} - \epsilon_k = v_F q$ independent of the particle/hole momentum k. Both particles and holes forming the excitation travel with the same velocity v_F, implying that the excitation does not disperse (i.e. decay).

commutator algebra. The most basic identity, from which all sorts of other formulae can be generated recursively, is the following:

$$\boxed{[\hat{A}, \hat{B}\hat{C}]_{\pm} = [\hat{A}, \hat{B}]_{\pm}\hat{C} \mp \hat{B}[\hat{A}, \hat{C}]_{-}.} \tag{2.37}$$

Iteration of this equation for boson operators $a, a^\dagger$ shows that

$$[a^\dagger, a^n] = -n a^{n-1}. \tag{2.38}$$

(Due to the fact that $a^2 = 0$ in the fermionic case, there is no fermion analog of this equation.) Taylor expansion then shows that, for any analytic function $F(a)$, $[a^\dagger, F(a)] = -F'(a)$. Similarly, another useful formula which follows from the above is the relation $a^\dagger F(aa^\dagger) = F(a^\dagger a)a^\dagger$, which is also verified by series expansion.

So far, we have merely rewritten parts of the Hamiltonian in terms of density operators. Ultimately, however, we wish to arrive at a representation whereby these operators, instead of the original electron operators, represent the fundamental degrees of freedom of the theory. Since the definition of the operators ρ involves the squares of two Fermi operators, we expect the density operators to resemble bosonic excitations. Thus, as a first and essential step towards the construction of the new picture, we explore the commutation relations between the operators $\hat{\rho}_{sq}$.

From the definition (2.35) and the auxiliary identity (2.37) it is straightforward to verify the commutation relation $[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] = \delta_{ss'} \sum_k (a^\dagger_{sk+q} a_{sk-q'} - a^\dagger_{sk+q+q'} a_{sk})$. As it stands, this relation is certainly not of much practical use. To make further progress, we must resort to a (not very restrictive) approximation. Ultimately we will want to compute some observables involving quantum averages taken on the ground state of the theory, $\langle\Omega| \ldots |\Omega\rangle$. To simplify the structure of the theory, we may thus replace the right-hand side of the relation by its ground state expectation value:

$$[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] \approx \delta_{ss'} \sum_k \langle\Omega| a^\dagger_{sk+q} a_{sk-q'} - a^\dagger_{sk+q+q'} a_{sk} |\Omega\rangle = \delta_{ss'}\delta_{q,-q'} \sum_k \langle\Omega|(\hat{n}_{sk+q} - \hat{n}_{sk})|\Omega\rangle\,,$$

where, as usual, $\hat{n}_{sk} = a^\dagger_{sk} a_{sk}$, and we have made use of the fact that $\langle\Omega| a^\dagger_{sk} a_{sk'} |\Omega\rangle = \delta_{kk'}$. Although this is an uncontrolled approximation, it is expected to become better the closer we stay to the zero-temperature ground state $|\Omega\rangle$ of the theory (i.e. at low excitation energies).

EXERCISE Try to critically assess the validity of the approximation. (For a comprehensive discussion, see the text by Giamarchi.[17])

At first glance, it would seem that the right-hand side of our simplified commutator relation actually vanishes. A simple shift of the summation index, $\sum_k \langle\Omega|\hat{n}_{sk+q}|\Omega\rangle \stackrel{?}{=} \sum_k \langle\Omega|\hat{n}_{sk}|\Omega\rangle$ indicates that the two terms contributing to the sum cancel. However, this argument is certainly too naive. It ignores the fact that our summation is limited by a cut-off momentum Γ. Since the shift $k \to k - q$ changes the cut-off, the interpretation above is invalid.

To obtain a more accurate result, let us consider the case $s = R$ and $q > 0$. We know that, in the ground state, all states with momentum $k < 0$ are occupied while all states with $k \geq 0$ are empty. This implies that

$$\begin{aligned}\sum_k \langle\Omega|(\hat{n}_{Rk+q} - \hat{n}_{Rk})|\Omega\rangle &= \sum_{-\Gamma<k\ -q} + \sum_{-q<k\ 0} + \sum_{0<k<\Gamma} \langle\Omega|(\hat{n}_{Rk+q} - \hat{n}_{Rk})|\Omega\rangle \\ &= \sum_{-q\ k\ 0} \langle\Omega|(\hat{n}_{Rk+q} - \hat{n}_{Rk})|\Omega\rangle = -\frac{qL}{2\pi}\,,\end{aligned}$$

where, with the last equality, we have used the fact that a momentum interval of size q contains $q/(2\pi/L)$ quantized momentum states. Similar reasoning for $s = L$ shows that the effective form of the commutator relation reads

$$[\hat{\rho}_{sq}, \hat{\rho}_{s'q'}] = -\delta_{ss'}\delta_{q,-q'}\sigma_s \frac{qL}{2\pi}. \tag{2.39}$$

Now, if it were not for the q-dependence of the right-hand side of this relation, we would indeed have found (approximate) bosonic commutation relations. Therefore, to make the connection to bosons explicit, let us define

$$\left.\begin{aligned} b_q &\equiv n_q\,\hat{\rho}_{\mathrm{L}q}, \qquad b_q^\dagger \equiv n_q \hat{\rho}_{\mathrm{L}(-q)}, \\ b_{-q} &\equiv n_q \hat{\rho}_{\mathrm{R}(-q)}, \quad b_{-q}^\dagger \equiv n_q \hat{\rho}_{\mathrm{R}q}, \end{aligned}\right\} \tag{2.40}$$

where $q > 0$ and $n_q \equiv (2\pi/Lq)^{1/2}$. It is easily confirmed that the newly defined operators b_q obey canonical commutation relations (exercise), i.e. we have indeed found that, apart from the scaling factors n_q, the density excitations of the system behave as bosonic "particles."

Expressed in terms of the operators b, the interaction part of the Hamiltonian takes the form (exercise)

$$V_{\mathrm{ee}} = \frac{1}{2\pi}\sum_{q>0} q\,(b_q \quad b_{-q}^\dagger)\begin{pmatrix} g_4 & g_2 \\ g_2 & g_4 \end{pmatrix}\begin{pmatrix} b_q^\dagger \\ b_{-q} \end{pmatrix}.$$

Notice that we have succeeded in representing a genuine two-body interaction, a contribution that usually renders a model unsolvable, in terms of a quadratic representation. However,

[17] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2004).

the free boson representation of the interaction term will be of little use until the **kinetic part of the Hamiltonian** $\hat{H}_0$ is represented in terms of the b operators. There are various ways of achieving this goal. The most straightforward route, a direct construction of a representation of $\hat{H}_0$ in terms of the Bose operators, is cumbersome in practice. However, there exists a more efficient way that is based on indirect reasoning. As follows from the discussion of Section 2.1, the properties of second quantized operators are fixed by their commutation relations.[18] So what we are going to do is search for an operator $\hat{H}_0'(b, b^\dagger)$ that has the same commutation relations with the boson operators $(b, b^\dagger)$ as the original kinetic energy operator $\hat{H}_0(a, a^\dagger)$. Using Eq. (2.34), the definition (2.35), and the auxiliary identity (2.37), it is straightforward to verify that $[\hat{H}_0, \hat{\rho}_{sq}] = q v_{\rm F} \sigma_s \hat{\rho}_{sq}$. On the other hand, using Eq. (2.39) one finds that the same commutation relations hold with the operator

$$\hat{H}_0' = \frac{2\pi v_F}{L} \sum_{qs} \hat{\rho}_{sq} \hat{\rho}_{s-q} \,,$$

i.e. $[\hat{H}_0', \hat{\rho}_{sq}] = q v_{\rm F} \sigma_s \hat{\rho}_{sq}$. Following the logic of our argument we thus identify $\hat{H}_0 = \hat{H}_0'$ (up to inessential constants) and substitute $\hat{H}_0'$ for the non-interacting Hamiltonian.

EXERCISE To gain some confidence in the identification $\hat{H}_0 = \hat{H}_0' + \text{const.}$, and to show that the undetermined constant actually equals zero, compute the energy expectation value of the state $|\ _{sq}\rangle \equiv \hat{\rho}_{sq} |\ \rangle$ both as $\langle\ _{sq} | \hat{H}_0 |\ _{sq}\rangle$ and as $\langle\ _{sq} | \hat{H}_0' |\ _{sq}\rangle$. Confirm that the two expressions coincide.

Finally, using Eq. (2.40) and adding the interaction contribution $V_{\rm ee}$ we arrive at the effective Hamiltonian

$$\hat{H} = \sum_{q>0} q \begin{pmatrix} b_q & b_{-q}^\dagger \end{pmatrix} \begin{pmatrix} v_{\rm F} + \frac{g_4}{2\pi} & \frac{g_2}{2\pi} \\ \frac{g_2}{2\pi} & v_{\rm F} + \frac{g_4}{2\pi} \end{pmatrix} \begin{pmatrix} b_q^\dagger \\ b_{-q} \end{pmatrix} . \tag{2.41}$$

We have thus succeeded in mapping the full interacting problem onto a *free* bosonic theory. The mapping $a \to \hat{\rho} \to b$ is our first example of a technique known as **bosonization**. Such techniques play an important role in 2(= 1 space + 1 time)-dimensional field theory in general. More sophisticated bosonization schemes will be discussed in Sections 4.3 and 9.4.4. Conversely, it is sometimes useful to represent a boson problem in terms of fermions via **fermionization**. One may wonder why it is indeed possible to effortlessly represent the low-lying excitations of a gas of fermions in terms of bosons. **Fermi–Bose transmutability** is indeed a peculiarity of one-dimensional quantum systems. Particles confined to a line cannot pass "around" each other. That means that the whole issue of sign factors arising from the interchange of particle coordinates does not arise, and much of the exclusion-type

[18] This argument can be made quantitative by group theoretical reasoning: Eq. (2.4) and (2.7) define the irreducible representation of an operator algebra – an *algebra* because [,] defines a product in the space of generators $\{a_\lambda, a_\lambda^\dagger\}$, a *representation* because the operators act in a vector space (namely Fock space $\mathcal{F}$), which is *irreducible* because all states $|\lambda_1, \ldots, \lambda_N\rangle \in \mathcal{F}$ can be reached by iterative application of operators onto a unique reference state (e.g. $|\Omega\rangle$). Under these conditions, Schur's lemma – to be discussed in more detail in Chapter 4 – states that two operators $\hat{A}_1$ and $\hat{A}_2$ having identical commutation relations with all $\{a_\lambda, a_\lambda^\dagger\}$ are equal up to a constant.

characteristics of the Fermi system are inactivated. A more systematic formulation of Fermi $\leftrightarrow$ Bose transformations will be discussed in Chapter 4.

Now, there is one last problem that needs to be overcome to actually solve the interacting problem. In Chapter 1, we learned how to interpret Hamiltonians of the structure $\sum_q b_q^\dagger b_q$ as superpositions of harmonic oscillators. However, in our present problem, terms of the type $b_q b_{-q}$ and $b_{-q}^\dagger b_q^\dagger$ appear. To return to familiar terrain, we need to eliminate these terms. However, before doing so, it is instructive to discuss the physical meaning of the problem.

Firstly, let us recall that the total number operator of a theory described by operators $b_\lambda^\dagger, b_\lambda$ is given by $\hat{N} = \sum_\lambda b_\lambda^\dagger b_\lambda$. Now, if the Hamiltonian has the form $\hat{H} = \sum_{\mu\nu} b_\mu^\dagger H_{\mu\nu} b_\nu$, the total number operator commutes with $\hat{H}$, i.e. $[\hat{N}, \hat{H}] = 0$ (exercise). This means that $\hat{H}$ and $\hat{N}$ can be simultaneously diagonalized, or, in more physical terms, that the Hamiltonian enjoys the feature of **particle number conservation**. More generally, any Hamiltonian in which operators appear as polynomials containing equal numbers of creation and annihilation operators (e.g. $b^\dagger b^\dagger bb, b^\dagger b^\dagger b^\dagger bbb$, etc.) has this property. This is because any operator of this structure creates as many particles as it annihilates. In problems where the total number of particles is conserved (e.g. the theory of interacting electrons in an isolated piece of metal), the Hamiltonian is bound to have this structure. Conversely, in situations where the number of excitations is not fixed (e.g. a theory of photons or phonons) particle number violating terms like bb or $b^\dagger b^\dagger$ can appear. Such a situation is realized in our present problem; the number of density excitations in an electron system is certainly not a conserved quantity which explains why contributions like $b_q b_{-q}$ appear in $\hat{H}$.

To eliminate the non-particle-number-conserving contributions we should, somehow, transform the matrix

$$K \equiv \begin{pmatrix} v_\mathrm{F} + \frac{g_4}{2\pi} & \frac{g_2}{2\pi} \\ \frac{g_2}{2\pi} & v_\mathrm{F} + \frac{g_4}{2\pi} \end{pmatrix},$$

to a diagonal structure. Transformations of K can be generated by transforming the operators b_q and $b_q^\dagger$ to a different representation. Specifically, with $\Psi_q \equiv (b_q^\dagger, b_{-q})^T$, we may define $\Psi'_q \equiv T^{-1}\Psi_q$, where T is a 2×2 matrix acting on the two components of Ψ. (Since K does not depend on q, T can be chosen to have the same property.) After the transformation, the Hamiltonian will have the form

$$H = \sum_{q>0} q\, \Psi_q^\dagger K \Psi_q \to \sum_{q>0} q\, \Psi_q'^\dagger \underbrace{T^\dagger K T}_{K'} \Psi'_q\,, \tag{2.42}$$

with a new matrix $K' \equiv T^\dagger K T$. We will seek for a transformation T that makes K' diagonal. However, an important point to be kept in mind is that not all 2×2 matrices T qualify as transformations. We must ensure that the transformed "vector" again has the structure $\Psi'_q \equiv \left(b_q'^\dagger, b'_{-q}\right)^T$, with a boson creation/annihilation operator in the first/second component – i.e. the commutation relations of the operators must be conserved by the transformation. Remembering that the algebraic properties of the operators b are specified through commutation relations, this condition can be cast in mathematical form by requiring that the commutator $\left[\Psi_{qi}, \Psi_{qj}^\dagger\right] = (-\sigma_3)_{ij} \stackrel{!}{=} \left[\Psi'_{qi}, \Psi_{qj}'^\dagger\right]$ be invariant under the transformation. Using the fact that $\Psi' = T^{-1}\Psi$, this condition is seen to be equivalent to the pseudo-unitarity condition, $T^\dagger \sigma_3 T \stackrel{!}{=} \sigma_3$.

With this background, we are now in a position to find a transformation that brings the matrix K' to a 2×2 diagonal form. Multiplication of the definition $K' = T^\dagger K T$ by σ_3 leads to

$$T^\dagger K T = K' \qquad \underbrace{\sigma_3 T^\dagger \sigma_3}_{T^{-1}} \sigma_3 K T = \sigma_3 K'.$$

This means that the matrix $\sigma_3 K'$ is obtained by a *similarity* transformation $T^{-1}(\cdots)T$ from the matrix $\sigma_3 K$, or, in other words, that the matrix $\sigma_3 K'$ contains the eigenvalues $\pm u$ of $\sigma_3 K$ on its diagonal. (That the eigenvalues sum to 0 follows from the fact that the trace vanishes, $\mathrm{tr}(\sigma_3 K) = 0$.) However, the eigenvalues of $\sigma_3 K$ are readily computed as

$$v_\rho = \frac{1}{2\pi}\left[(2\pi v_\mathrm{F} + g_4)^2 - g_2^2\right]^{1/2}. \tag{2.43}$$

Thus, with $\sigma_3 K' = \sigma_3 v_\rho$ we arrive at $K' = v_\rho \cdot \mathrm{id.}$, where "id." stands for the unit matrix.[19] Substitution of this result into Eq. (2.42) finally leads to the diagonal Hamiltonian $\hat{H} = v_\rho \sum_{q>0} q\, \Psi_q'^\dagger \Psi_q'$, or equivalently, making use of the identity $\Psi_q'^\dagger \Psi_q' = b_q^\dagger b_q + b_{-q}^\dagger b_{-q} + 1$,

$$\hat{H} = v_\rho \sum_q |q| b_q^\dagger b_q. \tag{2.44}$$

Here we have ignored an overall constant and omitted the prime on our new Bose operators.

Nicolai Nikolaevich Bogoliubov 1909–92
A theoretical physicist acclaimed for his works in nonlinear mechanics, statistical physics, theory of superfluidity and superconductivity, quantum field theory, renormalization group theory, proof of dispersion relations, and elementary particle theory.

In the literature, the transformation procedure outlined above is known as a **Bogoliubov transformation**. Transformations of this type are frequently applied in quantum magnetism (see below), superconductivity, or, more generally, all problems where the particle number is not conserved. Notice that the possibility to transform to a representation $\sim b^\dagger b$ does not imply that miraculously the theory has become particle number conserving. The new "quasi-particle" operators b are related to the original Bose operators through a transformation that mixes b and $b^\dagger$. While the quasi-particle number is conserved, the number of original density excitations is not.

Equations (2.43) and (2.44) represent our final solution of the problem of spinless interacting fermions in one dimension. We have suceeded in mapping the problem onto a form analogous to our previous results (1.34) and (1.39) for the phonon and the photon system, respectively. Indeed, all that has been said about those Hamiltonians applies equally to Eq. (2.44): the basic elementary excitations of the one-dimensional fermion system are waves, i.e. excitations with linear dispersion $\omega = v_\rho |q|$. In the present context, they are

[19] Explicit knowledge of the transformation matrix T, i.e. knowledge of the relation between the operators b and b', is not needed for our construction. However, for the sake of completeness, we mention that

$$T = \begin{matrix} \cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k \end{matrix}$$

with $\tanh(2\theta) = -g_2/(2\pi v_\mathrm{F} + g_4)$ represents a suitable parameterization.

termed **charge density waves (CDW)**. The Bose creation operators describing these excitations are, up to the Bogoliubov transformation, and a momentum dependent scaling factor $(2\pi/Lq)^{1/2}$, equivalent to the density operators of the electron gas. For a non-interacting system, $g_2 = g_4 = 0$, and the CDW propagates with the velocity of the free Fermi particles, $v_{\rm F}$. A fictitious interaction that does not couple particles of opposite Fermi momentum, $g_2 = 0$, $g_4 \neq 0$, speeds up the CDW. Heuristically, this can be interpreted as an "acceleration process" whereby a CDW pushes its own charge front. By contrast, interactions between left and right movers, $g_2 \neq 0$, diminish the velocity, i.e. due to the Coulomb interaction it is difficult for distortions of opposite velocities to penetrate each other. (Notice that, for a theory with $g_2 = 0$, no Bogoliubov transformation would be needed to diagonalize the Hamiltonian, i.e. in this case, undisturbed left- and right-moving waves would be the basic excitations of the theory.)

Our discussion above neglected the spin carried by the conduction electrons. Had we included the electron spin, the following picture would have emerged (see Problem 2.4): the long-range dynamics of the electron gas is governed by two independently propagating wave modes, the charge density wave discussed above, and a **spin density wave** (SDW).[20] The SDW carries a **spin current**, but is electrically neutral. As with the CDW, its dispersion is linear with an interaction-renormalized velocity, $v_{\rm s}$ (which, however, is generally larger than the velocity v_ρ of the CDW). To understand the consequences of this phenomenon, imagine an electron had been thrown into the system (e.g. by attaching a tunnel contact somewhere along the wire). As discussed above, a single electron does not represent a stable excitation of the one-dimensional electron gas. What will happen is that the spectral weight of the particle[21] disintegrates into a collective charge excitation and a spin excitation. The newly excited waves then propagate into the bulk of the system at different velocities $\pm v_\rho$ and $\pm v_{\rm s}$. In other words, the charge and the spin of the electron effectively "disintegrate" into two separate excitations, a phenomenon known as **spin–charge separation**. Spin–charge separation in one-dimensional metals exemplifies a mechanism frequently observed in condensed matter systems: the set of quantum numbers carried by elementary particles may get effectively absorbed by different excitation channels. One of the more spectacular manifestations of this effect is the appearance of fractionally charged excitations in quantum Hall systems, to be discussed in more detail in Chapter 9.

The theory of spin and charge density waves in one-dimensional conductors has a long history spanning four decades. However, despite the rigor of the theory its experimental verification has proved excruciatingly difficult! While various experiments are consistent with theory (for a review, see Ref.[17]), only recently have signatures of spin and charge density wave excitations been experimentally observed.

[20] One may think of the charge density of the electron gas $\rho = \rho_\uparrow + \rho_\downarrow$ as the sum of the densities of the spin up and spin down populations, respectively. The local spin density is then given by $\rho_{\rm s} \equiv \rho_\uparrow - \rho_\downarrow$. After what has been said above, it is perhaps not too surprising that fluctuations of these two quantities represent the dominant excitations of the electron gas. What *is* surprising, though, is that these two excitations do not interact with each other.

[21] For a precise definition of this term, see Chapter 7.