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Bosonization I: The Fermion–Boson Dictionary

"Bosonization" refers to the possibility of describing a theory of relativistic Dirac fermions obeying standard anticommutation rules by a boson field theory. While this may be possible in all dimensions, it has so far proved most useful in $d = 1$, where the bosonic version of the given fermionic theory is local and simple, and often simpler than the Fermi theory. This chapter should be viewed as a stepping stone toward a more thorough approach, for which references are given at the end.

In this chapter I will set up the bosonization machine, explaining its basic logic and the dictionary for transcribing a fermionic theory to a bosonic theory. The next chapter will be devoted to applications.

To my knowledge, bosonization, as described here, was first carried out by Lieb and Mattis [1] in their exact solution of the Luttinger model [2]. Later, Luther and Peschel [3] showed how to use it to find asymptotic (low momentum and energy) correlation functions for more generic interacting Fermi systems. It was independently discovered in particle physics by Coleman [4], and further developed by Mandelstam [5]. Much of what I know and use is inspired by the work of Luther and Peschel.

17.1 Preamble

Before getting into any details, I would first like to answer two questions. First, if bosonization applies only to relativistic Dirac fermions, why is it of any interest to condensed matter theory where relativity is not essential? Second, what is the magic by which bosonization helps us tame interacting field theories?

As for the first question, there are two ways in which Dirac fermions enter condensed matter physics. The first is in the study of two-dimensional Ising models, where we have already encountered them. Recall that if we use the transfer matrix approach and convert the classical problem on an $N \times N$ lattice to a quantum problem in one dimension we end up with a 2^N-dimensional Hilbert space, with a Pauli matrix at each of N sites. The two dimensions at each site represent the twofold choice of values open to the Ising spins. Consider now a spinless fermion degree of freedom at each site. Here too we have two choices: the fermion state is occupied or empty. There is some need for cleverness in going from the Pauli matrix problem to the fermion problem since Pauli matrices commute at different sites while fermions anticommute; this was provided by Jordan and Wigner. In the critical region the fermion is relativistic since one obtains all the symmetries of the continuum.

The second way in which Dirac fermions arise is familiar from our study of spinless fermions on a linear lattice, described by

$$H = -\sum_{n=-\infty}^{\infty} \psi^{\dagger}(n)\psi(n+1) + \text{h.c.} \tag{17.1}$$

In the above, the spinless fermion field obeys the standard anticommutation rules

$$\{\psi^{\dagger}(n), \psi(m)\} = \delta_{mn}, \tag{17.2}$$

with all other anticommutators vanishing.

Going to momentum states, the Hamiltonian becomes

$$H = -\int_{-\pi}^{\pi} \frac{dk}{2\pi} [\cos k]\, \psi^{\dagger}(k)\psi(k). \tag{17.3}$$

In the ground state we must fill all negative energy modes, that is, states between $\pm K_{\text{F}}$, where $K_{\text{F}} = \pi/2$. To study the low-energy properties of the system, we can focus on the modes near just the Fermi points, as shown in Figure 15.2. We find that they have $E = \pm k$, where k is measured from the respective Fermi points. These are the two components of the massless Dirac field. Any interaction between the primordial fermions can be described in terms of these two components at low energies.

Next, we ask how bosonization can make life easier. Say we have a problem where $H = H_0 + V$, where H_0 is the free Dirac Hamiltonian and V is a perturbation. Assume we can express all quantities of interest in terms of power series in V. In the interaction picture the series will involve the correlation functions of various operators evolving under H_0. Bosonization now tells us that the same series is reproduced by starting with $H = H_0^{\text{B}} + V^{\text{B}}$, where H_0^{B} is a massless free boson Hamiltonian and V^{B} is a bosonic operator that depends on V and is specified by the bosonization dictionary. Consider the special case $V = \rho^2$, where $\rho = \psi^{\dagger}(x)\psi(x)$, the Dirac charge density. This is a quartic interaction in the Fermi language and obviously non-trivial. But according to the dictionary, we must replace ρ by the bosonic operator $\frac{1}{\sqrt{\pi}}\partial_x\phi$, ϕ being the boson field. Thus, V is replaced by the *quadratic* interaction $\frac{1}{\pi}(\partial_x\phi)^2$. The bosonic version is trivial! I must add that this is not always the case; a simple mass term in the Fermi language becomes the formidable interaction $\cos\sqrt{4\pi}\phi$.

Let us now begin. I will first remind you of some basic facts about massless fermions and bosons in one dimension. This will be followed by the bosonization dictionary that relates interacting theories in one language to the other.

17.2 Massless Dirac Fermion

In one dimension, the Dirac equation

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{17.4}$$

will have as the Hamiltonian

$$H = \alpha P + \beta m, \tag{17.5}$$

where P is the momentum operator, and

$$\alpha = \sigma_3 = \gamma_5, \tag{17.6}$$

$$\beta = \sigma_2 = \gamma_0. \tag{17.7}$$

Let us focus on the massless case. There is nothing to diagonalize now: $\psi_\pm$, the upper and lower components of ψ, called *right and left movers*, are decoupled. In terms of the field operators obeying

$$\{\psi_\pm^\dagger(x),\ \psi_\pm(y)\} = \delta(x-y), \tag{17.8}$$

the second quantized Hamiltonian is

$$H = \int \psi^\dagger(x)(\alpha P)\psi(x)dx \tag{17.9}$$

$$= \int \psi_+^\dagger(x)(-i\partial_x)\psi_+(x)dx + \int \psi_-^\dagger(x)(i\partial_x)\psi_-(x)dx. \tag{17.10}$$

In terms of the Fourier transforms

$$\psi_\pm(p) = \int_{-\infty}^{\infty} \psi_\pm(x)e^{ipx}dx \tag{17.11}$$

obeying

$$\{\psi_\pm^\dagger(p),\ \psi_\pm(q)\} = 2\pi\delta(p-q), \tag{17.12}$$

we find that

$$H = \int \psi_+^\dagger(p)\ p\ \psi_+(p)\frac{dp}{2\pi} + \int \psi_-^\dagger(p)\ (-p)\psi_-(p)\frac{dp}{2\pi}. \tag{17.13}$$

From the above, it is clear that the right/left movers have energies $E = \pm p$ respectively. The Dirac sea is thus filled with right movers of negative momentum and left movers with positive momentum, as shown in Figure 17.1.

The inverse of Eq. (17.11) is

$$\psi_\pm(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\psi_\pm(p)e^{ipx}e^{-\frac{1}{2}\alpha|p|}, \tag{17.14}$$

where α is a convergence factor that will be sent to 0 at the end.

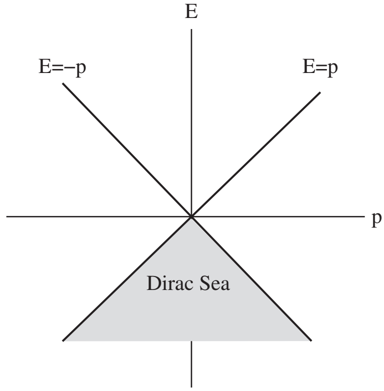

Figure 17.1 Relativistic fermion with right and left movers $E = \pm p$. The Dirac sea is filled with right movers of negative momentum and left movers with positive momentum. The two branches come from the linearized spectrum near the Fermi points $K = \pm K_{\rm F}$ of the non-relativistic fermion. (Only states on the lines $E = \pm p$ are occupied in the Fermi sea.)

Since the fields have trivial time evolution in this free-field theory, we can write down the Heisenberg operators at all times:

$$\psi_\pm(x\ t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \psi_\pm(p) e^{ip(x\mp t)} e^{-\frac{1}{2}\alpha|p|}. \tag{17.15}$$

Notice that $\psi_\pm$ is a function only of $x \mp t$.

Consider now the equal-time correlation function in the ground state:

$$\begin{aligned}\langle \psi_+(x)\psi_+^\dagger(0)\rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{1}{2}\alpha|p|} \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{-\frac{1}{2}\alpha|q|} e^{ipx} \underbrace{\langle \psi_+(p)\ \psi_+^\dagger(q)\rangle}_{2\pi\delta(p-q)\theta(q)} \\ &= \int_0^{\infty} \frac{dp}{2\pi} e^{ipx} e^{-\alpha|p|}. \end{aligned} \tag{17.16}$$

We have used the fact that a right mover can be created only for positive momenta since the Dirac sea is filled with negative momentum particles. So now we have

$$\langle \psi_+(x)\psi_+^\dagger(0)\rangle = \int_0^{\infty} \frac{dp}{2\pi} e^{-\alpha p} e^{ipx} \tag{17.17}$$

$$= \frac{1}{2\pi} \frac{1}{\alpha - ix}. \tag{17.18}$$

If we want the correlation function for unequal times, we just replace x by $x - t$ since we know that the right movers are functions of just this combination.

In the same way, we can show that

$$\langle \psi_\pm(x)\ \psi_\pm^\dagger(0)\rangle = \frac{\pm i/2\pi}{x \pm i\alpha}, \tag{17.19}$$

$$\langle \psi_\pm^\dagger(0)\ \psi_\pm(x)\rangle = \frac{\mp i/2\pi}{x \mp i\alpha}. \tag{17.20}$$

Note that

$$\langle \psi_\pm(x)\ \psi_\pm^\dagger(0) + \psi_\pm^\dagger(0)\ \psi_\pm(x)\rangle = \frac{\alpha/\pi}{x^2+\alpha^2} \tag{17.21}$$

$$\simeq \delta(x), \tag{17.22}$$

where in the last equation we are considering the limit of vanishing α.

Besides the Fermi field, there are bilinears in the field that occur often. Let us look at some key ones. The current density j_μ has components

$$j_0 = \psi^\dagger\psi \tag{17.23}$$

$$= \psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_-(x), \tag{17.24}$$

$$j_1 = \psi^\dagger\alpha\psi \tag{17.25}$$

$$= \psi_+^\dagger(x)\psi_+(x) - \psi_-^\dagger(x)\psi_-(x). \tag{17.26}$$

The axial current is given by $j^5_\mu = \varepsilon_{\mu\nu} j_\nu = (j_1, -j_0)$. The last bilinear is the "mass term"

$$\overline{\psi}\psi = \psi^\dagger(x)\beta\psi(x) \tag{17.27}$$

$$= -i\psi_+^\dagger(x)\psi_-(x) + i\psi_-^\dagger(x)\psi_+(x). \tag{17.28}$$

For later use, let us note that

$$\langle \overline{\psi}\psi(x)\ \ \overline{\psi}\psi(0)\rangle = \frac{1}{2\pi^2}\frac{1}{x^2+\alpha^2}. \tag{17.29}$$

The derivation of this result is left as an exercise. All you need are the anticommutation rules and the correlation functions from Eqs. (17.19) and (17.20).

17.2.1 Majorana Fermions

We close the section by recalling some facts about Majorana fermions. These may be viewed as Hermitian or real fermions. The Dirac field ψ_D can be expressed in terms of two *Hermitian* fields ψ and χ:

$$\psi_D = \frac{\psi + i\chi}{\sqrt{2}}, \tag{17.30}$$

$$\psi_D^\dagger = \frac{\psi - i\chi}{\sqrt{2}}. \tag{17.31}$$

(The components of the spinors ψ_D, ψ, and χ are implicit.) It is readily verified that

$$\{\psi_a(x), \psi_b(y)\} = \delta(x-y)\delta_{ab}, \tag{17.32}$$

where a and b label the two spinor components. There is a similar rule for χ. All other anticommutators vanish.

Exercise 17.2.1 *Show that*

$$\int \psi_D^\dagger(x)\psi_D(x)dx \equiv \int \psi_{Da}^\dagger(x)\psi_{Da}(x)dx = \tag{17.33}$$

$$= \int (i\psi_a\chi_a + \delta(0))dx \equiv \int (i\psi^T\chi + \delta(0))dx. \tag{17.34}$$

By computing the density of Dirac fermions in the vacuum, show that this means

$$\int :\psi_D^\dagger(x)\psi_D(x): dx = \int (i\psi^T\chi)dx. \tag{17.35}$$

If we write the massive Dirac Hamiltonian in terms of the Majorana fields defined above, we will get, with $\alpha = \sigma_3$ and $\beta = \sigma_2$,

$$H_D = \int [\psi_D^\dagger(\alpha P + \beta m)\psi_D]dx \tag{17.36}$$

$$= \frac{1}{2}\int [\psi^T(\alpha P + \beta m)\psi + \chi^T(\alpha P + \beta m)\chi]dx \tag{17.37}$$

$$+ \frac{1}{2}\int \left[i\psi^T(\alpha P + \beta m)\chi - i\chi^T(\alpha P + \beta m)\psi\right]dx.$$

You may check that the cross terms add to zero. (To make contact with the Majorana fermions from Chapter 9, we should change the representation of the α matrix so that it equals Pauli's σ_1. This change of variables with real coefficients is consistent with the Hermitian nature of the Majorana fields.)

So remember: one free Dirac fermion equals two Majorana fermions, just as one charged scalar field equals two real fields (not just in degrees of freedom, but at the level of H).

Exercise 17.2.2 *Using $\alpha = \sigma_3$ and $\beta = \sigma_2$ and the components in explicit form, verify that the non-interacting Hamiltonian for one Dirac fermion is the sum of the Hamiltonians for two Majorana fermions.*

17.3 Free Massless Scalar Field

The Hamiltonian for a massless scalar field is

$$H_B = \frac{1}{2}\int (\Pi^2 + (\partial_x\phi)^2)dx, \tag{17.38}$$

where Π and ϕ obey

$$[\phi(x), \Pi(y)] = i\delta(x-y). \tag{17.39}$$

The Schrödinger operators are expanded as follows:

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\sqrt{2|p|}} \left[\phi(p)e^{ipx} + \phi^\dagger(p)e^{-ipx}\right] e^{-\frac{1}{2}\alpha|p|}, \tag{17.40}$$

$$\Pi(x) = \int_{-\infty}^{\infty} \frac{dp|p|}{2\pi\sqrt{2|p|}} \left[-i\phi(p)e^{ipx} + i\phi^\dagger(p)e^{-ipx}\right] e^{-\frac{1}{2}\alpha|p|}, \tag{17.41}$$

where

$$[\phi(p), \phi^\dagger(p')] = 2\pi\delta(p-p'). \tag{17.42}$$

Due to the convergence factors, ϕ and Π will obey

$$[\phi(x), \Pi(y)] = \frac{i\alpha/\pi}{\alpha^2 + (x-y)^2} \tag{17.43}$$

$$\simeq i\delta(x-y). \tag{17.44}$$

The Hamiltonian now takes the form:

$$H = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \phi^\dagger(p)\phi(p)|p|. \tag{17.45}$$

Exercise 17.3.1 *Verify Eq. (17.45).*

We now introduce right and left movers $\phi_\pm$:

$$\phi_\pm(x) = \frac{1}{2}\left[\phi(x) \mp \int_{-\infty}^{x} \Pi(x')dx'\right] \tag{17.46}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dp}{2\pi\sqrt{2|p|}} e^{-\frac{1}{2}\alpha|p|} \left[\phi(p)(1 \pm |p|/p)e^{ipx} + \text{h.c.}\right]$$

$$= \pm\int_{0}^{\pm\infty} \frac{dp}{2\pi\sqrt{2|p|}} \left[e^{ipx}\phi(p) + \text{h.c.}\right] e^{-\frac{1}{2}\alpha|p|}. \tag{17.47}$$

I leave it to you to verify, using Eq. (17.46), that

$$\left[\phi_\pm(x), \phi_\pm(y)\right] = \pm\frac{i}{4}\varepsilon(x-y) \equiv \pm\frac{i}{4}\text{sgn}(x-y), \tag{17.48}$$

$$\left[\phi_+(x), \phi_-(y)\right] = \frac{i}{4}. \tag{17.49}$$

Exercise 17.3.2 *Verify Eqs. (17.48) and (17.49) starting with Eq. (17.46). If you started with Eq. (17.47), you would find that because of the convergence factors, a rounded-out step function will arise in place of $\varepsilon(x-y)$, and this will become a step function as $\alpha \to 0$.*

If we use the Heisenberg equations of motion for $\phi(p)$ and $\phi^\dagger(p)$, we will find that $\phi_\pm$ are functions only of $x \mp t$.

We must next work out some correlation functions in this theory. It is claimed that

$$G_\pm(x) = \langle \phi_\pm(x)\phi_\pm(0) - \phi_\pm^2(0)\rangle \tag{17.50}$$

$$= \frac{1}{4\pi} \ln \frac{\alpha}{\alpha \mp ix}, \tag{17.51}$$

$$G(x) = \langle \phi(x)\phi(0) - \phi^2(0)\rangle \tag{17.52}$$

$$= \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2 + x^2}. \tag{17.53}$$

I will now establish one of them, leaving the rest as exercises. Consider

$$G_+(x) = \int_0^\infty \frac{dp}{2\pi\sqrt{2|p|}} e^{-\frac{1}{2}\alpha|p|} \int_0^\infty \frac{dq}{2\pi\sqrt{2|q|}} e^{-\frac{1}{2}\alpha|q|} \langle(\phi(p)\phi^\dagger(q)\rangle(e^{ipx} - 1)$$

$$= \int_0^\infty \frac{dp}{4\pi|p|}(e^{ipx} - 1)e^{-\alpha p} \tag{17.54}$$

$$= \frac{1}{4\pi} \ln \frac{\alpha}{\alpha - ix}, \tag{17.55}$$

where the last line comes from looking up a table of integrals. If you cannot find this particular form of the result, I suggest you first differentiate both sides with respect to x, thereby eliminating the $1/|p|$ factor. Now the integral is easily shown to be $i/(4\pi(\alpha - ix))$. Next, integrate this result with respect to x, with the boundary condition $G_+(0) = 0$.

Finally, we consider a class of operators one sees a lot of in two-dimensional (spacetime) theories. These are exponentials of the scalar field. Consider first

$$G_\beta(x) \equiv \langle e^{i\beta\phi(x)}\, e^{-i\beta\phi(0)}\rangle. \tag{17.56}$$

For the correlator to be non-zero, the sum of the factors multiplying ϕ in the exponentials has to vanish. This is because the theory (the Hamiltonian of the massless scalar field) is invariant under a constant shift in ϕ. To evaluate this correlator, we need the following identity:

$$e^A \cdot e^B =: e^{A+B} : e^{\langle AB + \frac{A^2+B^2}{2}\rangle}, \tag{17.57}$$

where the normal-ordered operator $: A :$ has all its destruction operators to the right and creation operators to the left, as well as the fact that the vacuum expectation value of a normal-ordered exponential operator is just 1. All other terms in the series annihilate the vacuum state on the left or right or both. Thus,

$$\langle : e^\Omega : \rangle = 1. \tag{17.58}$$

Exercise 17.3.3 *If you want to amuse yourself by proving Eq. (17.57), here is a possible route. Start with the more familiar identity (which we will not prove):*

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{17.59}$$

$$= e^B e^A e^{\frac{1}{2}[A,B]}, \tag{17.60}$$

provided $[A,B]$ *commutes with A and B. Using this, first write* $e^A = e^{A^+ + A^-}$, *where* $A^\pm$ *are the creation and destruction parts of A, in normal-ordered form. Now turn to* e^{A+B}, *and separate the exponentials using the identity above. Next, normal-order each part using this formula again, and finally normal-order the whole thing. (The last step is needed because :A::B: is not itself normal ordered.) Finally, remember that all commutators are c-numbers and therefore equal to their vacuum expectation values.*

We now use Eqs. (17.57) and (17.58) to evaluate G_β:

$$G_\beta(x) = \langle : e^{i\beta(\phi(x)-\phi(0))} : \rangle e^{\beta^2[\langle \phi(x)\phi(0) - \frac{\phi^2(0)+\phi^2(x)}{2}\rangle]} \tag{17.61}$$

$$= e^{\beta^2 \frac{1}{4\pi} \ln \frac{\alpha^2}{\alpha^2+x^2}} \tag{17.62}$$

$$= \left(\frac{\alpha^2}{\alpha^2 + x^2} \right)^{\beta^2/4\pi}. \tag{17.63}$$

Notice two things. First, by varying β we can get operators with a continuum of power-law decays of correlations. Next, as we send α to 0, the correlator vanishes. To avoid this we must begin with operators suitably boosted or renormalized. The thing to do in the above example is to consider the renormalized operator

$$[e^{i\beta\phi}]_{\mathrm{R}} = (\alpha\mu)^{\frac{-\beta^2}{4\pi}} e^{i\beta\phi}, \tag{17.64}$$

where μ is an arbitrary mass. This operator will have finite correlations in the limit of zero α: if we give it less of a boost, it dies; more, and it blows up.

One can similarly show, using Eqs. (17.50) and (17.53), that

$$\langle e^{i\beta\phi_\pm(x)} e^{-i\beta\phi_\pm(0)} \rangle = \left(\frac{\alpha}{\alpha \mp ix} \right)^{\beta^2/4\pi}. \tag{17.65}$$

17.3.1 The Dual Field θ

So far we have focused on the combination

$$\phi = \phi_+ + \phi_-. \tag{17.66}$$

In some calculations one needs correlations of the *dual field*,

$$\theta = \phi_- - \phi_+. \tag{17.67}$$

From Eq. (17.46),

$$\theta(x) = \int_{-\infty}^{x} \Pi(x')dx', \tag{17.68}$$

$$\Pi(x) = \frac{d\theta}{dx}. \tag{17.69}$$

The correlations of the dual field are just the same as those of ϕ:

$$\langle e^{i\beta\theta(x)} e^{-i\beta\theta(0)} \rangle = \left(\frac{\alpha^2}{\alpha^2 + x^2} \right)^{\beta^2/4\pi}. \tag{17.70}$$

Here is one way to derive Eq. (17.70):

$$\begin{aligned}
\langle e^{i\beta\theta(x)} e^{-i\beta\theta(0)} \rangle &= \langle e^{i\beta(\phi_-(x)-\phi_+(x))} e^{-i\beta(\phi_-(0)-\phi_+(0))} \rangle \\
&= \langle e^{i\beta\phi_-(x)} e^{-i\beta\phi_-(0)} \rangle \langle e^{-i\beta\phi_+(x)} e^{i\beta\phi_+(0)} \rangle \\
&= e^{\beta^2 G_-(x)} e^{\beta^2 G_+(x)} \\
&= \left(\frac{\alpha^2}{\alpha^2 + x^2} \right)^{\beta^2/4\pi},
\end{aligned} \tag{17.71}$$

where I have not shown the (canceling) phase factors coming from separating and recombining exponentials of $\phi_\pm$.

17.4 Bosonization Dictionary

So far we have dealt with massless Fermi and Bose theories and the behavior of various correlation functions in each. Now we are ready to discuss the rules for trading the Fermi theory for the Bose theory. The most important formula is this:

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_\pm(x)}. \tag{17.72}$$

This is not an operator identity: no combination of boson operators can change the fermion number the way ψ can. The equation above really means that any correlation function of the Fermi field, calculated in the Fermi vacuum with the given (α) cut-off, is reproduced by the correlator of the bosonic operator given on the right-hand side, if computed in the bosonic vacuum with the same momentum cut-off. Given this equivalence, we can replace any interaction term made out of the Fermi field by the corresponding bosonic counterpart. Sometimes this will require some care, but this is the general idea.

Substituting Eq. (17.46) in Eq. (17.72), we find

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp\left[\pm i\sqrt{\pi} \left[\phi(x) \mp \int_{-\infty}^{x} \Pi(x')dx' \right] \right]. \tag{17.73}$$

The integral of Π plays the role of the Jordan–Wigner string that ensures the global anticommutation rules of fermions, as first shown by Mandelstam [5].

There are several ways to convince you of the correctness of the master formula Eq. (17.72). First, consider the correlation

$$\langle \psi_+(x)\psi_+^\dagger(0)\rangle = \frac{1}{2\pi}\frac{1}{\alpha - ix}. \tag{17.74}$$

Let us see this reproduced by the bosonic version:

$$\left\langle \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{4\pi}\phi_+(x)} \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{4\pi}\phi_+(0)} \right\rangle \tag{17.75}$$

$$= \frac{1}{2\pi\alpha}\langle : e^{i\sqrt{4\pi}\phi_+(x)} e^{-i\sqrt{4\pi}\phi_+(0)} :\rangle e^{4\pi\langle \phi_+(x)\phi_+(0) - \phi_+^2\rangle} \tag{17.76}$$

$$= \frac{1}{2\pi\alpha} e^{4\pi G_+(x)} \tag{17.77}$$

$$= \frac{1}{2\pi\alpha}\frac{\alpha}{\alpha - ix}. \tag{17.78}$$

In the above we have used Eq. (17.58), the normal-ordering formula Eq. (17.57), the definition of G_+ from Eq. (17.50), and its actual value from Eq. (17.51).

It is possible to verify in the same spirit that the bosonized version of the Fermi field obeys all the anticommutation rules (with delta functions of width α). I leave this to the more adventurous ones among you. Instead, I will now consider some composite operators and show the care needed in dealing with their bosonization. The first of these is

$$\overline{\psi}\psi = -\frac{1}{\pi\alpha}\cos\sqrt{4\pi}\phi. \tag{17.79}$$

The proof involves just the use of Eq. (17.60), and goes as follows:

$$\begin{aligned} \overline{\psi}\psi(x) &= -i\psi_+^\dagger(x)\psi_-(x) + \text{h.c.} \\ &= \frac{1}{2\pi\alpha}\left[e^{-i\sqrt{4\pi}\phi_+(x)} e^{-i\sqrt{4\pi}\phi_-(x)}(-i) + \text{h.c.}\right] \\ &= \frac{1}{2\pi\alpha}\left(e^{-i\sqrt{4\pi}\phi(x)} e^{\frac{1}{2}4\pi(-1)\frac{i}{4}}(-i) + \text{h.c.}\right) \qquad (17.80) \\ &= -\frac{1}{\pi\alpha}\cos\sqrt{4\pi}\phi. \qquad (17.81) \end{aligned}$$

The factor $\frac{i}{4}$ in the exponent arises from the commutator of the right and left movers, Eq. (17.49).

It can similarly be shown that

$$\overline{\psi}i\gamma^5\psi = -\left[\psi_+^\dagger(x)\psi_-(x) + \psi_-^\dagger(x)\psi_+(x)\right] \tag{17.82}$$

$$= \frac{1}{\pi\alpha}\sin\sqrt{4\pi}\phi. \tag{17.83}$$

In the above manipulations we brought together two operators at the same point. Each one has been judiciously scaled to give sensible matrix elements (neither zero nor infinite) acting on the vacuum. There is no guarantee that a product of two such well-behaved operators at the same point is itself well behaved. A simple test is to see if the product has a finite matrix element in the vacuum as the points approach each other. In the example above, this was the case; in fact, the mean value of the composite operator is zero since its factors create and destroy different (right- or left-moving) fermions. This is not the case for the next item: the operator $\psi_+^\dagger(x)\psi_+(x)$, say for $x=0$. We define it by a limiting process called *point splitting* as follows:

$$\begin{aligned}
\psi_+^\dagger(0)\psi_+(0) &= \lim_{x\to 0}\frac{1}{2\pi\alpha}e^{-i\sqrt{4\pi}\phi_+(x)}e^{i\sqrt{4\pi}\phi_+(0)} \\
&= \lim_{x\to 0}\frac{1}{2\pi\alpha} : e^{-i\sqrt{4\pi}\phi_+(x)}e^{i\sqrt{4\pi}\phi_+(0)} : e^{4\pi G_+(x)} \\
&= \lim_{x\to 0}\frac{i}{2\pi(x+i\alpha)} : 1 - i\sqrt{4\pi}\frac{\partial\phi_+}{\partial x}x + \cdots : \qquad (17.84)\\
&= \lim_{x\to 0}\frac{i}{2\pi x} + \frac{1}{\sqrt{\pi}}\frac{\partial\phi_+}{\partial x} + \cdots \qquad (17.85)
\end{aligned}$$

These manipulations need some explanation. We perform a Taylor expansion only within the normal-ordering symbols because only the normal-ordered operators have nice (differentiable) matrix elements. Thus, terms of higher order in x and sitting within the symbol are indeed small and can be dropped as $x\to 0$. Consider next the $x+i\alpha$ in the denominator. Is it permissible to drop the α in comparison to x, even though x itself is being sent to 0? Yes. We must always treat any distance x in the continuum theory as being much larger than α, which is to be sent to 0 whenever possible. Finally, note that the density operator in question has an infinite c-number part which is displayed in front. This reflects the fact that the vacuum density of right movers is infinite due to the Dirac sea. If we define a normal-ordered density, i.e., take away the singular vacuum average from it, we obtain

$$: \psi_+^\dagger(x)\psi_+(x) : = \frac{1}{\sqrt{\pi}}\frac{\partial\phi_+}{\partial x}. \qquad (17.86)$$

A similar result obtains for the left-mover density. Combining the two, we get some very famous formulae in bosonization:

$$\begin{aligned}
j_0 &= \frac{1}{\sqrt{\pi}}\frac{\partial\phi}{\partial x}, \qquad (17.87)\\
j_1 &= \frac{1}{\sqrt{\pi}}\frac{\partial(\phi_+-\phi_-)}{\partial x} \qquad (17.88)\\
&= -\frac{\partial_x\theta}{\sqrt{\pi}} = -\frac{\Pi}{\sqrt{\pi}}. \qquad (17.89)
\end{aligned}$$

For the Lagrangian formalism, we may assemble these into

$$j_\mu = \frac{\varepsilon_{\mu\nu}}{\sqrt{\pi}}\partial_\nu\phi. \qquad (17.90)$$

We close this section with two more results. First, a very useful but odd-looking relation:

$$\left[\frac{-1}{\pi\alpha}\cos\sqrt{4\pi}\phi\right]^2 = -\frac{1}{\pi}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2\pi^2\alpha^2}\cos\sqrt{16\pi}\phi, \tag{17.91}$$

dropping c-numbers.

Here is a sketch of the derivation.

$$\left[\frac{-1}{\pi\alpha}\cos\sqrt{4\pi}\phi(0)\right]^2 = \frac{1}{4\pi^2\alpha^2}\lim_{x\to 0}\left[e^{i\sqrt{4\pi}\phi(x)} + cc\right]\cdot\left[e^{i\sqrt{4\pi}\phi(0)} + cc\right]. \tag{17.92}$$

Now we combine exponentials easily because everything commutes. We find that

$$\left[\frac{-1}{\pi\alpha}\cos\sqrt{4\pi}\phi(0)\right]^2 \tag{17.93}$$

$$= \frac{1}{2\pi^2\alpha^2}\lim_{x\to 0}\left[\cos(\sqrt{4\pi}(\phi(x)+\phi(0))) + \cos(\sqrt{4\pi}(\phi(x)-\phi(0)))\right]. \tag{17.94}$$

In the first cosine we can simply double the angle to $\sqrt{16\pi}\phi(0)$. In the second, we want to do a Taylor expansion, but can only do it within a normal-ordered operator. So we proceed as follows, using Eq. (17.57) along the way:

$$\begin{aligned} &\frac{1}{2\pi^2\alpha^2}\lim_{x\to 0}\cos(\sqrt{4\pi}(\phi(x)-\phi(0)) \\ &= \lim_{x\to 0}\frac{1}{2\pi^2\alpha^2} : \cos(\sqrt{4\pi}(\phi(x)-\phi(0)) : \frac{\alpha^2}{x^2+\alpha^2} \qquad (17.95) \\ &= \lim_{x\to 0}\frac{1}{2\pi^2\alpha^2} : 1 - \frac{x^2}{2}(4\pi)(\partial_x\phi)^2 + \cdots : \frac{\alpha^2}{x^2+\alpha^2} \\ &= -\frac{1}{\pi}\left(\frac{\partial\phi}{\partial x}\right)^2 + c\text{-number}, \qquad (17.96) \end{aligned}$$

where in the last line you must remember that $x \gg \alpha$ even at small x. Substituting this into Eq. (17.94), we arrive at Eq. (17.91).

Similar arguments lead to

$$\left[\frac{1}{\pi\alpha}\sin\sqrt{4\pi}\phi\right]^2 = -\frac{1}{\pi}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2\pi^2\alpha^2}\cos\sqrt{16\pi}\phi. \tag{17.97}$$

In the field theory literature you will not see the second term mentioned. The reason is that at weak coupling this operator is highly irrelevant (or non-renormalizable). The reason for our keeping it is that in the presence of strong interactions it will become relevant.

Finally, having seen the dictionary reproduce various fermionic operators in terms of bosons, we may ask "What about the Hamiltonian?" Indeed, the dictionary may be used to show that

$$H_{\mathrm{F}} = \int \Big(\psi_+^\dagger(x)(-i\partial_x)\psi_+(x) + \psi_-^\dagger(x)(i\partial_x)\psi_-(x)\Big)\,dx$$
$$= \frac{1}{2}\int (\Pi^2 + (\partial_x\phi)^2)dx = H_{\mathrm{B}}. \tag{17.98}$$

Exercise 17.4.1 *Prove Eq. (17.98). I suggest you:*

- *use the symmetric derivatives; for example,*

$$\psi_+^\dagger(x)(\partial_x)\psi_+(x) = \lim_{\varepsilon\to 0}\psi_+^\dagger(x)\left(\frac{\psi_+(x+\varepsilon)-\psi_+(x-\varepsilon)}{2\varepsilon}\right); \tag{17.99}$$

- *expand bosonic exponentials to quadratic order in* ε*;*
- *remember that just as* $x \gg \alpha$*, so is* $\varepsilon \gg \alpha$ *in combinations like* $\varepsilon \pm i\alpha$*; drop total derivatives and c-numbers.*

17.5 Relativistic Bosonization for the Lagrangians

Often one uses bosonization in a relativistic theory. Here is the dictionary in *Euclidean* space with the notation defined for free fields:

$$Z_{\mathrm{F}} = \int [d\bar\psi][d\psi]e^{-S_0(\psi)} = \int [d\bar\psi][d\psi]e^{-\int \bar\psi \partial\!\!\!/ \psi d^2x}, \tag{17.100}$$

$$Z_{\mathrm{B}} = \int [d\phi]e^{-S_0(\phi)} = \int [d\phi]e^{-\int \frac{1}{2}(\nabla\phi)^2 d^2x}, \tag{17.101}$$

$$\bar\psi\partial\!\!\!/\psi \to \frac{1}{2}(\nabla\phi)^2 = \frac{1}{2}\Big[(\partial_\tau\phi)^2 + \partial_x\phi)^2\Big], \tag{17.102}$$

$$\bar\psi\gamma^\mu\psi \to \frac{\varepsilon^{\mu\nu}}{\sqrt{\pi}}\partial_\nu\phi \quad (=j^\mu), \tag{17.103}$$

$$\bar\psi\psi \to -\Lambda\cos\sqrt{4\pi}\phi, \tag{17.104}$$

$$\bar\psi i\gamma^5\psi \to \Lambda\sin\sqrt{4\pi}\phi, \tag{17.105}$$

$$(\bar\psi\psi)^2 = \Big[-\Lambda\cos\sqrt{4\pi}\phi\Big]^2 = -\frac{1}{2\pi}(\nabla\phi)^2. \tag{17.106}$$

Several points are worth noting:

- In the relativistic equations we make the replacement

$$\frac{1}{\pi\alpha} \to \Lambda, \tag{17.107}$$

  where Λ is the cut-off in two-dimensional Euclidean momentum, in contrast to $1/\alpha$, which was a momentum cut-off on spatial momenta.
- In the last equation the highly irrelevant $\cos\sqrt{16\pi}\phi$ has been dropped and we have 2π and not π in the denominator because the point-splitting is done in space and time and there is a compensating sum over *two* squared derivatives.

- In Eq. (17.100), I integrate $e^{-S_0(\psi)}$ and not $e^{+S_0(\psi)}$ as in earlier chapters where I wanted to emphasize that the sign meant nothing for Grassmann actions. Here I use the e^{-S_0} for both to simplify the boson–fermion dictionary.

References and Further Reading

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[3] A. Luther and I. Peschel, Physical Review B, **12**, 3908 (1975).
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18

Bosonization II: Selected Applications

We now pass from this rather sterile business of deriving the bosonization formulas to actually using them. Of the countless applications, I have chosen a few that I am most familiar with. While my treatment of the subject will not be exhaustive, it should prepare you to read more material dealing with the subject.

The applications are to the massless Schwinger and Thirring models, the uniform- and random-bond Ising models, and the Tomonaga–Luttinger and Hubbard models. There is an enormous body of literature devoted to these models. I will simply focus on those aspects that illustrate bosonization in the simplest possible terms.

18.1 Massless Schwinger and Thirring Models

The first two examples are the easiest since the Dirac fermion is present from the outset. In later examples it will arise after some manipulations and approximations.

18.1.1 The Massless Schwinger Model

This model was invented by Schwinger [1] to describe electrodynamics in two dimensions. The Euclidean Lagrangian density is

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi - e_0 j^\mu A_\mu + \frac{1}{2}(\varepsilon^{\mu\nu}\partial_\mu A_\nu)^2. \tag{18.1}$$

Schwinger solved this by functional methods; we can now solve it by bosonization. Writing

$$j^\mu A_\mu = \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\phi A_\mu = -\phi\frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu A_\mu, \tag{18.2}$$

we can complete the square on the A integral and find the bosonic Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{e_0^2}{\pi}\phi^2. \tag{18.3}$$

This means that there is a scalar pole (in the Minkowski space propagator) at

$$m^2 = \frac{2e_0^2}{\pi}. \tag{18.4}$$

Here is what is going on. In one space dimension there is no photon. In the gauge $A_0 = 0$, we just have an instantaneous electrostatic potential A_1 between fermions. This Coulomb interaction yields a linear potential or constant force because the flux cannot spread out in $d = 1$. The density oscillations (sound) which would have been massless are now massive due to the long-range interaction. Schwinger's point was that gauge invariance did not guarantee a massless electromagnetic field.

Note for now that if we add a fermion mass term, the problem cannot be solved exactly because we are adding a $\cos\sqrt{4\pi}\phi$ term.

18.1.2 Massless Thirring Model

The Thirring model [2] describes a current–current interaction:

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi - \frac{g}{2}j^{\mu}j_{\mu}. \tag{18.5}$$

Upon bosonizing, this becomes

$$\mathcal{L} = \frac{1}{2}\left(1+\frac{g}{\pi}\right)(\nabla\phi)^2. \tag{18.6}$$

This model was a milestone because it exhibited correlation functions that decayed with a g-dependent power. Consider, for example, the $\bar{\psi}\psi - \bar{\psi}\psi$ correlation at equal time. In the non-interacting theory it has to fall as

$$\langle\bar{\psi}(r)\psi(r)\bar{\psi}(0)\psi(0)\rangle \simeq \frac{1}{r^2} \tag{18.7}$$

just from dimensional analysis: $[\psi] = \frac{1}{2}$ in momentum units. In the bosonized version this would be reproduced as follows (in the Hamiltonian version):

$$\begin{aligned}\langle\bar{\psi}(r)\psi(r)\bar{\psi}(0)\psi(0)\rangle &= \frac{1}{\pi^2\alpha^2}\langle\cos\sqrt{4\pi}\phi(r)\cos\sqrt{4\pi}\phi(0)\rangle \\ &= \frac{1}{2\pi^2\alpha^2}\left(\frac{\alpha^2}{r^2}\right)^{4\pi/4\pi} \simeq \frac{1}{r^2}.\end{aligned} \tag{18.8}$$

(In the path integral version Λ would replace $1/(\pi\alpha)$.) This formula is valid if the kinetic term has a coefficient of $\frac{1}{2}$, whereas now it is $\frac{1}{2}\left(1+\frac{g}{\pi}\right)$ due to interactions. So we define a new field

$$\phi' = \sqrt{\left(1+\frac{g}{\pi}\right)}\phi, \tag{18.9}$$

in terms of which

$$\mathcal{L} = \frac{1}{2}(\nabla\phi')^2, \tag{18.10}$$

$$\bar{\psi}\psi = -\frac{1}{\pi\alpha}\cos\sqrt{\frac{4\pi}{1+\frac{g}{\pi}}}\phi', \tag{18.11}$$

$$\langle\bar{\psi}(r)\psi(r)\bar{\psi}(0)\psi(0)\rangle = \frac{1}{2\pi^2\alpha^2}\left(\frac{\alpha^2}{r^2}\right)^{4\pi/4\pi(1+\frac{g}{\pi})} \simeq \frac{1}{r^\gamma}, \text{ where}$$
$$\gamma = \frac{2}{(1+\frac{g}{\pi})}. \tag{18.12}$$

The thing to notice is that the anomalous power or dimension of the correlation function varies continuously with the interaction strength. Once again, we see how in a massless theory the correlations can decay with a power not dictated by the engineering dimension of the operator: the cut-off, which has to be introduced to make sense of the theory (now in the guise of α), serves as the additional dimensional parameter.

The *massive Thirring* model is defined by adding $-m\bar{\psi}\psi$, which leads to the following bosonized Euclidean Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\left(1+\frac{g}{\pi}\right)(\nabla\phi)^2 - \frac{m}{\pi\alpha}\cos\sqrt{4\pi}\phi \tag{18.13}$$
$$= \frac{1}{2}\left(1+\frac{g}{\pi}\right)(\nabla\phi)^2 - m\Lambda\cos\sqrt{4\pi}\phi. \tag{18.14}$$

We shall return to this model in the next chapter. We now move on to two applications of bosonization to condensed matter: the correlation functions of the Ising model at criticality and of the random-bond Ising model whose bonds fluctuate from site to site around their critical value. In the latter case we have to find the correlation function averaged over bond realizations. We will see how to do this using what is called the replica trick.

18.2 Ising Correlations at Criticality

Let us recall some key features of the $d = 2$ Ising model. The partition function is

$$Z = \sum_{s=\pm 1} e^{K\sum_{\langle ij\rangle} s_i s_j}, \tag{18.15}$$

where $\langle i,j\rangle$ tells us that the Ising spins $s_i = \pm 1$ and $s_j = \pm 1$ are nearest neighbors on the square lattice. The sum in the exponent is over bonds of the square lattice.

The correlation function

$$G(r) = \langle s_r s_0\rangle, \tag{18.16}$$

where 0 is the origin and r a point a distance r away, is known to fall at the critical point as

$$G(r) \simeq \frac{1}{r^\eta} = \frac{1}{r^{\frac{1}{4}}}. \tag{18.17}$$

This power is universal. This exponent of $\eta = \frac{1}{4}$ is rather difficult to derive and the reason will become clear as we go along. I will now describe a trick due to Itzykson and Zuber [3] that uses bosonization to circumvent this.

Let us recall the extreme anisotropic τ-continuum limit of Fradkin and Susskind [4],

$$K_x = \tau, \tag{18.18}$$

$$K_\tau^* = \lambda\tau \quad (\tau \to 0), \tag{18.19}$$

which leads to the transfer matrix

$$T = e^{-\tau H}, \quad \text{where} \tag{18.20}$$

$$H = -\lambda \sum \sigma_1(m) - \sum \sigma_3(m)\sigma_3(m+1). \tag{18.21}$$

The idea of Fradkin and Susskind is that anisotropy will change the metric but not the exponent for decay or any other universal quantity.

Next, we follow Schultz, Mattis, and Lieb [5] and trade the Pauli matrices for full-fledged Fermi operators defined by

$$\psi_1(n) = \frac{1}{\sqrt{2}} \left(\prod_{-\infty}^{n-1} \sigma_1 \right) \sigma_2(n), \tag{18.22}$$

$$\psi_2(n) = \frac{1}{\sqrt{2}} \left(\prod_{-\infty}^{n-1} \sigma_1 \right) \sigma_3(n). \tag{18.23}$$

We are now considering an infinite spatial lattice, and the "string" of σ_1's comes from the far left to the point $n-1$. The *Majorana* fermions obey

$$\{\psi_i(n), \psi_j(m)\} = \delta_{ij}\delta_{mn}. \tag{18.24}$$

Let us imagine that our lattice has a spacing a. Define continuum operators $\psi_c = \psi/\sqrt{a}$ that obey Dirac δ-function anticommutation rules as $a \to 0$. In terms of these, we get in the continuum limit the following continuum Hamiltonian $H_c = H/a$:

$$H_c = \frac{1}{2}\int \psi^T(\alpha P + \beta m)\psi\, dx, \qquad m = (1-\lambda)/a, \tag{18.25}$$

where α is now σ_1, as mentioned earlier.

We have seen in Chapter 8 how this quadratic Hamiltonian is diagonalized. By filling all the negative energy levels we get the ground-state energy E_0. This energy (per unit spatial volume) is essentially the free energy per site of the square lattice model.

Let us turn instead to the two-point correlation functions. Now, it may seem that in a free-field theory this should be trivial. But it is not, because we want the two-point function of the spins, which are non-local functions of the Fermi field.

Let us find the equal-time correlation of two spins a distance n apart in space. (The power law for decay should be the same in all directions even though length scales are not.) Thus, we need to look at

$$\begin{aligned}\langle 0|\sigma_3(0)\sigma_3(n)|0\rangle &= \langle 0|\sigma_3(0)\sigma_3(1)\sigma_3(1)\cdots\sigma_3(n-1)\sigma_3(n)|0\rangle \\ &= \langle 0|[2i\psi_1(0)\psi_2(1)\cdot 2i\psi_1(1)\psi_2(2)\cdots \\ &\quad \cdots\times 2i\psi_1(n-1)\psi_2(n)]|0\rangle. \end{aligned}\tag{18.26}$$

We find that *the two-point function of spins is a 2n-point function of fermions*. This becomes very hard to evaluate if we want the limit of large n: we must evaluate a Pfaffian of arbitrarily large size. We are, however, presently interested in obtaining just the power law of the asymptotic decay of the spin–spin correlation.

Bosonization cannot be invoked since it applies only to Dirac fermions, so we follow the trick of Itzykson and Zuber [3]. First, note that apart from the end factors, $\psi_1(0)$ at the left and $\psi_2(n)$ at the right, we have the product over sites of

$$2i\psi_2(i)\psi_1(i) = -ie^{\frac{i\pi}{2}[2i\psi_2(i)\psi_1(i)]} \tag{18.27}$$

$$= e^{\frac{i\pi}{2}[2i\psi_2(i)\psi_1(i)-1]}. \tag{18.28}$$

This equation follows from the fact that $2i\psi_2(i)\psi_1(i)$ is just like a Pauli matrix (with square unity) for which

$$\sigma_1 = (-i)e^{i\frac{\pi}{2}\sigma_1}. \tag{18.29}$$

The exponent in Eq. (18.28) is just $\frac{i\pi}{2}(\overline{\psi}\psi - 1)$. I will drop the 1 since it makes no difference to the decay of the correlation function. When we form the product over sites it becomes a sum, and in the continuum limit the integral of $\overline{\psi}\psi$ between 0 and R, where $R = na$ is the distance between the points in laboratory units. There is no simple way to evaluate

$$G(R) \simeq \langle 0|e^{\frac{i\pi}{2}\int_0^R \overline{\psi}(x)\psi(x)dx}|0\rangle. \tag{18.30}$$

Consider now an auxiliary problem, where we have made two non-interacting copies of the Ising system, with spins called s and t, and associated Pauli matrices σ and τ and Majorana fermions ψ and χ. It is clear that

$$\begin{aligned}\langle s_n t_n s_0 t_0\rangle &= \langle s_n s_0\rangle\langle t_n t_0\rangle \\ &= [G(n)]^2, \end{aligned}\tag{18.31}$$

since the thermal averages proceed independently and identically for the two sectors. The trick is to find G^2 and then take the square root. Let us see how this works. First, we will be dealing with products of the following terms:

$$2i\psi_2\psi_1 2i\chi_2\chi_1 = -[2i\chi_1\psi_1]\cdot[2i\chi_2\psi_2] \tag{18.32}$$

$$= e^{\frac{i\pi}{2}[2i\psi_1\chi_1 + 2i\psi_2\chi_2]} \tag{18.33}$$

$$= e^{i\pi :\psi_D^\dagger\psi_D:}. \tag{18.34}$$

The last step needs some explanation. Let us form a Dirac fermion

$$\psi_{\rm D} = \frac{\psi + i\chi}{\sqrt{2}} \tag{18.35}$$

and consider its charge density:

$$\psi_{\rm D}^\dagger \psi_{\rm D} = \frac{1}{2}(\psi_1 - i\chi_1)(\psi_1 + i\chi_1) + (1 \to 2) \tag{18.36}$$

$$= i\psi_1\chi_1 + i\psi_2\chi_2 + 1 \tag{18.37}$$

$$:\psi_{\rm D}^\dagger \psi_{\rm D}: = i\psi_1\chi_1 + i\psi_2\chi_2, \tag{18.38}$$

where I have used the fact that the vacuum density of the Dirac fermions is 1 per site: half for the right movers, half for the left movers. (Recall that in momentum space half the states are filled, which translates into half per site in real space.)

What about the fact that the Dirac fermion that comes out of the Ising model has a first quantized Hamiltonian $H = \alpha P + \beta m$, where $\alpha = \sigma_1$ and $\beta = \sigma_2$, whereas the one used in bosonization has $\alpha = \sigma_3$ and $\beta = \sigma_2$? It does not matter: the two are connected by a unitary transformation (a $\frac{\pi}{2}$ rotation generated by σ_2), and $\psi_{\rm D}^\dagger \psi_{\rm D}$ is invariant under this.

We now reveal our punch line: in view of the above,

$$G^2(R) = \langle 0| e^{i\pi \int_0^R :\psi_{\rm D}^\dagger(x)\psi_{\rm D}(x):dx} |0\rangle \tag{18.39}$$

$$= \langle 0| e^{\int_0^R i\sqrt{\pi}\,\partial_x \phi dx} |0\rangle \tag{18.40}$$

$$= \langle 0| e^{i\sqrt{\pi}\phi(R)} e^{-i\sqrt{\pi}\phi(0)} |0\rangle \tag{18.41}$$

$$\simeq \frac{1}{R^{\frac{1}{2}}}, \tag{18.42}$$

where I have recalled Eq. (17.63). *Thanks to bosonization, a non-local Green's function in the Fermi language has become a local two-point function in the bosonic language.* Several points of explanation are needed. First, we have used Eq. (17.87) in going from the first to the second equation in the above sequence. Next, we have used the fact that at the critical point the Fermi theory has no mass. Thus, the bosonic ground state in which the bosonic correlator is evaluated is the free-field vacuum. Lastly, we have used Eq. (17.63) to evaluate the desired two-point function. (I have ignored α compared to R in the denominator and dropped the power of α in the numerator since I just want the R dependence.) Taking the square root, we find the desired decay law $G(R) \simeq R^{-\frac{1}{4}}$.

I have been careless about the end points, where the product does not follow the pattern. If this is taken into account, one finds that we must use $\sin\sqrt{\pi}\phi$ in place of $e^{i\sqrt{\pi}\phi}$. This does not, however, change the critical exponent. If one tries the Itzykson–Zuber trick away from criticality one finds that one has to find the correlation function of the same operator but in the theory with an interaction $\cos\sqrt{4\pi}\phi$, which is the bosonized version of the harmless-looking mass term in the free Dirac theory.

To conclude, the following were the highlights of our derivation of $G(R)$:

- The critical theory of the Ising model in the extreme anisotropic τ-continuum limit is a massless Majorana theory.
- The two-point function of spins a distance R apart is given by the average of the exponential of the integral of a Majorana fermion bilinear from 0 to R.
- By considering the square of G, we made the integrand referred to above into the normal-ordered Dirac charge density.
- By bosonizing the latter into the derivative of ϕ, we got rid of the integral in the exponent and were left with just a two-point function of $e^{i\sqrt{\pi}\phi}$'s coming from the end points of the integration.
- By evaluating this in the free-field theory we found that G^2 falls off like $R^{-\frac{1}{2}}$. We then took the square root of this answer.

18.3 Random-Bond Ising Model

Consider an Ising model in which the coupling between neighboring spins is not uniformly K, but randomly chosen at each bond from an ensemble. This can happen in real systems due to vacancies, lattice imperfections, and so on. We should therefore imagine that each sample is different and translationally non-invariant. The study of the $d = 2$ Ising model with such a complication was pioneered by Dotsenko and Dotsenko [6] (referred to as DD hereafter) in a very influential paper. I will now describe their work, as well as further contributions by others. You will see bosonization at work once more.

First, let us understand what we want to calculate in a random system. The behavior of an individual system with bonds chosen in a sample-specific way from the ensemble of possibilities is not interesting, unless by luck we are dealing with a property that is sample independent. (The free energy per site in the infinite volume limit is one such object.) In general, what one wants are physical quantities, first calculated sample by sample *and then* averaged over samples. This is called a *quenched average*, and is a lot more difficult problem than the *annealed average* in which one treats the bond strength as another statistical variable in thermal equilibrium, just like the Ising spins themselves. Which one should one use? If the bonds are frozen into some given values over the period of the measurements, we must take them as a fixed external environment and do the quenched average. If they fluctuate ergodically over the period of measurement, we must do the annealed average. The DD problem deals with quenched averages. In this case one must work with the averaged free energy $\overline{f}$ obtained by averaging $\ln Z$ over all samples. The temperature-derivative of $\overline{f}$ gives the average internal energy, and so on. (As mentioned above, it is known that in the infinite-volume limit, each sample will give the same f. This is not true for all quantities.) Similarly, one can take two spins a distance R apart and find the correlator G sample by sample. This will depend on the absolute values of the coordinates, since there is no translational invariance. However, the ensemble average $\overline{G}$ will depend only on R. Besides these mean values, one can calculate the fluctuations around these mean values. Given the distance R and a temperature, there is a unique number $G(R)$ in a pure system describing the correlation. In our case there is probability $P(G(R))$ that $G(R)$ will have this or that value. We will return to this point at the end.

We have seen that the Ising model is described by a non-interacting Majorana field theory. We can take this Hamiltonian and write Z as a Euclidean path integral over Grassmann numbers as follows [Eq. (9.71)]:

$$Z_{\mathrm{M}}(K) = \int [d\psi] \exp\left[- \int \frac{1}{2} \overline{\psi} (\not{\partial} + m) \psi \; d^2x \right]. \tag{18.43}$$

In the above, the mass m is determined by λ or equivalently the temperature. It vanishes at the critical temperature. We are assuming we are close enough to criticality for this continuum theory to be valid. Suppose now that the bonds, instead of being uniform, vary from point to point on the two-dimensional lattice, never straying too far from criticality. This means that $m = m(x)$ varies with the two-dimensional coordinate x, and $Z_{\mathrm{M}} = Z_{\mathrm{M}}(m(x))$ is therefore a functional of $m(x)$. Let us assume that the probability distribution for m is a Gaussian at each site:

$$P(m(x)) = \prod_x e^{-(m(x)-m_0)^2/2g^2}. \tag{18.44}$$

Hereafter we will focus on the case of zero mean: $m_0 = 0$. Thus, each bond fluctuates symmetrically around the critical value. To find $\overline{f}$ we must calculate

$$\overline{f} = \int P(m(x)) \ln Z_{\mathrm{M}}(m(x)) dm(x). \tag{18.45}$$

Since we are averaging $\ln Z$ and not Z, we see that the problem is not as easy as that of adding an extra thermal variable $m(x)$. We circumvent this using what is called the *replica trick*. We use

$$\ln Z = \lim_{n \to 0} \frac{Z^n - 1}{n}. \tag{18.46}$$

In what follows, we will drop the minus one in the numerator since it adds a constant to the answer, and also drop the factor of inverse n since it multiplies the answer by a factor without changing any of the critical properties. In short, in Eq. (18.45) we can replace $\ln Z$ by Z^n (and of course send n to zero at the end). But Z^n is just the partition function of n replicas of the original model. Thus,

$$\begin{aligned} \overline{f} &= \int \left[\prod_1^n d\psi_i \right] \exp\left[- \int \sum_1^n \frac{1}{2} \overline{\psi}_i (\not{\partial} + m(x)) \psi_i d^2x \right] e^{-m^2(x)/2g^2} dm(x) \\ &= \int \left[\prod_1^n d\psi_i \right] \exp\left[\int \left(-\frac{1}{2} \sum_1^n \overline{\psi}_i (\not{\partial}) \psi_i + \frac{g^2}{8} \left(\sum_1^n \overline{\psi}_i \psi_i \right)^2 \right) d^2x \right]. \end{aligned} \tag{18.47}$$

Thus, the random model has been traded for an interacting but translationally invariant theory, called the n-component Gross–Neveu model [7, 8]. The above is a shortened derivation of the DD result. It is understood that all calculations are performed for general

n, and that in any analytic expression where n occurs, the limit $n \to 0$ is taken. The value of the DD work is that it shows in detail that this crazy replica procedure is indeed doing the ensemble average we want to do.

Now DD proceed to deduce two results:

- The specific heat will have a $\ln\ln$ divergence instead of the $\ln$ divergence of the pure system. To derive this, one must also explore the case $m_0 \neq 0$.
- The average two-point function $\overline{G(R)}$ falls essentially like R^0 as compared to the $R^{-1/4}$ in the pure system.

While the first result seemed reasonable, the second did not for the following reason. It is known (and we will see) that when $n = 0$, the Gross–Neveu model is essentially a free-field theory at large distances, the interactions falling logarithmically. It is known in that in such asymptotically free theories correlations are usually that of a free field up to logarithms. Thus, we can accept the change in the specific heat from log to log-log, but not the change of the decay exponent from $\frac{1}{4}$ to 0. It was, however, difficult to see what had gone wrong in the rather formidable calculation of DD, which involved an average like Eq. (18.30), difficult enough in free-field theory, in an interacting theory.

I decided to approach the problem in a different way [9]. Recall how, in the pure case, by considering the square of the correlation, we could convert the problem, via bosonization, to the evaluation of a two-point function. Let us try the same trick here. Consider *any one sample* with some given set of bonds. On it, imagine making two copies of the Ising system. Then, following the reasoning from the last section.

$$G^2(0,R,m(x)) = \frac{\int [d\psi_{\mathrm{D}}][d\overline{\psi}_{\mathrm{D}}] \exp\left[-\int d^2x \overline{\psi_{\mathrm{D}}}(\partial\!\!\!/ + m(x))\psi_{\mathrm{D}}\right] \exp\left[i\pi \int_0^R : \psi_{\mathrm{D}}^\dagger \psi_{\mathrm{D}} : dx\right]}{Z_{\mathrm{D}}(m(x))}. \tag{18.48}$$

In the above, G remembers that one spin was at the origin and the other at R (in both copies). In principle one must move this pair over the lattice maintaining this separation R. However, this is obviated by the subsequent replica averaging which restores translation and rotational invariance.

The good news is that, due to the doubling, we have a Dirac fermion. The bad news is that the normalizing partition function downstairs is itself a functional of $m(x)$, which makes it hard to average G^2. So, we multiply top and bottom by Z_{D}^{n-1} and set $n = 0$. This gets rid of the denominator and adds $n-1$ copies upstairs. We then have

$$G^2(0,R,m(x)) \tag{18.49}$$

$$= \int \left[\prod_1^n d\psi_i d\overline{\psi_i}\right] \exp\left[-\int \sum_1^n \overline{\psi}_i(\partial\!\!\!/ + m(x))\psi_i d^2x\right] e^{i\pi \int_0^R \psi_1^\dagger \psi_1 dx}, \tag{18.50}$$

where the subscript 1 labels the species we started with and all fermions are understood to be Dirac. If we now do the Gaussian average over $m(x)$, we just complete the squares on the mass term and obtain

$$\overline{G^2(R)} \tag{18.51}$$

$$= \int \left[\prod_1^n d\psi_i \overline{\psi_i}\right] \exp\left[\int \left[\sum_1^n -\overline{\psi}_i(\partial\!\!\!/)\psi_i + \frac{g^2}{2}\left(\sum_1^n \overline{\psi}_i\psi_i\right)^2\right] d^2x\right] e^{i\pi\int_0^R \psi_1^\dagger \psi_1 dx}. \tag{18.52}$$

Let us now bosonize this theory using the results from Section 17.5 to obtain:

$$\begin{aligned}\overline{G^2(R)} &= \int \prod_{i=1}^n d\phi_i \exp\left[\int d^2x \sum_1^n -\frac{1}{2}(\nabla\phi_i)^2 + \frac{g^2\Lambda^2}{2}\left[\sum_1^n \cos(\sqrt{4\pi}\phi_i)\right]^2\right] \\ &\times \exp\left[i\sqrt{\pi}(\phi_1(R)-\phi_1(0))\right].\end{aligned} \tag{18.53}$$

Consider the square of the sum over cosines. The diagonal terms can be lumped with the free-field term using Eq. (17.106):

$$\left[\Lambda \cos\sqrt{4\pi}\phi\right]^2 = -\frac{1}{2\pi}(\nabla\phi)^2. \tag{18.54}$$

(The relativistic formula ignores the $\cos\sqrt{16\pi}\phi$ term, which is fortunately highly irrelevant in the present weak coupling analysis.) In terms of the new field

$$\phi' = \left(1+\frac{g^2}{2\pi}\right)^{\frac{1}{2}}\phi, \tag{18.55}$$

once again called ϕ in what follows,

$$\overline{G^2(R)} = \left\langle \exp\left[i\sqrt{\frac{\pi}{1+g^2/2\pi}}\phi_1(R)\right]\exp\left[-i\sqrt{\frac{\pi}{1+g^2/2\pi}}\phi_1(0)\right]\right\rangle_g, \tag{18.56}$$

where the subscript g tells us that the average is taken with respect to the vacuum of an interacting field theory with action

$$S = \int d^2x \left(\sum_1^n -\frac{1}{2}(\nabla\phi_i)^2 + \frac{g^2\Lambda^2}{2}\left[\sum_i\sum_{j\neq i}\cos\left(\sqrt{\frac{4\pi}{1+g^2/2\pi}}\phi_i\right)\cos\left(\sqrt{\frac{4\pi}{1+g^2/2\pi}}\phi_j\right)\right]\right). \tag{18.57}$$

Unlike in the homogeneous Ising model, where we had a two-point function to evaluate in a free-field theory, we have here an interacting theory. Since g^2 measures the width of the bond distribution, perhaps we can work first with small g in a perturbation expansion? For example, if $g^2 = 0.001$ we could read off the answer using perturbation theory. Unfortunately this is not possible. The problem is that the coupling in this theory cannot be a constant, it has to be function $g(\Lambda)$ because there are ultraviolet divergences. These divergences give Λ-dependent answers for quantities of interest, and to neutralize this unwanted dependence we must choose g as a function of Λ, i.e., we must renormalize.

The first step is to compute the β-function:

$$\beta(g) = \frac{dg}{d\ln\Lambda}. \tag{18.58}$$

This computation, best done in the fermionic version, involves finding, to any given order, the contributions the eliminated modes make to the interaction between the surviving modes. To second order in g^2 one draws the three possible one-loop graphs and integrates the loop momenta from the old Λ to the new. For the n-component Gross–Neveu model [7, 8], one knows that

$$\beta(g) = (1-n)\frac{g^3}{2\pi} + \text{ higher order.} \tag{18.59}$$

Typically $n \geq 2$, and this leads to a theory where the coupling grows in the infrared, but here, with $n = 0$, it is the opposite. If the initial bare coupling is $g(a)$, where $a = 1/\Lambda$ is the lattice size, the coupling at scale R is obtained by integrating

$$\frac{dg}{d\ln\Lambda} = \frac{g^3}{2\pi} \tag{18.60}$$

from $\Lambda = 1/a$ to $\Lambda = 1/R$ to obtain

$$g^2(R) = \frac{g^2(a)}{1 + \frac{g^2(a)}{\pi}\ln(R/a)} \tag{18.61}$$

$$\simeq \frac{\pi}{\ln R/a} \quad \text{for } R/a \to \infty. \tag{18.62}$$

This means the following. If we naively perturb the theory defined at scale a to find a quantity like the correlation function at scale R, we will find that the effective parameter is not $g(a)$ but $g^2(a)\ln R/a$. This is because the result Eq. (18.61) will appear as a badly behaved power series,

$$g^2(R) = g^2(a)\left(1 - \frac{g^2(a)}{\pi}\ln(R/a) + \cdots\right). \tag{18.63}$$

What the RG does for us is to sum the series and allow us to use a coupling at scale R that is actually very small for large R/a. Not only will the effective coupling be small, there will be no large logs when we describe physics at scale R.

To exploit this, we have to follow the familiar route of integrating the Callan–Symanzik equations to relate $\overline{G^2(R, g(\Lambda), \Lambda = 1/a)}$ computed with the initial coupling and cut-off to $\overline{G^2(R, g(1/R), \Lambda = 1/R)}$.

So let us recall the solution given in Eqs. (14.108)–(14.113):

$$\overline{G^2(R, g(\Lambda), \Lambda)} = \exp\left[\int_{g(\Lambda)}^{g(1/R)} \frac{\gamma(g)}{\beta(g)} dg\right] \overline{G^2(R, g(1/R), 1/R)}. \tag{18.64}$$

Now, the dimensionless function $\overline{G^2(R,g(1/R),1/R)} = \overline{\langle s_R s_0\rangle^2}$ is a function only of $R \cdot \Lambda(R) = R \cdot R^{-1} = 1$ and $g(R) \simeq 1/\ln R$, which vanishes as $R \to \infty$. So the leading R dependence is in the exponential integral.

We already have $\beta(g)$, and we just need $\gamma(g)$ defined as

$$\gamma(g) = \frac{d \ln Z(g(\Lambda),\Lambda)}{d \ln \Lambda}, \tag{18.65}$$

where Z is the factor that multiplies the given correlation function and makes it independent of Λ.

The correlation function of interest is

$$\overline{G^2(R)} = \left\langle \exp\left[i\sqrt{\frac{\pi}{1+g^2/2\pi}}\phi_1(R)\right] \exp\left[-i\sqrt{\frac{\pi}{1+g^2/2\pi}}\phi_1(0)\right]\right\rangle_g, \tag{18.66}$$

where the subscript g means it is evaluated with coupling g. In particular, let $g(\Lambda)$ be the coupling in the lattice of size $a = 1/\Lambda$. Now,

$$\langle e^{i\beta(\phi(x)-\phi(0))}\rangle = \left(\frac{\alpha^2}{\alpha^2+x^2}\right)^{\beta^2/4\pi} = \left(\frac{1}{\pi^2 x^2 \Lambda^2}\right)^{\beta^2/4\pi}, \tag{18.67}$$

which follows from Eq. (17.63) and $\alpha = 1/(\Lambda\pi)$.

If we ignore the interaction term [the double sum over cosines in Eq. (18.57)], we find that

$$\overline{G^2(R)} = \left[\frac{1}{\pi R \Lambda}\right]^{\frac{1}{2(1+g^2/2\pi)}} \cdot (1+O(g^4)). \tag{18.68}$$

The term in square brackets comes from using Eq. (18.67), valid for the free-field theory, and its g dependence comes from explicit factors of g in the definition of the operators. The corrections due to the interactions begin at order g^4 because the diagonal terms in the double sum have been pulled out and the off-diagonal terms do not contribute to correlation in question due to the constraint that the sum of all the exponents must add up to zero for each boson.

We see that $\overline{G^2(R)}$ can be made independent of the cut-off if we pick some arbitrary mass μ and multiply it by

$$\left[\frac{\Lambda}{\mu}\right]^{\frac{1}{2(1+g^2/2\pi)}} \simeq \left[\frac{\Lambda}{\mu}\right]^{\frac{1}{2}-\frac{g^2}{4\pi}} \quad \text{to order } g^2 \tag{18.69}$$

$$\equiv \left[\frac{\Lambda}{\mu}\right]^{\gamma}, \quad \text{which means} \tag{18.70}$$

$$\gamma = \frac{1}{2} - \frac{g^2}{4\pi}. \tag{18.71}$$

Doing the integral in Eq. (18.64), it is easy to obtain (dropping corrections that fall as inverse powers of $\ln R$)

$$\overline{G^2(R)} = \overline{\langle s_R s_0 \rangle^2} \sim \frac{(\ln R)^{1/4}}{R^{1/2}}, \tag{18.72}$$

where the $\frac{1}{2}$ and the $-\frac{g^2}{4\pi}$ in γ [Eq. (18.71)] contribute to $R^{-\frac{1}{2}}$ and $(\ln R)^{1/4}$ respectively.

Exercise 18.3.1 *Do the g integral in Eq. (18.64) using the known expressions or $\beta(g)$ and $\gamma(g)$, and derive Eq. (18.72).*

We now use the fact that the mean of the square is an upper bound on the square of the mean to obtain

$$\overline{\langle s_R s_0 \rangle} \leq \frac{(\ln R)^{1/8}}{R^{1/4}}. \tag{18.73}$$

Thus, we find that the DD formula $G(R) \simeq R^0$ cannot be right since it violates this bound. It is also nice to see the kind of logs you expect in an asymptotically free theory.

Several developments have taken place since this work was done. First, I learned that Shalayev [12] had independently done this, without using bosonization. Next, in my paper I had claimed that if my arguments were repeated for higher moments one would find that the average of the $2n$th power of G would be the nth power of $\bar{G^2}$. A. W. W. Ludwig pointed out [10, 11] that this was wrong: the error came from using the $e^{i\sqrt{\pi}\phi}$ in place of $\sin\sqrt{\pi}\phi$. Although this made no difference to the preceding derivation of $\bar{G^2}$, it does affect the higher moments. Ludwig in fact carried out the very impressive task of obtaining the full probability distribution $P(G(R))$.

Andreichenko and collaborators did a numerical study [13] to confirm the correctness of my bound and some additional predictions made by Shalayev. For more technical details of my derivation given above, see the excellent book by Itzykson and Drouffe [14].

18.4 Non-Relativistic Lattice Fermions in $d = 1$

We now turn to a family of problems where the fermion is present from the beginning instead of arising from a treatment of Ising spins. However, the fermion is non-relativistic to begin with and the Dirac fermion arises in the low-energy approximation. Some excellent sources are Emery [15], Sachdev [16], and Giamarchi [17].

Here is a road map for what follows so you don't fail to see the forest because of the trees.

We will explore many aspects of the following model of non-relativistic fermions hopping on a lattice in $d = 1$:

$$H = H_0 + H_{\mathrm{I}}$$

$$= -\frac{1}{2}\sum_j \psi^\dagger(j+1)\psi(j) + \text{h.c.}$$
$$+\Delta \sum_j \left(\psi^\dagger(j)\psi(j) - \frac{1}{2}\right)\left(\psi^\dagger(j+1)\psi(j+1) - \frac{1}{2}\right). \tag{18.74}$$

I will refer to this as the Tomonaga–Luttinger (TL) model, although these authors [21, 22] only considered the low-energy continuum version of it. It is also related by the Jordan–Wigner transformation to what is called the XXZ spin chain.

Let me remind you of what we know from our previous encounter with this model in Section 15.3.

In real space it was clear that as $\Delta \to \infty$, the particles would occupy one or the other sublattice to avoid having nearest neighbors. Any movement of charge would produce nearest neighbors and cost an energy of order Δ. This is the gapped CDW state.

For weak coupling, we went to momentum space using

$$\psi(j) = \int_{-\pi}^{\pi} \psi(K) e^{iKj} \frac{dK}{2\pi}, \tag{18.75}$$

and found the kinetic energy

$$H_0 = \int_{-\pi}^{\pi} \psi^\dagger(K)\psi(K)(-\cos K)\frac{dK}{2\pi}. \tag{18.76}$$

The Fermi sea was made of filled negative energy states with $-\frac{\pi}{2} < K < \frac{\pi}{2}$. The Fermi "surface" was made of two points $R = K_F = \frac{\pi}{2}$ and $L = -K_F = -\frac{\pi}{2}$. I will limit myself to half-filling, where $K_F = \frac{\pi}{2}$, until I turn to the Hubbard model.

Keeping only modes within $\pm\Lambda$ of the Fermi points $\pm K_F$ (see Figure 15.2), we found that

$$H_0 = \sum_{i=L,R} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \psi_i^\dagger(k)\psi_i(k)k, \tag{18.77}$$

where

$$k = |K| - K_F, \tag{18.78}$$
$$i = L, R \quad \text{(left or right).} \tag{18.79}$$

(Notice that the k above is the magnitude $|K|$ minus K_F. This definition is most suited for going to higher dimensions, where the energy grows with the radial momentum. Soon we will trade this for a k measured from the nearest Fermi point.)

We then found an RG transformation that left the corresponding action S_0 invariant. We considered the most general interaction and found that there remained just one marginal interaction at tree level, namely u, which scattered particles from opposite Fermi points. (In our specific model, $u \propto \Delta$.) A one-loop calculation showed $\beta(u) = 0$ due to the cancellation between the ZS$'$ and BCS diagrams describing CDW and superconducting instabilities. It was then stated that $\beta(u)$ vanished to all orders, implying a line of fixed points. It was not clear from that analysis how the system would ever escape the fixed line and reach the gapped CDW state, as it had to at strong coupling.

We now resume that tale. We rederive some of these old results of the fermionic RG *using bosonization*, and then go beyond. In particular, we

- compute the varying exponents in several correlation functions along the fixed line;
- explain how we escape the fixed line as some operators that were irrelevant in the fermionic weak coupling RG become relevant;
- explain the nature of the gapped states to which these relevant perturbations take us; and
- map the model to that of a spin chain using the Jordan–Wigner transformation and interpret these results in spin language.

Since this chapter is long, I will also furnish a synopsis of the details to follow so that as you go through the material, you know where we are in our odyssey.

The first step to bosonization is to unearth a Dirac fermion. We will do this by focusing on the states near the Fermi surface, but this time in real space, by truncating the expansion Eq. (18.75) to states within $\pm\Lambda$ of $K = \pm K_{\mathrm{F}} = \pm\frac{\pi}{2}$:

$$\psi(j) = \int_{-\pi}^{\pi} \psi(K) e^{iKj} \frac{dK}{2\pi} \tag{18.80}$$

$$\simeq \int_{-\Lambda}^{\Lambda} \psi(K_{\mathrm{F}}+k) e^{iK_{\mathrm{F}}j} e^{ikj} \frac{dk}{2\pi} + \int_{-\Lambda}^{\Lambda} \psi(-K_{\mathrm{F}}+k) e^{-iK_{\mathrm{F}}j} e^{ikj} \frac{dk}{2\pi} \tag{18.81}$$

$$\equiv a^{\frac{1}{2}} \left[e^{iK_{\mathrm{F}}j} \psi_+(x=aj) + e^{-iK_{\mathrm{F}}j} \psi_-(x=aj) \right] \tag{18.82}$$

$$= a^{\frac{1}{2}} \left[e^{i\frac{\pi}{2}j} \psi_+(x) + e^{-i\frac{\pi}{2}j} \psi_-(x) \right] \quad \text{since } K_{\mathrm{F}} = \tfrac{\pi}{2} \text{ here.} \tag{18.83}$$

Observe that the k above is measured from the Fermi points $\pm K_{\mathrm{F}}$. The subscript $\pm$ labels the Fermi point (R or L) on which the low-energy field is centered. The lattice spacing a converts position j on the lattice to position $x = ja$ in the continuum, and the factor $a^{\frac{1}{2}}$ relates continuum Fermi fields $\psi_\pm$ with Dirac-δ anticommutators to lattice fermions with Kronecker-δ anticommutators. The fields $\psi_\pm(x)$ have only small momenta ($|k| < \Lambda$) in their mode expansion. For the field $\psi_\pm$, the energy goes up (down) with k.

Substituting in Eq. (18.74), we will find that

$$H_{\mathrm{c}} = \frac{H_0}{a} = \int dx \left[\psi_+^\dagger(x)(-i\partial_x)\psi_+(x) + \psi_-^\dagger(x)(i\partial_x)\psi_-(x) \right]$$

$$= \int \psi^\dagger(x) \alpha P \psi(x) dx, \tag{18.84}$$

where H_{c} is the continuum version of H. This paves the way for bosonization.

We will then express the interaction in terms of $\psi_{\pm}$ and bosonize it to get the *sine-Gordon model*:

$$H_c K = \int dx \left(\frac{1}{2} \left[K\Pi^2 + \frac{1}{K}(\partial_x \phi)^2 \right] + \frac{y}{2\pi^2\alpha^2} \cos\sqrt{16\pi}\phi \right), \tag{18.85}$$

$$K = \frac{1}{\sqrt{1+\frac{4\Delta}{\pi}}}, \tag{18.86}$$

$$y = K \cdot \Delta = \frac{\Delta}{\sqrt{1+\frac{4\Delta}{\pi}}}. \tag{18.87}$$

The interpolating steps will soon be provided in pitiless detail.

We will then analyze this bosonized Hamiltonian. *Although at this stage its two parameters y and K are functions of* Δ*, we will consider a two-parameter family of models in which y and*

$$x = 2 - 4K \tag{18.88}$$

are independent. The RG flows in the (x, y) plane will describe the fate of each starting point. The original model will be described by a one-parameter curve $(x(\Delta), y(\Delta))$ of starting points. The curve is reliably known only for small Δ. The nature of various fixed points, the fixed line, and phases will be examined.

We will then interpret the same flows and fixed points in terms of the spin-$\frac{1}{2}$ *Heisenberg chain*,

$$H = \sum_j S_x(j+1)S_x(j) + S_y(j+1)S_y(j) + \Delta \cdot S_z(j+1)S_z(j), \tag{18.89}$$

related to our spinless fermion Hamiltonian of Eq. (18.74) by a Jordan–Wigner transformation.

Finally, we will consider the *Hubbard model* with on-site interaction of spin-up and spin-down fermions. We will find that the inclusion of spin is far from being a cosmetic change. It will dramatize the gruesome fate of the fermion, which gets torn limb from limb when interactions are turned on.

18.4.1 Deriving the Sine-Gordon Hamiltonian

The first essential ingredient in bosonization is the massless Dirac fermion, which is lurking within our non-relativistic fermion. To extract it, we first write the non-interacting Hamiltonian in terms of the low-energy Dirac fields $\psi_{\pm}$ using Eq. (18.83):

$$H_0 = -\frac{1}{2}\sum_j \psi^\dagger(j+1)\psi(j) + \text{h.c.} \tag{18.90}$$

$$= -\frac{1}{2}a\sum_j\left[-ie^{-i\frac{\pi}{2}j}\psi_+^\dagger(x=ja+a)+ie^{i\frac{\pi}{2}j}\psi_-^\dagger(x=ja+a)\right]$$
$$\times\left[e^{i\frac{\pi}{2}j}\psi_+(x=ja)+e^{-i\frac{\pi}{2}j}\psi_-(x=ja)\right]+\text{h.c.} \tag{18.91}$$
$$= \frac{a}{2}\sum_j\left[i\psi_+^\dagger(x)\psi_+(x)-i\psi_-^\dagger(x)\psi_-(x)+ia\frac{\partial\psi_+^\dagger(x)}{\partial x}\psi_+(x)-ia\frac{\partial\psi_-^\dagger(x)}{\partial x}\psi_-(x)\right]$$
$$+\text{h.c.}+\text{ignorable terms and terms oscillating at }\pm 2K_F, \tag{18.92}$$
$$H_{0c} = \frac{H_0}{a} = \int dx\left[\psi_+^\dagger(x)(-i\partial_x)\psi_+(x)+\psi_-^\dagger(x)(i\partial_x)\psi_-(x)\right], \tag{18.93}$$

where H_{0c} is the non-interacting continuum Hamiltonian and I have integrated by parts and used $a\sum_j \to \int dx$.

Now look at the interaction

$$H_I = \Delta\sum_j\left(\psi^\dagger(j)\psi(j)-\frac{1}{2}\right)\left(\psi^\dagger(j+1)\psi(j+1)-\frac{1}{2}\right) \tag{18.94}$$
$$\equiv \Delta\sum_j :\psi^\dagger(j)\psi(j)::\psi^\dagger(j+1)\psi(j+1):. \tag{18.95}$$

We may set

$$\psi^\dagger(j)\psi(j)-\frac{1}{2} = :\psi^\dagger(j)\psi(j):, \tag{18.96}$$

because we have half a fermion per site in the vacuum. Let us combine all this with the expansion of the lattice fields in terms of the smooth continuum fields for $K_F = \frac{\pi}{2}$ [Eq. (18.83)] to obtain

$$H_{Ic} = \frac{H_I}{a}$$
$$= a\Delta\sum_j\left[:\psi_+^\dagger(x)\psi_+(x)+\psi_-^\dagger(x)\psi_-:+(-1)^j(\psi_+^\dagger(x)\psi_-(x)+\psi_-^\dagger(x)\psi_+(x))\right]$$
$$\times\left[:\psi_+^\dagger(x)\psi_+(x)+\psi_-^\dagger(x)\psi_-(x):-(-1)^j(\psi_+^\dagger(x)\psi_-(x)+\psi_-^\dagger(x)\psi_+(x))\right] \tag{18.97}$$
$$= a\Delta\sum_j\left[\frac{1}{\sqrt{\pi}}\partial_x\phi\right]^2-\left[\psi_+^\dagger(x)\psi_-(x)+\psi_-^\dagger(x)\psi_+(x)\right]^2+\ (-1)^j\text{ oscillations}$$
$$= \Delta\int dx\left[\frac{(\partial_x\phi)^2}{\pi}-\left[\frac{1}{\pi\alpha}\sin\sqrt{4\pi}\phi)\right]^2\right] \tag{18.98}$$
$$= \Delta\int dx\left[\frac{2(\partial_x\phi)^2}{\pi}+\frac{1}{2\pi^2\alpha^2}\cos\sqrt{16\pi}\phi)\right]\quad\text{using Eq. (17.97).} \tag{18.99}$$

Notice that we ignore the change in $\psi(x)$ from site j to $j+1$ (down by a power of a), but not that of the factor $(-1)^j$, which oscillates on the lattice scale. We are also using the fact that at half-filling, the potentially oscillatory factor $e^{4K_F j}$, which comes from the product of the second terms in each of the brackets in Eq. (18.97), becomes $(-1)^{2j} = 1$. This is the umklapp term which describes the process $RR \leftrightarrow LL$ with momentum change equal to a reciprocal lattice vector.

This brings us to the continuum Hamiltonian in bosonized form,

$$H_c = \int dx \left(\frac{1}{2} \left[\Pi^2 + \left(1 + \frac{4\Delta}{\pi}\right) (\partial_x \phi)^2 \right] + \frac{\Delta}{2\pi^2 \alpha^2} \cos\sqrt{16\pi}\phi \right). \tag{18.100}$$

At this stage we introduce the *Luttinger parameter*

$$K = \left[1 + \frac{4\Delta}{\pi} \right]^{-\frac{1}{2}}, \tag{18.101}$$

in terms of which

$$H_c K = \int dx \left(\frac{1}{2} \left[K\Pi^2 + \frac{1}{K} (\partial_x \phi)^2 \right] + \frac{y}{2\pi^2 \alpha^2} \cos\sqrt{16\pi}\phi \right), \tag{18.102}$$

$$y = K \cdot \Delta = \frac{\Delta}{\sqrt{1 + \frac{4\Delta}{\pi}}}. \tag{18.103}$$

The rescaling of H_c by K, which we ignore, can be easily incorporated as another parameter, a velocity.

We will take the view that y and K are two free parameters, rather than functions of a single underlying Δ. The TL model will be a one-parameter curve in this two-dimensional plane.

Let us now define a new field and momentum:

$$\phi' = \frac{1}{\sqrt{K}} \phi, \tag{18.104}$$

$$\Pi' = \sqrt{K} \Pi, \tag{18.105}$$

which still obey canonical commutation rules because they were scaled oppositely. By contrast, the ϕ is a c-number in the path integral and can be rescaled as Eq. (18.55). The Hamiltonian now becomes (upon dropping the primes)

$$H_c = \int dx \left[\frac{1}{2} \left[\Pi^2 + (\partial_x \phi)^2 \right] + \frac{y}{2\pi^2 \alpha^2} \cos\sqrt{16\pi K}\phi \right], \tag{18.106}$$

which is a special case of the *sine-Gordon model* whose canonical form is

$$H_{SG} = \int dx \left[\frac{1}{2} \left[\Pi^2 + (\partial_x \phi)^2 \right] + \frac{y}{2\pi^2 \alpha^2} \cos\beta\phi \right]. \tag{18.107}$$

In the Luttinger model analysis,

$$\beta^2 = 16\pi K. \tag{18.108}$$

We will also use a related parameter (unfortunately also called x),

$$x = 2 - 4K = 2\left(1 - \frac{\beta^2}{8\pi}\right), \tag{18.109}$$

because the physics changes dramatically with the sign of x. It is most natural to envisage the physics in the (x, y) plane.

18.4.2 Renormalization Group Analysis of the Sine-Gordon Model

We see that the model describes a massless scalar field plus the cosine interaction due to the umklapp process ($RR \leftrightarrow LL$). It is parametrized by K and y. We need to know what the umklapp term does to the massless boson.

The answer depends on K, which determines whether or not the umklapp term is relevant. For the RG analysis it is convenient to go from the Hamiltonian in Eq. (18.106) to the Euclidean action

$$S = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{y\Lambda^2}{2}\cos\beta\phi\right) d^2x \tag{18.110}$$

and the path integral over $e^{-S(\phi)}$. Notice that we use the Lorentz-invariant bosonization formulas of Section 17.5. The replacement

$$\frac{1}{\pi\alpha} = \Lambda \tag{18.111}$$

trades the spatial momentum cut-off $1/\alpha$ for Λ, the cut-off on k, the magnitude of the two-dimensional Euclidean momentum $\boldsymbol{k}$. The evolution of y will be found by integrating out a thin shell of momenta near the cut-off $k = \Lambda$.

Let us write ϕ as a sum of slow and fast modes,

$$\phi = \phi_s + \phi_f \equiv \phi(0 \le k \le \Lambda(1-dt)) + \phi(\Lambda(1-dt) < k \le \Lambda). \tag{18.112}$$

The free-field action separates as well:

$$S_0 = \int \left[\frac{1}{2}(\nabla\phi_s)^2 + \frac{1}{2}(\nabla\phi_f)^2\right] d^2x. \tag{18.113}$$

The RG that leaves S_0 invariant involves integrating out ϕ_f, followed by the rescaling of spacetime coordinates:

$$d^2x = s^2 d^2x', \tag{18.114}$$

$$\frac{d}{dx} = \frac{1}{s}\frac{d}{dx'}, \tag{18.115}$$

$$\phi(x) = \phi'(x'). \tag{18.116}$$

Now we introduce the interaction, integrate out $\phi_{\rm f}$ as usual, and see happens to the coupling y of the slow modes that remain. Here is the abridged analysis:

$$Z = \int d\phi_{\rm s} \int d\phi_{\rm f} \exp\left[-\int\left[\frac{1}{2}(\nabla\phi_{\rm s})^2 + \frac{1}{2}(\nabla\phi_{\rm f})^2\right]d^2x - \frac{y\Lambda^2}{2}\int d^2x \cos\beta(\phi_{\rm s}+\phi_{\rm f})\right]$$

$$= \int d\phi_{\rm s} \exp\left[-\int \frac{1}{2}(\nabla\phi_{\rm s})^2 d^2x\right]\left\langle \exp\left[-\frac{y\Lambda^2}{2}\int d^2x \cos\beta(\phi_{\rm s}+\phi_{\rm f})\right]\right\rangle_{\rm f} \tag{18.117}$$

$$\simeq \int d\phi_{\rm s} \exp\left[-\int\left(\frac{1}{2}(\nabla\phi_{\rm s})^2 + \frac{y\Lambda^2}{2}\cos\beta\phi_{\rm s}\langle\cos\beta\phi_{\rm f}\rangle_{\rm f}\right)d^2x\right], \tag{18.118}$$

where $\langle\cdots\rangle_{\rm f}$ is the average over fast modes and we are using the leading term in the cumulant expansion ($\langle e^A\rangle \simeq e^{\langle A\rangle}$); the $\sin\beta\phi_{\rm s}\sin\beta\phi_{\rm f}$ term is ignored because it has zero average over fast modes. The average $\langle\cdots\rangle_{\rm f}$ above is *only over the sliver of width* Λdt.

To perform the average we first set $A = i\beta\phi,\ B = 0$ in Eq. (17.57) to deduce that

$$\langle e^{i\beta\phi}\rangle = e^{-\frac{1}{2}\beta^2\langle\phi^2\rangle}. \tag{18.119}$$

Using this result, we find that

$$\langle\cos(\beta\phi_{\rm f})\rangle = e^{-\frac{1}{2}\beta^2\langle\phi_{\rm f}^2\rangle} \tag{18.120}$$

$$= \exp\left[-\frac{\beta^2}{2}\int_{\Lambda(1-dt)}^{\Lambda}\frac{kdkd\theta}{4\pi^2}\frac{1}{k^2}\right] \tag{18.121}$$

$$= 1 - \frac{\beta^2}{4\pi}dt. \tag{18.122}$$

Now we rescale the coordinates as per Eq. (18.114),

$$d^2x = s^2 d^2x' = (1+2dt)d^2x', \tag{18.123}$$

to obtain (on dropping primes)

$$\frac{y\Lambda^2}{2}\int d^2x \cos\beta\phi \to \frac{y\Lambda^2}{2}\left(1+\left(2-\frac{\beta^2}{4\pi}\right)dt\right)\int d^2x\cos\beta\phi,$$

$$\frac{dy}{dt} = \left[2-\frac{\beta^2}{4\pi}\right]y \tag{18.124}$$

$$= (2-4K)y \ \text{ because} \tag{18.125}$$

$$\beta^2 = 16\pi K \quad \text{in the Luttinger model.} \tag{18.126}$$

Thus, we find that the umklapp term is

$$\text{irrelevant for } K > \tfrac{1}{2} \text{ or } \beta^2 > 8\pi, \tag{18.127}$$

$$\text{relevant for } K < \tfrac{1}{2} \text{ or } \beta^2 < 8\pi. \tag{18.128}$$

We rescaled x but not Λ, which just stood there. Are we not supposed to rescale all dimensionful quantities when we change units? The short answer is that in the Wilson approach the cut-off remains fixed because we use the cut-off as the unit of measurement. We could call it Λ or we could call it 1. If we begin with the ball of radius 10^{10} GeV and keep integrating away, in *laboratory units* then of course Λ_{lab} is being steadily reduced, but in rescaled units it will be fixed. It is this fixed value we are denoting by Λ above.

As a check, consider a Gaussian theory with action

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_\Lambda)^2 + \frac{1}{2}m^2\phi_\Lambda^2 \right], \tag{18.129}$$

where m is the mass in lab units and Λ is the cut-off on the momentum content of ϕ_Λ. Suppose we integrate out modes between Λ/s and Λ. We are left with

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_{\Lambda/s})^2 + \frac{1}{2}m^2\phi_{\Lambda/s}^2 \right], \tag{18.130}$$

which tells us that in lab units the theory with the reduced cut-off Λ/s continues to describe a particle of the same mass m, and asymptotic correlations will fall as e^{-mx}. There has been no change of units.

Let us now repeat this, but starting with the mass term expressed in terms of some initial cut-off Λ and a dimensionless parameter r_0:

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_\Lambda)^2 + \frac{1}{2}r_0\Lambda^2\phi_\Lambda^2 \right]. \tag{18.131}$$

Upon mode elimination this becomes

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_{\Lambda/s})^2 + \frac{1}{2}r_0\Lambda^2\phi_{\Lambda/s}^2 \right]. \tag{18.132}$$

We now change units:

$$k = \frac{k'}{s}, \tag{18.133}$$

$$x = sx', \tag{18.134}$$

$$\frac{d}{dx} = \frac{1}{s}\frac{d}{dx'}. \tag{18.135}$$

In these new units the momentum now goes all the way to Λ and we end up with

$$S = \int d^2x' \left[\frac{1}{2}(\nabla'\phi_\Lambda)^2 + \frac{1}{2}r_0 s^2\Lambda^2\phi_\Lambda^2 \right] \tag{18.136}$$

$$\stackrel{\text{def}}{=} \int d^2x' \left[\frac{1}{2}(\nabla'\phi_\Lambda)^2 + \frac{1}{2}r_{0s}\Lambda^2\phi_\Lambda^2 \right]. \tag{18.137}$$

We see that, under the RG,

$$r_0 \to r_{0s} = r_0 s^2. \tag{18.138}$$

(We could also lump the s^2 with Λ^2 in Eq. (18.136) and identify s^2 times Λ^2 in the new units with the Λ^2_{lab} original laboratory units, thereby showing that the m^2 in laboratory units is fixed at $r_0\Lambda^2_{\text{lab}}$.)

18.4.3 Tomonaga–Luttinger Liquid: ($K > \frac{1}{2}, y = 0$)

We consider the line of fixed points $y = 0$ and focus on the sector $K > \frac{1}{2}$ where the perturbation $y\cos\sqrt{16\pi K}\phi$ is irrelevant. In terms of a variable

$$x = 2 - 4K, \tag{18.139}$$

the region where the cosine is irrelevant is

$$x = 2 - 4K < 0. \tag{18.140}$$

Not only does this line $y = 0$ for $x < 0$ describe the models with $y = 0$, it also describes models which flow to $y = 0$ under the RG. Later we will see what range of y will flow into this line under RG. In studying this line we are studying all systems in the basin of attraction of this line. Remember, however, that if you begin at some (K, y) in this basin, you will end up at $(K^*, 0)$, where $K^* \neq K$ in general. (Equivalently, $(x, y) \to (x^*, 0)$ after the RG.) So the K in what follows is in general the final K^* of a system that started away from the fixed line and got sucked into it.

For $x > 0$, the line is unstable to perturbations and the system must be tuned to stay on it. Also bear in mind that we have assumed exactly half-filling; otherwise, the umklapp term is not allowed: $e^{4iK_{\text{F}}n}$ oscillates and averages to zero unless $K_{\text{F}} = \frac{\pi}{2}$. What if we are just a little off $K_{\text{F}} = \frac{\pi}{2}$? Then the oscillations will be very slow in space to begin with, but after a lot of RG iterations, the oscillations will become rapid in the new lattice units and the seemingly relevant growth will fizzle away.

The line of fixed points $(K > \frac{1}{2}, y = 0) \equiv (x < 0, y = 0)$ is ubiquitous and appears in many guises and with different interpretations. Here it describes a fermionic liquid state called the Tomonaga–Luttinger (TL) liquid. The name was coined by Haldane [19, 20], who explored its properties and exposed the generality of the notion. It is the $d = 1$ version of Landau theory. Recall that Landau's Fermi liquid is parametrized by the F function, or its harmonics $u_m \equiv F_m$. Even if we cannot calculate the u_m from some underlying theory, we can measure them in some experiments and use them to describe others in terms of these measured values. The main point is that many low-energy quantities can be described by a few Landau parameters. Likewise, K and a velocity parameter, which I have suppressed, fully define all aspects of the fermionic system – response functions, thermodynamics, correlation functions – in the infrared.

The line of fixed points has one striking property: exponents that vary continuously with K. (This is not so for the Landau Fermi liquid, which has canonical power laws as F varies.) I will show this now, and as a by-product, establish the claim made earlier that the fermion pole at $\omega = k$ (in Minkowski space) is immediately destroyed by the smallest interaction, i.e., the smallest departure from $K = 1$.

Consider $\langle \psi^\dagger(x)\psi(0)\rangle$. Without interactions, we had

$$H = \int dx \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x \phi)^2\right] dx, \tag{18.141}$$

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i\sqrt{4\pi}\phi_\pm(x)}, \quad \text{where} \tag{18.142}$$

$$\phi_\pm(x) = \frac{1}{2}\left[\phi(x) \mp \int_{-\infty}^{x} \Pi(x')dx'\right] \equiv \frac{1}{2}(\phi \mp \theta), \tag{18.143}$$

and where the *dual field*

$$\theta(x) = \int_{-\infty}^{x} \Pi(x')dx'. \tag{18.144}$$

With interactions, we had

$$H = \int dx \left[\frac{K}{2}\Pi^2 + \frac{1}{2K}(\partial_x \phi)^2\right] dx. \tag{18.145}$$

Introducing the rescaled variables of the interacting theory,

$$\phi = K^{\frac{1}{2}}\phi', \quad \Pi = K^{-\frac{1}{2}}\Pi', \quad \theta = K^{-\frac{1}{2}}\theta', \tag{18.146}$$

in terms of which the kinetic energy has the standard coefficient of $\frac{1}{2}$, and recalling that

$$\phi = \phi_+ + \phi_-, \tag{18.147}$$

$$\theta = \phi_- - \phi_+, \tag{18.148}$$

one finds that

$$\psi_\pm(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp \pm i\sqrt{\pi}\left[(K^{\frac{1}{2}} \pm K^{-\frac{1}{2}})\phi'_+ + (K^{\frac{1}{2}} \mp K^{-\frac{1}{2}})\phi'_-\right]. \tag{18.149}$$

Exercise 18.4.1 *Derive Eq. (18.149).*

It is now a routine exercise to show that

$$\langle \psi^\dagger{}_\pm(x)\psi_\pm(0)\rangle \simeq \left[\frac{1}{\alpha \mp ix}\right]^{\frac{(K\pm 1)^2}{4K}} \cdot \left[\frac{1}{\alpha \pm ix}\right]^{\frac{(K\mp 1)^2}{4K}} \tag{18.150}$$

$$= \frac{1}{\alpha \mp ix} \cdot \left[\frac{1}{\alpha^2 + x^2}\right]^{\gamma}, \tag{18.151}$$

$$\gamma = \frac{(K-1)^2}{4K}. \tag{18.152}$$

Exercise 18.4.2 *Derive Eq. (18.150).*

For unequal-time correlations, we just need to remember that $\psi_\pm$ are functions of $x \mp t$ to obtain

$$\langle \psi^\dagger{}_\pm(x,t)\psi_\pm(0)\rangle \simeq \frac{1}{\alpha \mp i(x \mp t)} \cdot \left[\frac{1}{\alpha^2 + x^2 - t^2}\right]^\gamma . \tag{18.153}$$

We see that the decay power varies with K. Upon Fourier transforming to (ω, k), we see that as soon as $K \neq 1$, the pole (in Minkowski space)

$$G(\omega,k) \simeq \frac{1}{\omega - k} \tag{18.154}$$

morphs into a cut using just dimensional analysis: $G(\omega,k)$ has fractional dimension in ω or k:

$$G \simeq (\omega,k)^{\frac{K^2-4K+1}{2K}}. \tag{18.155}$$

There is a huge body of literature on the response functions at non-zero T, ω, and q that you are now ready to explore. For example, one can show that in the TL liquid the occupation number $n(k)$ has not a jump at k_F, but a kink:

$$n(k) = n(k_\mathrm{F}) + c\ \mathrm{sgn}(k - k_\mathrm{F})|k - k_\mathrm{F}|^\delta, \tag{18.156}$$

$$\delta = \frac{K + K^{-1} - 2}{4}. \tag{18.157}$$

18.5 Kosterlitz–Thouless Flow

Let us now find the basin of attraction of the fixed TL line in the (x,y) plane and the manner in which a transition to a gapped phase occurs when we cross the boundary of this basin. We have seen from

$$\frac{dy}{dt} = (2 - 4K)y \tag{18.158}$$

that on the axis labeled by

$$x = (2 - 4K) \tag{18.159}$$

y is relevant or irrelevant for $x > 0$ or $x < 0$ respectively. So in the (x,y) plane we expect flow lines to terminate on or leave the x-axis in the y-direction as x goes from being negative to positive. The flow slows down as we approach $K = \frac{1}{2}$ $(x = 0)$ and then reverses sign, as depicted in Figure 18.1. How do these lines change direction as we cross this point? What is the full story in the (x,y) plane?

For this we turn to the celebrated RG flow devised by Kosterlitz and Thouless [26] in their analysis of the phase transition in the XY model of planar spins. Recall from Chapter 10 that there too we have a line of fixed points with a T-dependent exponent. As $T \to \infty$ the decay had be exponential based on the high-T series. This decay cannot be brought about by spin waves, the small fluctuations about the constant field described by a

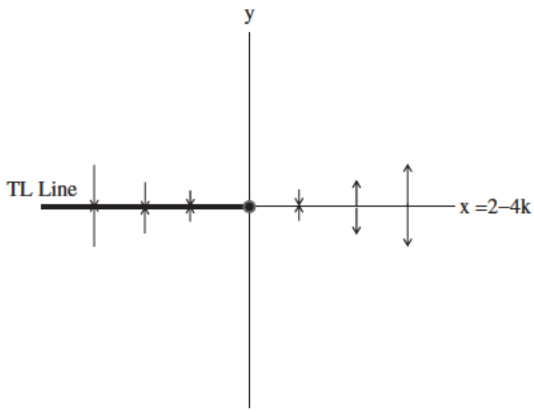

Figure 18.1 The line of fixed points of the sine-Gordon model as a function of β in $\cos\beta\phi$, or the Luttinger parameter K defined by $\beta^2 = 16\pi K$, or the parameter $x = 2-4K = 2\left(1-\frac{\beta^2}{8\pi}\right)$. The cos is increasingly irrelevant for $x < 0$ and increasingly relevant for $x > 0$. The size of the arrows indicates the rate of flow into or away from the fixed line. The dark line describes the Luttinger liquid.

Gaussian action that ignores the periodic nature of the angle θ. The transition to the phase with exponential decay is driven by vortices and antivortices, which are configurations in which the angle θ changes by $\pm 2\pi$ as we go around their cores. At low T these are tightly bound into vortex–antivortex pairs. The fugacity (likelihood of appearing in the sum over configurations) for free vortices and antivortices is described by the cosine interaction with coupling y, which changes (with T) from irrelevance to relevance and vortices and antivortices go from being bound to being free. The same sine-Gordon model describes this transition.

To zero in on the point $(x = 0, y = 0)$, let us rewrite Eq. (18.158) as

$$\frac{dy}{dt} = xy \tag{18.160}$$

and observe that the flow is quadratic in small quantities, so we need the flow of x, or essentially K, to the same order. The only way to renormalize K is by field renormalization, which begins at second order in y. Dropping all constants, we begin with the pair

$$\frac{dy}{dt} = xy, \tag{18.161}$$

$$\frac{dx}{dt} = y^2. \tag{18.162}$$

In this flow, one easily finds that

$$y^2(t_1) - x^2(t_1) = y^2(t_2) - x^2(t_2), \tag{18.163}$$

that is, the flow is along hyperbolas. Of special interest are its asymptotes,

$$x = \pm y. \tag{18.164}$$

Looking at Figure 18.2, for $y > 0$, the line $x = -y$ in the second quadrant separates flows into the massless fixed line from the ones that flow to massive or gapped theories. The reflected asymptote $x = y$ in the third quadrant defines the basin of attraction of the TL line for $y < 0$. We focus on the $y > 0$ case since the mathematics is identical in the two cases. The physics is different, as will be explained later.

Let us start at the far left at a point

$$y^2(0) - x^2(0) = \delta \tag{18.165}$$

just above the separatrix $x = -y$. This means that at any generic t,

$$y^2(t) - x^2(t) = \delta. \tag{18.166}$$

We want to know how $\xi(\delta)$ diverges as we approach the separatrix that flows into the fixed point at the origin.

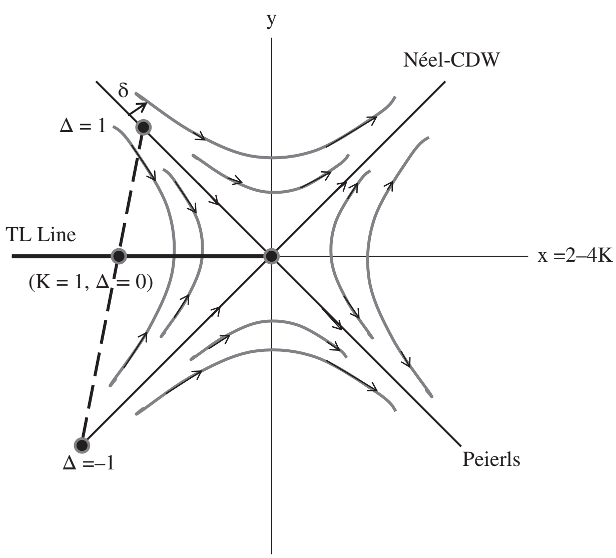

Figure 18.2 The Kosterlitz–Thouless (KT) flow. The origin is at ($x \stackrel{\text{def}}{=} 2 - 4K = 0, y = 0$). The TL liquid is found on the x-axis for $x < 0$ or $K > \frac{1}{2}$. The point $K = 1, y = 0$ describing a free fermion lies on this line. The dotted line passing through it is a *schematic* of the TL model as its sole parameter Δ is varied. That the point $\Delta = 1$ is the last of the massless phase and flows under RG to the origin we know from the exact solution. Larger values of Δ approach this end point and veer away to a gapped CDW phase. The point $\Delta = -1$ marks the other end of the gapless phase after which the flow is to the Peierls phase. The correlation length diverges as $e^{\pi/\sqrt{\delta}}$ when we approach either separatix.

This is determined by the flow

$$\frac{dx}{dt} = y^2(t) = (\delta + x^2(t)), \tag{18.167}$$

with a solution

$$t = \frac{1}{\sqrt{\delta}}\left[\arctan\frac{x(t)}{\delta} - \arctan\frac{x(0)}{\delta}\right], \tag{18.168}$$

$$t \simeq \frac{\pi}{\sqrt{\delta}}, \tag{18.169}$$

assuming that we start at the far left and end at the far right.

Since the rescaling factor $s = e^t$, it follows that $\xi(t) = e^{-t}\xi(0)$. Assuming that for large t the correlation length $\xi(t) \to \mathcal{O}(1)$ (because we are essentially on the line $x = y$, far from the critical point at the origin),

$$\xi(0) = e^t \xi(t) \simeq \exp\left[\frac{\pi}{\sqrt{\delta}}\right], \tag{18.170}$$

implying the *exponential* divergence of the correlation length as $\delta \to 0$.

What if we start *on* the line $x = y$? The solution to

$$\frac{dx}{dt} = x^2 \tag{18.171}$$

is

$$x(t) = \frac{x(0)}{1 - x(0)t} = -\frac{|x(0)|}{1 + |x(0)|t} \simeq -\frac{1}{t}, \tag{18.172}$$

which is the logarithmic, marginally irrelevant flow we saw earlier in ϕ_4^4. On the other side, if we begin on the separatrix $x = y$, the solution

$$x(t) = \frac{x(0)}{1 - x(0)t} \tag{18.173}$$

will grow to large values because now $x(0) > 0$. At some point the weak coupling analysis will fail.

Besides these flows, there are the relevant lines flowing away from the fixed line for $x > 0$. The behavior of various regions is shown in Figure 18.2. Although the analysis was for small x and y, it is assumed that the overall topology will survive, though the flow lines could deviate from what was shown above.

18.6 Analysis of the KT Flow Diagram

Figure 18.2 is worth more than the usual thousand words. As mentioned before, the separatrix $y = -x$ in the second quadrant, flowing into the origin, defines the domain of attraction of the fixed line for $y > 0$, $x < 0$. When we cross it, $\xi \simeq e^{1/\sqrt{\delta}}$, where the deviation δ is shown in the figure. If we start just above the separatrix, we initially flow along it toward the origin and then veer away along the separatrix $x = y$, to a state with a hefty gap. If we follow the original model along a curve parameterized by Δ, the point where it intersects the separatrix $y = -x$ is when $\Delta = 1$. This because we know from the exact solution that the gap develops for $\Delta > 1$.

What is behind this gap? The state we are headed for has a large positive y and that means we want, based on Eq. (18.102),

$$\cos\sqrt{16\pi}\phi = \frac{1}{2}(1 - 2\sin^2\sqrt{4\pi}\phi) \tag{18.174}$$

to be maximally negative, i.e.,

$$\sin^2\sqrt{4\pi}\phi = 1 \tag{18.175}$$

$$\sin\sqrt{4\pi}\phi = \pm 1. \tag{18.176}$$

Thus there are two ground states. In them,

$$\langle \sin\sqrt{4\pi}\phi\rangle \simeq \langle \psi_+^\dagger(x)\psi_-(x) + \psi_-^\dagger(x)\psi_+(x)\rangle = \langle i\bar{\psi}\gamma^5\psi\rangle = \pm\mathcal{D}_{\text{CDW}}, \tag{18.177}$$

where the CDW order parameter $\mathcal{D}_{\text{CDW}}$ describes a variable that connects the left and right Fermi points, and oscillates as $(-1)^j$. Indeed, from Eq. (18.99),

$$:\psi^\dagger(x)\psi(x): = :\psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_-(x): + (-1)^j(\psi_+^\dagger(x)\psi_-(x) + \psi_-^\dagger(x)\psi_+(x)), \tag{18.178}$$

we see that the fermion charge density has one part that is smooth and one that oscillates as $(-1)^j$, and it is the latter which has developed a condensate or expectation value. This was our early conclusion based on looking at the nearest-neighbor interaction at very large coupling. As $\Delta \to \infty$, one sublattice is occupied and the other is empty to get rid of the nearest-neighbor repulsion. In such a state it costs energy to move the charge, forcing it to have a nearest neighbor. That is the gap.

Suppose we turn on a negative Δ on the fixed line. There is another separatrix $x = y$ in the third quadrant that defines the domain of attraction of the TL fixed line. The exact solution tells us that this end point corresponds to $\Delta = -1$. If we go below, we first flow toward the origin and then off to large negative values of y. This takes us to the *Peierls state*. What happens here?

Because $y < 0$, we want

$$\cos\sqrt{16\pi}\phi = \frac{1}{2}(-1 + 2\cos^2\sqrt{4\pi}\phi) \tag{18.179}$$

to be maximally positive, i.e.,

$$\cos^2\sqrt{4\pi}\phi = 1 \tag{18.180}$$

$$\cos\sqrt{4\pi}\phi = \pm 1. \tag{18.181}$$

Thus there are two "Peierls" ground states.

To interpret the physics of the Peierls state we recall that

$$\langle \psi^\dagger(j+1)\psi(j) + \text{h.c.}\rangle = \langle(\psi_R^\dagger(x)(-i)^j + \psi_L^\dagger(x)(i)^j)(j \to j+1)\rangle + \text{h.c.}$$

$$= (-1)^j \langle(-i\psi_R^\dagger(x)\psi_L(x) + \text{h.c.})\rangle + \text{NOP} \quad (18.182)$$

$$= \frac{(-1)^j}{2\pi\alpha}\langle \cos\sqrt{4\pi}\phi\rangle \equiv (-1)^j \mathcal{D}_\text{P}. \quad (18.183)$$

I have dropped the non-oscillatory part (NOP) $\psi_R^\dagger(-i\partial_x)\psi_R + \psi_L^\dagger(+i\partial_x)\psi_L$ and emphasized only that in the Peierls state the kinetic energy alternates as $(-1)^j\mathcal{D}_\text{P}$.

The dotted line in the figure shows our original model with just one parameter Δ. For small Δ we can start at a reliably known point in the (x,y) plane and follow the flow to the x-axis. As $y \to 0$, x will move to the right. In general, we cannot precisely relate Δ to the parameters K (or x) and y due to renormalization effects. However, we can say, based on the Yang and Yang solution [23], that the gapless liquid phase is bounded by $|\Delta| < 1$.

Finally, on the $x > 0$ side, we can go directly to the CDW and Peierls phases starting with arbitrarily small y, as shown in Figure 18.2.

The $y \to -y$ symmetry of the KT flow diagram is consistent with the fact that

$$H(\Delta) = -\frac{1}{2}\sum_j \psi^\dagger(j+1)\psi(j) + \text{h.c.}$$

$$+\Delta \sum_j \left(\psi^\dagger(j)\psi(j) - \frac{1}{2}\right)\left(\psi^\dagger(j+1)\psi(j+1) - \frac{1}{2}\right) \quad (18.184)$$

is unitarily equivalent to $-H(-\Delta)$:

$$U^\dagger H(\Delta) U = -H(-\Delta), \quad (18.185)$$

where, under U,

$$\psi(j) \to (-1)^j \psi(j). \quad (18.186)$$

This reverses the sign of the hopping term leading to Eq. (18.185). Despite this unitary equivalence under $\Delta \to -\Delta$, the physics can be very different: e.g., CDW versus Peierls as $|\Delta| \to \infty$.

The transition from a metal to insulator driven by interaction is generally very hard to analyze with any exactitude. The preceding model is one of the rare examples, albeit in $d = 1$. Since Yang and Yang and Baxter have established many exact results (such as the expression for the CDW order parameter as a function of Δ), we can interpret them in the light of the metal insulator transition. One such study is [28]. Despite the use of continuum methods, many exact results are derived about conductivity as well as some surprising results on the effect of a random potential. Other illustrations of bosonization can be found in [29–32]; the list is not exhaustive or even representative – however, once you get your hands on these you can follow the leads given therein to find more. For the application of bosonization to a single impurity problem, see Kane and Fisher [33].

18.7 The XXZ Spin Chain

The model of spinless fermions we have solved is mathematically identical to the spin-$\frac{1}{2}$ chain with

$$H_{\rm XXZ} = \sum_j \left[S_x(j)S_x(j+1) + S_y(j)S_y(j+1) + \Delta S_z(j)S_z(j+1)\right]. \tag{18.187}$$

The following Jordan–Wigner transformation relates the two:

$$S_z(j) = \psi^\dagger(j)\psi(j) - \frac{1}{2}, \tag{18.188}$$

$$S_+(j) = (-1)^j \psi^\dagger(j) \exp\left[i\pi \sum_{k<j} \psi^\dagger(k)\psi(k)\right] = S_-^\dagger(j), \tag{18.189}$$

where the $(-1)^j$ is introduced to give the kinetic term the same sign as in the Luttinger model, with a minimum at zero momentum.

We can bodily lift our results from the fermion problem to the spin chain. In particular, both have a gapless region that gives way to broken symmetry states with an order parameter at momentum $2K_{\rm F} = \pi$. The gapless region is bounded by $\Delta = \pm 1$, as we know from the exact solutions of Yang and Yang [23] and Baxter [24, 25], who solved the XYZ model with different couplings for the three terms by relating $H_{\rm XYZ}$ to the transfer matrix of the eight-vertex model. (I remind you once again that we can relate K to Δ only at weak coupling. As we begin with larger values of Δ, the parameters $K(\Delta)$ or $x(\Delta)$ will get renormalized as the irrelevant coupling y renormalizes to 0. The flow in the (x,y) planes is not vertical, not known exactly, and the definition of y is sensitive to how we cut off the theory, i.e., α.)

In the spin language, the CDW state when $y \to +\infty$ corresponds to a state with $\langle S_z \rangle \simeq (-1)^j$ because $S_z(j) = n_j - \frac{1}{2}$. In the limit $y \to -\infty$, we have the *spin-Peierls* state in which the average bond energy $\langle S_+(j)S_-(j+1) + \text{h.c.}\rangle$ oscillates as $(-1)^j$.

While we can borrow these results from the mapping to the TL model, correlation functions are a different matter. Whereas $S_z - S_z$ correlations are easy because S_z is just a fermion bilinear, correlation functions of $S_\pm$ are non-local in the fermion language and involve the dual field θ.

Consider, for example, the simplest case $\Delta = 0$ and the correlator

$$\langle S_+(0)S_-(j)\rangle$$
$$= \langle \psi^\dagger(0) \exp\left[i\pi \sum_{k=0}^{j-1} \psi^\dagger(k)\psi(k)\right] \psi(j)\rangle \tag{18.190}$$

$$\simeq (-1)^j \psi^\dagger(0) e^{i\sqrt{\pi}(\phi(x)-\phi(0))+ik_{\mathrm{F}}x}\psi(x) \tag{18.191}$$

$$= a(-1)^j \left[\psi_+^\dagger(0)+\psi_-^\dagger(0)\right] e^{i\sqrt{\pi}(\phi(x)-\phi(0))+ik_{\mathrm{F}}x}\left[\psi_+(x)e^{ik_{\mathrm{F}}x}+\psi_-(x)e^{-ik_{\mathrm{F}}x})\right], \tag{18.192}$$

where I have canceled the string to the left of $j=0$ and used

$$i\pi\sum_k^{j-1}\psi^\dagger(k)\psi(k) = i\int_0^x \sqrt{\pi}\ \partial_x\phi dx + \left[\frac{i\pi j}{2} = iK_{\mathrm{F}}j = ik_{\mathrm{F}}x\right], \tag{18.193}$$

where $k_{\mathrm{F}} = K_{\mathrm{F}}/a$ is the dimensional Fermi momentum.

Now we have, from Eq. (18.189) (upon ignoring the factor $\frac{a}{2\pi\alpha}$),

$$\begin{aligned}\langle S_+(0)S_-(j)\rangle &\simeq (-1)^j\left[e^{-i\sqrt{\pi}(\phi(0)-\theta(0))}+e^{i\sqrt{\pi}(\phi(0)+\theta(0))}\right]\\ &\quad\times e^{i\sqrt{\pi}(\phi(x)-\phi(0))}e^{ik_{\mathrm{F}}x}\left[e^{i\sqrt{\pi}(\phi(x)-\theta(x))}e^{ik_{\mathrm{F}}x}+e^{-i\sqrt{\pi}(\phi(x)+\theta(x))}e^{-ik_{\mathrm{F}}x}\right]\\ &= (-1)^j\langle e^{i\sqrt{\pi}(\theta(0)-\theta(x))}\rangle\left\langle\left(1+e^{-i\sqrt{4\pi}\phi(0)}\right)\left(1+e^{i\sqrt{4\pi}\phi(x)}e^{2ik_{\mathrm{F}}x}\right)\right\rangle\\ &= (-1)^j\left[\frac{\alpha^2}{\alpha^2+x^2}\right]^{1/4}\left[1+\frac{(-1)^j}{x^2}\right].\end{aligned} \tag{18.194}$$

At $K\neq 1$, the leading term will be

$$\langle S_+(0)S_-(j)\rangle \simeq (-1)^j\frac{1}{x^{(1/2K)}}, \tag{18.195}$$

whereas to leading order the $S_z - S_z$ correlation that goes as $(-1)^j$ is

$$\langle S_z(0)S_z(j)\rangle \simeq (-1)^j\frac{1}{x^{2K}}. \tag{18.196}$$

We see that at $K=\frac{1}{2}$, we have the isotropic Heisenberg chain, described by the origin in Figure 18.2. (This result does not follow from weak-coupling bosonization, which is reliable only near $K=1$. Rather, we take K as a phenomenological parameter.) The main message is that the origin describes the isotropic Heisenberg antiferromagnet as we approach it from the second quadrant on the separatrix $y=-x$. This problem was originally solved by Bethe, who introduced the famous Bethe ansatz.

18.8 Hubbard Model

Now we consider fermions with spin. Usually, the inclusion of spin causes some predictable changes. This is not so here.

The Hubbard model has a non-interacting part,

$$H_0 = -\frac{1}{2}\sum_{s,n}\left[\psi_s^\dagger(n)\psi_s(n+1)+\mathrm{h.c.}\right]+\mu\sum_{s,n}\psi_s^\dagger(n)\psi_s(n), \tag{18.197}$$

where $s = \uparrow, \downarrow$ are two possible spin orientations. *We do not assume* $K_F = \frac{\pi}{2}$ *at this point*, and use a general chemical potential μ.

Following the usual route, we get two copies of the spinless model:

$$H_0 = \sum_s \int_{-\pi}^{\pi} (\mu - \cos k)\psi_s^\dagger(k)\psi_s(k)\frac{dk}{2\pi}, \tag{18.198}$$

and the continuum version

$$H_c = \sin K_F \sum_s \int dx(\psi_{s-}^\dagger(x)(i\partial_x)\psi_{s-}(x) + \psi_{s+}^\dagger(x)(-i\partial_x)\psi_{s+}(x)). \tag{18.199}$$

Let us now turn on the Hubbard interaction,

$$H_{\text{int}} = U\sum_n \psi_\uparrow^\dagger(n)\psi_\uparrow(n)\psi_\downarrow^\dagger(n)\psi_\downarrow(n), \tag{18.200}$$

where $\psi_\uparrow$, $\psi_\downarrow$ stand for the original non-relativistic fermion. The Hubbard interaction is just the extreme short-range version of the screened Coulomb potential between fermions. Due to the Pauli principle, only opposite-spin electrons can occupy the same site. One can extend the model to include nearest-neighbor interactions, but we won't do so here.

Let us now express this interaction in terms of the Dirac fields. We get, in obvious notation,

$$\begin{aligned}
&\psi_\uparrow^\dagger(n)\psi_\uparrow(n)\psi_\downarrow^\dagger(n)\psi_\downarrow(n) \\
&\quad = (\psi_{\uparrow+}^\dagger(n)\psi_{\uparrow+}(n) + \psi_{\uparrow-}^\dagger(n)\psi_{\uparrow-}(n) + (\psi_{\uparrow+}^\dagger(n)\psi_{\uparrow-}(n)e^{-2iK_F n} + \text{h.c.})) \\
&\qquad \times(\uparrow\to\downarrow).
\end{aligned} \tag{18.201}$$

If we expand out the products and keep only the parts with no rapidly oscillating factors (momentum conservation), we will, for generic K_F, get the following terms:

$$H_{\text{int}} = U(j_{0\uparrow}j_{0\downarrow}) + U(\psi_{\uparrow+}^\dagger(n)\psi_{\uparrow-}(n)\psi_{\downarrow-}^\dagger(n)\psi_{\downarrow+}(n) + \text{h.c.}). \tag{18.202}$$

If we now bosonize these terms as per the dictionary, we get, in the continuum (dropping the subscript c for continuum),

$$H = \int dx \frac{1}{2}\left[\Pi_\uparrow^2 + (\partial\phi_\uparrow)^2 + (\uparrow\to\downarrow)\right] + U\left[\frac{\partial\phi_\uparrow\partial\phi_\downarrow}{\pi} + \frac{1}{\pi^2\alpha^2}\cos\sqrt{4\pi}(\phi_\uparrow - \phi_\downarrow)\right]. \tag{18.203}$$

We can now separate the theory into two parts by introducing charge and spin fields ϕ_c and ϕ_s:

$$\phi_{c/s} = \frac{\phi_\uparrow \pm \phi_\downarrow}{\sqrt{2}}. \tag{18.204}$$

This will give us

$$H = H_c + H_s, \tag{18.205}$$

$$K_c \cdot H_c = \int \frac{1}{2}\left[K_c \Pi_c^2 + \frac{1}{K_c}(\partial\phi_c)^2\right] dx, \tag{18.206}$$

$$K_s \cdot H_s = \int \left(\frac{1}{2}\left[K_s \Pi_s^2 + \frac{1}{K_s}(\partial\phi_s)^2\right] + \frac{U}{\pi^2\alpha^2}\cos\sqrt{8\pi}\phi_s\right) dx, \tag{18.207}$$

$$K_{c/s}^2 = \frac{1}{1 \pm \frac{U}{\pi}}. \tag{18.208}$$

It is obvious that the charge sector is gapless and described by a quadratic Hamiltonian. This means that there will be no gap to creating charge excitations, the system will be metallic. The fate of the spin sector needs some work. Upon rescaling the kinetic term to standard form we find the cosine interaction

$$\cos\beta\phi_s = \cos\sqrt{\frac{8\pi}{\sqrt{1 - U/\pi}}}\phi_s. \tag{18.209}$$

We can now see that for weak positive U, this interaction does not produce any gap because $\beta^2 > 8\pi$, while for weak negative U, it does because $\beta^2 < 8\pi$. The exact solution of Lieb and Wu [18] and the following physical argument explain the spin gap for $U < 0$. If there is an attraction between opposite spin electrons, they will tend to form on-site, singlet pairs. To make a spin excitation, we must break a pair, and this will cost us, i.e., there will be a gap in the spin sector.

The fact that $K_s \neq K_c$ means that charge and spin move at different velocities. This *spin–charge separation* cannot be understood in terms of interacting electrons whose charge and spin would be irrevocably bound. This is more evidence of the demise of the quasiparticle, adiabatically connected to the primordial fermion.

In the special case of half-filling, another term comes in. If we look at Eq. (18.201), we see that in the case of half-filling, since $K_F = \pi/2$, the factors $e^{\pm 4iK_F n}$ are not rapidly oscillating, but simply equal to unity. Thus, two previously neglected terms in which two right movers are destroyed and two left movers are created, and vice versa, come into play. (This is an umklapp process, in which lattice momentum is conserved modulo 2π [19,20]). I leave it to you to verify that the bosonized form of this interaction, after rescaling of the charge field in the manner described above for the spin field, is another $\cos\beta_c\phi_c$, with

$$\beta_c = \sqrt{\frac{8\pi}{\sqrt{1 + U/\pi}}}. \tag{18.210}$$

Thus we find that the situation is exactly reversed in the charge sector: there is a gap in repulsive case, and no gap in the attractive case. To see what is happening, think of very large positive U. Now there will be one electron per site at half-filling, unable to move without stepping on someone else's toes, i.e., there is a charge gap of order U if you try to move the charge. But the spin can do whatever it wants with no cost. If U were very large and negative, there would be tightly bound pairs on half the sites. These doubly charged objects can be moved without cost. There will, however, be a cost for breaking the spin pair.

18.9 Conclusions

I have tried to show you how to use bosonization to solve a variety of problems. The formalism is straightforward, but has some potential pitfalls which I avoided because I know of them. So before I let you go, I need to inform you.

In this treatment we always work in infinite volume from the beginning and are cavalier about boundary conditions at spatial infinity. The Fermi fields expressed in terms of boson fields are meant to be used for computing correlation functions and not as operator identities. After all, no combinations of bosonic operators ϕ or Π can change the fermion number the way ψ or $\psi^\dagger$ can. But of course, this was never claimed.

There is a more comprehensive and careful development in which such an operator correspondence may be set up, starting with finite volume. In these treatments the mode expansions for $\phi(x)$ and $\Pi(x)$ have additional terms (of the form $\frac{x}{L}$) that vanish as the system size $L \to \infty$. Next, in our scheme we had $[\phi_+(x), \phi_-(y)] = \frac{i}{4}$, which was needed to ensure some anticommutators, while in the more careful treatments $[\phi_+(x), \phi_-(y)] = 0$, a feature that is central to conformal field theory, which treats right and left movers completely independently. In these treatments there are compensating *Klein factors*, which are operators tacked on to ensure that different species of fermions anticommute. (We did not need them in the problems I discussed since the factors come in canceling pairs.)

The excellent article by van Delft and Schoeller [34] devotes an appendix to the differences between what is presented here (called the field-theory approach) and what they call the constructive approach. Other online articles I have benefited from are due to Voit [35], Schulz [36], and Miranda [37]. A rigorous treatment may be found in Heidenreich *et al.* [38]. A more intuitive review is due to Fisher and Glazman [39].

In addition, I have found lucid introductions in the books by Itzykson and Drouffe (vol. 1) [14], Fradkin [40], Sachdev [16], Giamarchi [17], and Guiliani and Vignale [41].

There is a development called *non-Abelian bosonization*, due to Witten [42], in which the internal symmetries of the model are explicitly preserved. For example, if we are considering an N-component Gross–Neveu model, the $U(N)$ symmetry is not explicit if we bosonize each component with its own field ϕ_i. In non-Abelian bosonization, $U(N)$ group elements replace the ϕ_i and the symmetry is explicit. For a review, see [43].

Haldane expanded bosonization to $d = 2$ [44]. For an application, see [45].

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