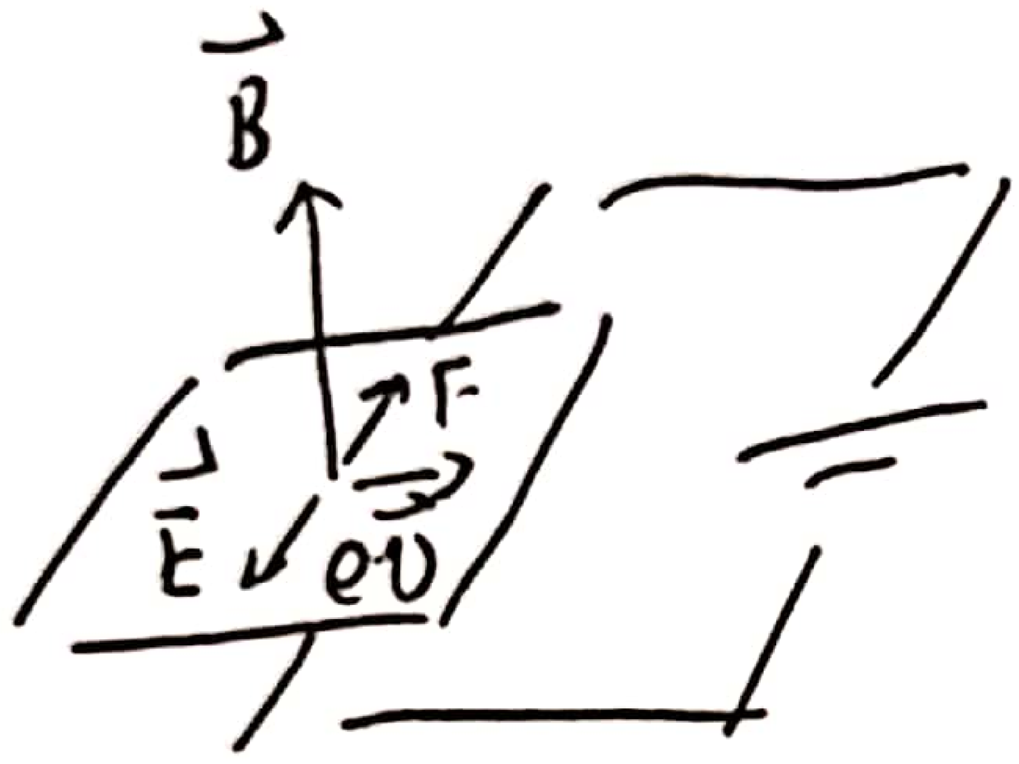
Maxwell equation

$$\mathcal{L} = \dot{\vec{A}}^2 - (\nabla \times \vec{A})^2 = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \Rightarrow.$$

$$\Updownarrow$$

$$\mathcal{L} = p^2 - x^2.$$

在2D情况下，



QHE.

$\vec{B} \perp$ 平面 $\vec{B} = (0, 0, B)$.

$\vec{E} \parallel$ 平面. $\vec{E} = (E_x, E_y, 0)$.

in 2d case.

$$\mathcal{L} = E_x^2 + E_y^2 - B_z^2$$

in (1+1)d case.

$$\mathcal{L} = E_x^2 - B_z^2.$$

$$\vec{E} = \dot{\vec{A}} \qquad \vec{B} = \nabla \times \vec{A}.$$

$$\mathcal{L} = \dot{A}^2 - (\partial \times A)^2.$$

$(1+1)$ $\vec{A} \to A$

(1+1)d case, E_x 僅一分量, 故A為標量.

1d 波動方程. (聲子).

$$\mathcal{L} = \frac{1}{2}\left((\partial_t \phi)^2 - v^2 (\partial_x \phi)^2 \right)$$

XY model. 超流.

波色化.

$$\mathcal{L} = \frac{1}{2}\left((\partial_t \theta)^2 - (\nabla\theta)^2 \right) = \frac{1}{2}\left((\partial_t \theta)^2 - (\partial_x \theta)^2 \right).$$

僅需 $A \overset{\text{def}}{=} \phi$ or θ.

2D / (1+1)D 系統特殊之處.

②. 連續性方程.

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} j = 0.$$

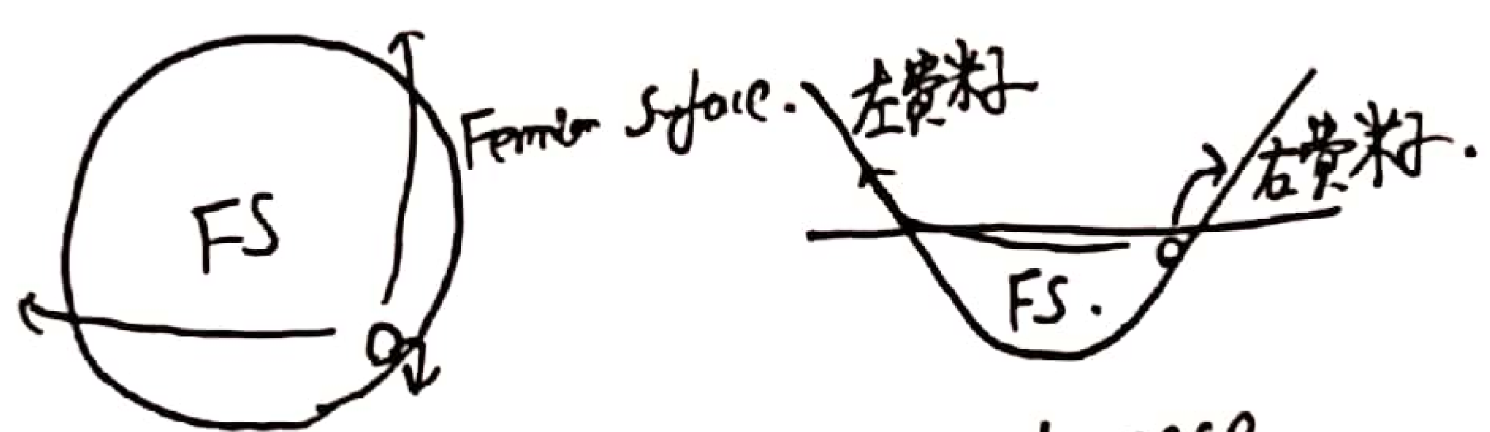
通解 $\begin{cases} \rho = \partial_x \phi \\ j = -\partial_t \phi \end{cases}$

③. 複變函數.

④. 手征性.

波動/擴散方程. 解僅唯二.

會給出兩種不同的費米子 (左、右費米子).



2d case. 1d case.

⑤. 低維系統、統計行為

可做粒子交換 方才有波色子費米子之分.

而在低維體係中. 會出現 任意子.

$d \geq 3$. 僅有費米子. 波色子. 無法做交換.

(自旋統計 by Pauli.)

⑥. 線性譜、線性激發

⑦. Jordan-Wigner 變換.

费米子、波色子算子.

Boson. $b(x)$ $\quad n_b = b^+ b$.

Fermion. $f(x)$. $\quad n_f = f^+ f$.

$\left(b(x) b^+(y) - b^+(y) b(x) = \delta(x-y) \right)$

考慮. $[b(x), b^+(y) b(y)] = b(x) b^+(y) b(y) - b^+(y) b(y) b(x)$.

$= \left(\delta(x-y) + b^+(y) b(x) \right) b(y) - b^+(y) b(y) b(x)$

$= \delta(x-y) b(y) + b^+(y) b(x) b(y) - b^+(y) b(y) b(x)$

$= \delta(x-y) b(x)$.

OR. $[n_{b(y)}, b(x)] = -\delta(x-y) b(x)$

類似本徵值問題

考慮 ~~$[f(x), f(y)]$~~

$[n_{f(x)}, f(y)] = f^+(x) f(x) f(y) - f(y) f^+(x) f(x)$

$= f^+(x) f(x) f(y) - \left(\delta(x-y) - f^+(x) f(y) \right) f(x)$

$= -\delta(x-y) f(x) + f^+(x) f(x) f(y) + f^+(x) f(y) f(x)$

$= -\delta(x-y) f(y)$.

從而 ~~$[n(x), f$~~

$[n(x), \psi(y)] = -\delta(x-y) \psi(y)$、

對费米子、波色子均成立.

定義. $b(x) = e^{i\phi(x)}$.

$[n(x), b(y)] = -\delta(x-y) b(y)$.

代入. $b(y) = e^{i\phi(y)}$. (其中 $\phi(y)$ 為算子, 與 n 不對易).

$n(x) e^{i\phi(y)} - e^{i\phi(y)} n(x) = -\delta(x-y) e^{i\phi(y)}$.

$\therefore e^{-i\phi(y)} n(x) e^{i\phi(y)} \underset{-n(x)}{\cancel{n(x)}} = -\delta(x-y)$.

$= n(x) - i[\phi(y), n(x)] - n(x) = -\delta(x-y)$ $\quad \Rightarrow \quad [n(x), \phi(y)] = i\delta(x-y)$.

(Backer Campbell - Hausdorff 公式.

$e^{-A} B e^{A} =$)

Dirac 1927. QED quantization.

$\psi = \sum_n C_n \varphi_n$.

$H = E = \langle \psi | H | \psi \rangle \overset{=}{\Rightarrow} t_{nm} C_n^* C_m$.

$i\dot{C}_n = \frac{\partial H}{\partial C_n^*} \Rightarrow \begin{cases} i\dot{C}_n^* = -\frac{\partial H}{\partial C_n} \\ \dot{C}_n = \frac{\partial H}{\partial (iC_n^*)} \end{cases} \Leftrightarrow \begin{cases} P_n = C_n \\ q_n = iC_n \end{cases}$

$\Leftrightarrow \dot{C}_n = \frac{\partial H}{\partial iC_n^*}$

若 $C_n = \sqrt{r} e^{i\theta}$，則 r 與 θ 對偶
與 $[n(x), \phi(y)] = i\delta(x-y)$ 異曲同工.

或. $[b(x), b^\dagger(x)] = \delta(x-y)$.

令 $\begin{cases} b = \cancel{\sqrt{n}} e^{i\phi}\sqrt{n} \\ b^\dagger = \sqrt{n} e^{-i\phi} \end{cases}$ $b^\dagger b = \sqrt{n(x)} e^{-i\phi(x)} e^{i\phi(y)} \sqrt{n(y)}$

$\sim n$.

$b^\dagger b = \cancel{e^{i\phi} n e^{-i\phi}}$

$\sim e^{i\phi} n e^{-i\phi}$

$\therefore e^{i\phi} n e^{-i\phi} - n = \delta(x-y)$.

$\Leftrightarrow [\phi, n] = i\delta(x-y)$.

費米子 $\xrightarrow{n, \phi 場}$ 玻色子、

1934. Bloch. 1d Fermions have the same type of low-energy excitations as a harmonic chain

1950s. 朝永振一郎 Remarks of Bloch's ~~method~~ method of Sound waves applied to many-Fermions problem.

1964. Luttinger 嚴格可解的多體模型.

1981. Haldane. Celebrating Haldane's luttinger liquid theory.

Reference.
Shanker Chapter 17. 18.
B. Simons. Chapter 2.
X. G. Wen. edge state's hydrodynamics description

源. Jordan-Wigner 變換.
Bloch's Sound Wave.

$\psi \sim e^{i\phi}$. φ 為玻色場. $\psi = \frac{\eta}{\sqrt{4\pi a}} e^{i\phi}$. η: Klein operator. 減一粒子 使粒子數不變(波函)
a: 截斷.

Jordan-Wigner 變換

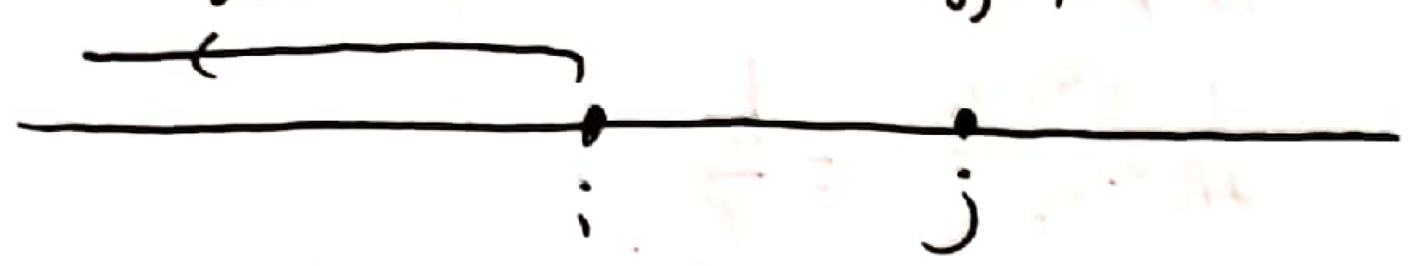
lattice model.

$f_i = e^{i\theta_i} b_i$.

$$\begin{cases} f_i f_j = e^{i\theta_i} b_i e^{i\theta_j} b_j = e^{i\theta_i} e^{i\theta_j} (e^{-i\theta_j} b_i e^{i\theta_j}) b_j. \\ f_j f_i = e^{i\theta_j} b_j e^{i\theta_i} b_i = e^{i\theta_j} e^{i\theta_i} (e^{-i\theta_i} b_j e^{i\theta_i}) b_i. \end{cases}$$

$i<j$

θ_i 只與路徑有關. b_j 與路徑無關.



$\theta_i = \int_{-\infty}^{i} b^+(x) b(x) dx$,

$\cancel{[\theta_i, \theta_j]}$ $[\theta_i, b_j] = 0$.

則
$$\begin{cases} f_j f_i = e^{i\theta_i} e^{i\theta_j}. \\ f_i f_j = e^{i\theta_i} e^{i\theta_j} e^{i\theta}. \quad \cancel{\theta =} \end{cases}$$

$\Downarrow$

$$\boxed{\begin{aligned} & f_i f_j + f_j f_i \\ & = \cancel{0}\, e^{i\theta_i} e^{i\theta_j} (e^{i\theta}+1) \\ & = 0 \Rightarrow \theta = \pi. \end{aligned}}$$

即 $f^2 = 0$. $e^{i\theta b} e^{i\theta b} = 0$.

$$\begin{aligned} & e^{-i\theta b^+ b}\, b\, e^{i\theta b^+ b} |n\rangle \\ & = e^{i\theta b} e^{in\theta} e^{-i\theta b^+ b} b |n\rangle \\ & = \cancel{\sqrt{n} e^{i\theta\pi}} \sqrt{n} e^{in\theta} e^{-i\theta b^+ b} |n\rangle \\ & = \sqrt{n} e^{in\theta - i\theta(n-1)} |n-1\rangle \\ & = e^{i\theta} b |n\rangle \end{aligned}$$

∴ if $f(x) = e^{i\pi \int_{-\infty}^{x} b^+(y) b(y) dy} \cdot b(y)$

相位. string. 與過去有關的全部經歷. 改變統計性質.

$\psi(x) = e^{i\theta \int_{-\infty}^{x} f^+(y) f(y) dy} f(x)$.

$$\begin{cases} f_i = e^{i\theta_i} b_i \\ f_i^2 = 0. \end{cases} \Rightarrow e^{i\theta_i} b_i e^{i\theta_i} b_i \propto b_i^2 \sim 0.$$

Anyon.

$f(x) f(y) = e^{i\theta} f(y) f(x) \Rightarrow \theta = \pi$. $f(x) f(y) = -f(y) f(x)$.

若θ為任意相位. ⇒ 產生任意子.

if. $f = e^{i\theta} b \Rightarrow H_F \to H_B \to Z_F = Z_B$.

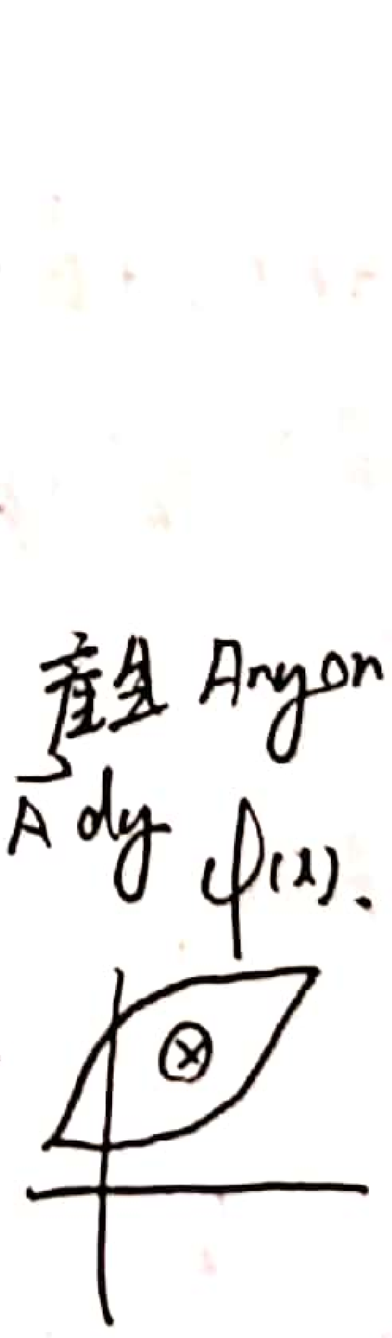
嚴格證明由 Haldane duality 給出.

Bloch 觀察.
$$\begin{cases} C_L = \frac{\pi}{3} k_B \frac{k_B T}{\hbar c_s} & \text{聲子.} \\ C_L = \frac{\pi}{3} k_B \frac{k_B T}{\hbar v_F}. \end{cases}$$

在QHE中. anyon in magnetic 產生 Anyon.

$\psi(x) \Rightarrow e^{i\int_{-\infty}^{x} A\, dy} \psi(x)$.

類似於 AB 效應.



總結.

① $\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} j = 0 \xrightarrow{\text{in 1d case}} \rho = \partial_x \phi,\ j = -\partial_t\phi. \propto \nabla\theta$. (僅對於1d case).

② 密度與相位的共軛量. $[n,\phi] = i\delta(x-x')$.

③ string 可以改變統計性質. $f(x) = e^{i\theta\int_{-\infty}^{x}\rho(x)dx} b(x) = \cancel{e^{i\theta\int}}\ e^{i\theta\int_{-\infty}^{x}\rho(x)dx}\sqrt{n}\,e^{i\theta(x)}$

$= e^{i(\theta\int_{-\infty}^{x}\rho(x)dx + \theta(x))}$.

密度漲落有gap. 相位漲落無gap.

相位場 $\phi = \pi\int_{-\infty}^{x}$

$$\rho = \rho_0 + \delta\rho. = \pi\rho_0 x + \pi\int_{-\infty}^{x}\delta\rho(y)dy + \theta(x)$$

$$= \underbrace{k_F x}_{\text{與費米面有關}} + \underbrace{\pi\int_{-\infty}^{x}\delta\rho(y)dy}_{\text{改變統計性質}} + \underbrace{\theta(x)}_{\text{相位漲落}}.$$

$\left(\rho_0 = \frac{N}{L} = n = \frac{1}{L}\sum_k = \frac{1}{2\pi}\int_{-k_F}^{k_F}dk = \frac{k_F}{\pi}.\right)$

1d case Maxwell equation

$$\begin{cases}\frac{\partial E}{\partial x} = -\frac{\partial H}{\partial t}\\ \frac{\partial E}{\partial t} = -\frac{\partial H}{\partial x}.\end{cases} \Rightarrow \begin{cases}E = \dot{A},\ H = -\partial_x A\\ E = \partial_t\phi.\ H = -(\frac{\partial\phi}{\partial t})\end{cases}$$

(Current algebra).

Bosonization: 以玻色場描述費米子. $\psi \sim e^{i\phi}$.

(高維處理思路 $\int D\bar\psi D\psi DA\, e^{-S[\bar\psi,\psi,A]} = \int DA\, e^{-S_{eff}[A]}$、積掉費米場、等效玻色場)

Lattice Model.

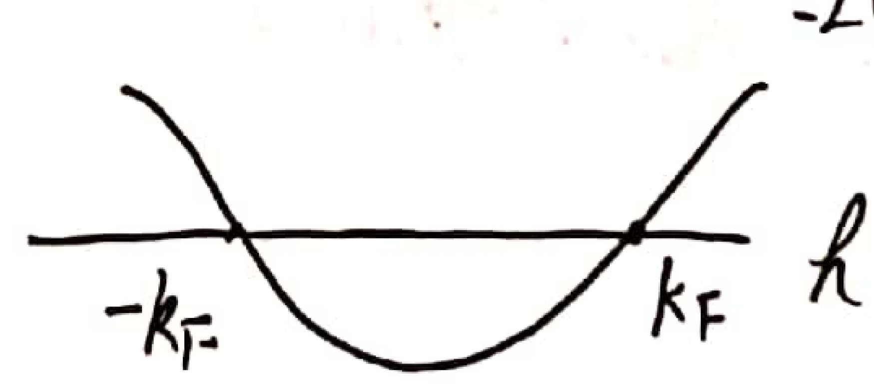
$$H = -t\sum_n(\psi_n^\dagger\psi_{n+1} + h.c.) + h\psi_n^\dagger\psi_n$$

$$= \sum_k(-2t\cos k + h)\psi_k^\dagger\psi_k.$$

$\updownarrow$

$$\int dx\,\psi^\dagger(-\frac{\partial^2}{\partial x^2} + h)\psi.$$

$-2t\cos k$. 以費米面為零勢能面, 在費米面附近的 Dirac eq.



$$H = \sum_{k\approx k_F} v_k(k-k_F)\psi_R^\dagger(k)\psi_R(k) + \sum_{k\approx -k_F} v_k(k+k_F)\psi_L^\dagger(k)\psi_L(k).$$

令 $k = k_F + q$.

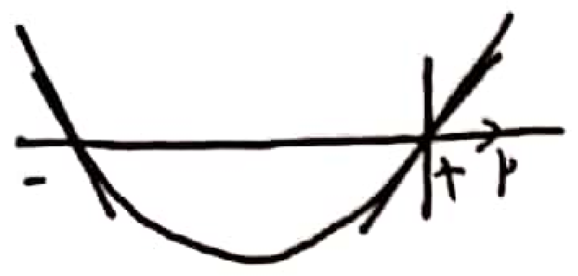
$$H_{eff} = \psi_q^\dagger\begin{pmatrix}v_q & 0\\ 0 & -v_q\end{pmatrix}\psi_q. \Leftarrow \psi_q = \begin{pmatrix}\psi_R(q)\\ \psi_L(q)\end{pmatrix} \Leftarrow \begin{aligned}H_R &= \sum_{q\sim 0} vq\,\psi_R^\dagger(q)\psi_R(q)\\ H_L &= -\sum_{q\sim 0} vq\,\psi_L^\dagger(q)\psi_L(q).\end{aligned}$$

Not an operator identity.

玻色化並非恆等變換。但從關聯函數考慮，費米關聯與玻色子關聯一致

費米關聯全部用玻色子關聯代替，結果不變。

$\psi = \frac{1}{\sqrt{2\pi a}} e^{\pm i\sqrt{4\pi}\phi(x)}$.



$$\psi_\pm(x) = \frac{1}{L}\sum_k \psi_\pm(k)\, \cancel{e^{\pm ikx}}\, e^{ikx} e^{-\frac{1}{2}\alpha|k|}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \psi_\pm(k) e^{ikx} e^{-\frac{1}{2}\alpha|k|}$$ 截斷，保證收斂。

$\{\psi_+(x), \psi_-(y)\} = 0$ 顯然

現計算 $\cancel{\langle\phi^+(x)\phi(y)\rangle}$ $\langle\psi_+(x)\psi_+^\dagger(0)\rangle = \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty} dp\, dq\, e^{ipx+iq0} e^{-\frac{1}{2}\alpha|p|-\frac{1}{2}\alpha|q|} \underbrace{\langle\psi_+(k)\psi_+^\dagger(q)\rangle}_{\text{in ground state}}$

$\sim 2\pi\delta(p-q)$.

$$= \cancel{\left(\frac{1}{2\pi}\right)^2\int_0^\infty e^{ipx-\alpha|p|}}$$

$$= \cancel{\frac{1}{2\pi}\int_0^\infty e^{ipx}} - \alpha|p|\,dp = \frac{1}{2\pi}\left(\frac{1}{\alpha-ix}\right)$$

$$\frac{1}{2\pi}\int_0^\infty e^{ipx-\alpha|p|}dp$$

$$\left(\begin{array}{l} C_k^+|\circledcirc\rangle = 0. \text{ if } |k|<k_F. \\ C_k^+|\circledcirc\rangle \neq 0 = |\underset{k}{\circledcirc}\rangle \text{ if } |k|>k_F. \end{array}\right)$$

$\therefore \langle\psi_+(x)\psi_+^\dagger(0)\rangle = \frac{1}{2\pi}\frac{1}{\alpha-ix}$.

$$\langle\psi_+^\dagger(0)\psi_+(x)\rangle = \left(\frac{1}{2\pi}\right)^2\int dp\, dq\, e^{ipx+iq0} e^{-\frac{1}{2}\alpha|p|-\frac{1}{2}\alpha|q|} \underbrace{\langle\psi_+^\dagger(q)\psi_+(0)\rangle}_{k=q<0} \sim 2\pi\delta(k-q).$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} dp\, e^{ipx+\frac{1}{2}\alpha p}.$$

$$\delta(x) = \left\{\begin{array}{l}\langle\psi_+^\dagger(0)\psi_+(x)\rangle \\ + \\ \langle\psi_+(x)\psi_+^\dagger(0)\rangle\end{array}\right. = \frac{1}{2\pi}\frac{1}{\alpha-ix} + \frac{1}{2\pi}\frac{1}{\alpha+ix} = \frac{\alpha/\pi}{\alpha^2+x^2} = \delta(x). \quad \text{if } x \gg \alpha.$$

$\Rightarrow \langle\psi(x)\psi^\dagger(0)\rangle = \frac{1}{2\pi}\left(\frac{1}{\alpha-ix}\right)$

令 $\psi(x) = e^{i\phi(x)}$. $\Leftrightarrow \langle e^{i\phi(x)} e^{-i\phi(0)}\rangle \sim \frac{1}{2\pi}\left(\frac{1}{\alpha-ix}\right)$.

(在XY Model中. $\langle e^{i\theta(x)} e^{-i\theta(0)}\rangle \sim \frac{1}{|x|^?}$

$\cancel{\leftarrow e}$ $\cancel{\leftarrow \phi}$ $= \langle\phi(x)\phi^\dagger(y)\rangle = g(x-y) \sim \frac{1}{|x-y|^?}$)

討論 scalar (Boson) field 行為

$$H_B = \frac{1}{2}\int\left(\pi^2 + (\partial_x\phi)^2\right)dx.$$

$$\left\{\begin{array}{l}\mathcal{L} = \frac{1}{2}\int(\partial_t\phi)^2 - (\partial_x\phi)^2 \\ \pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \partial_t\phi.\end{array}\right.$$

$\Leftrightarrow [\phi, \partial_t\phi] = i\delta(x-x')$

運動方程 $\partial_t^2\phi - \partial_x^2\phi = 0$

解具有標準形式.

$$\begin{cases} \phi = \int \frac{dp}{2\pi\sqrt{2|p|}}\left(\phi(p)e^{ipx} + \phi^{+}(p)e^{-ipx}\right)\cancel{e^{-\frac{1}{2}|\alpha|}}\, e^{-\frac{1}{2}\alpha|p|} \\ \pi \underset{=\partial_t\phi}{=} \int \frac{dp\,|p|}{2\pi\sqrt{2|p|}}\left(-i\phi(p)e^{ipx} + i\phi^{+}(p)e^{-ipx}\right)e^{-\frac{1}{2}\alpha|p|}. \end{cases}$$

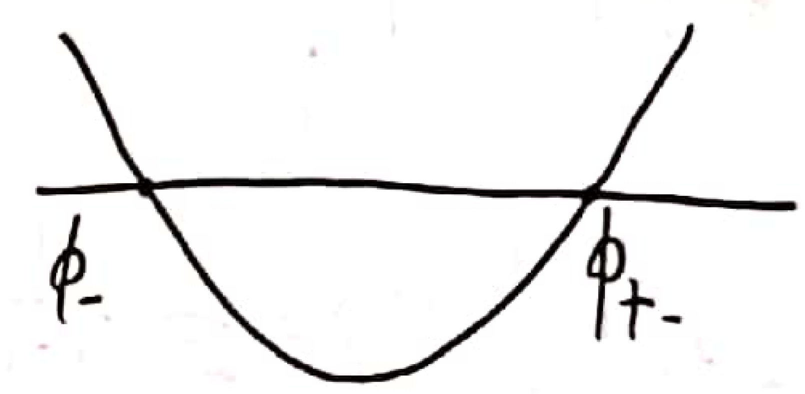
$px = px - |p|t.$

$$\begin{cases} [\phi(p)\ \ \phi^{+}(p')] = 2\pi\delta(p-p'). \\ [\phi(x)\ .\ \pi(y)] = \frac{i\alpha/\pi}{\alpha^2+(x-y)^2} = \delta(x-y). \end{cases}$$

回代 $H_B = \frac{1}{2}\int(\pi^2 + (\partial_x\phi)^2)\,dx = \int\frac{dp}{2\pi}|p|\,\phi^{+}(p)\,\phi(p)$

構造 right mover 與 left mover.

$$\begin{cases} \phi_{+}(x) = \frac{1}{2}\left(\phi(x) - \int_{-\infty}^{x}\pi(x')dx'\right) \\ \phi_{-}(x) = \frac{1}{2}\left(\phi(x) + \int_{-\infty}^{x}\pi(x')dx'\right) \end{cases} \sim \begin{cases} \phi_{+} + \phi_{-} = \phi. \\ \phi_{+} - \phi_{-} = -\int_{-\infty}^{x}\pi(x')dx. \end{cases}$$



$$\phi_{+} = \frac{1}{2}\int_{0}^{+\infty}\frac{dp}{2\pi\sqrt{2|p|}}\left(e^{ipx}\phi(p) + e^{-ipx}\phi^{+}(p)\right)e^{-\frac{1}{2}\alpha|p|}$$

$$\phi_{-} = \frac{1}{2}\int_{-\infty}^{0}\frac{dp}{2\pi\sqrt{2|p|}}\left(e^{ipx}\phi(p) + e^{-ipx}\phi^{+}(p)\right)e^{-\frac{1}{2}\alpha|p|}$$

$$\left(e^{i\pi\int_{-\infty}^{x}\rho(x')dx' + i\theta} \quad \tilde{\pi} \quad \tilde{\phi} \quad k_F x + \pi\int_{-\infty}^{x}\delta\rho(x')dx'. \right)$$

$$\phi = \phi_{+} + \phi_{-}$$

$$\psi(x) = e^{i\phi(x)}$$

$\psi(x)$有 $\begin{cases} \psi(x)\psi(y) + \psi(y)\psi(x) = 0. & \text{對易關係} \\ \langle\psi_{\pm}(x)\psi_{\pm}^{+}(0)\rangle = \frac{1}{2\pi}\frac{1}{\alpha \pm ix}. & \text{關聯函數.} \end{cases}$

$$e^{i\phi(x)}e^{i\phi(y)} + e^{i\phi(y)}e^{i\phi(x)} = 0.$$

考慮恒等式.

$$e^{A}e^{B} = \cancel{e^{A}e^{B}}\ e^{A+B+\frac{1}{2}[A,B]}.$$

$$\therefore e^{i\phi(x)+i\phi(y)-\underbrace{\frac{1}{2}[\phi(x),\phi(y)]}_{i\frac{\pi}{2}}} + e^{i\phi(x)+i\phi(y)+\frac{1}{2}[\phi(x),\phi(y)]} = 0.$$

即 $e^{i\theta+i\frac{\pi}{2}} + e^{i\theta-i\frac{\pi}{2}} = 0$

注. $[\phi(x), \phi(y)] = \pi\,\mathrm{sgn}(x-y).$ $(x \neq y)$.

以上計算、考慮 $x \neq y$ 且 x, y 相距甚遠