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The Two-Dimensional Ising Model

7.1 Ode to the Model

The two-dimensional Ising model is a necessary rite of passage in our transition from basics to hardcore topics. Like the harmonic oscillator in quantum mechanics, it is the easiest example of a completely solvable problem. By studying it one can learn many valuable lessons, notably about perturbative methods, the concept of duality, and the exact modeling of a phase transition.

Phase transitions will occupy much of this book, and we will return to study them in detail. For now, let us focus on the magnetic transition in the Ising model. The most detailed book on this subject is the one by B. McCoy and T. T. Wu [1].

On a square lattice with N columns and M rows, we define the model by

$$Z = \sum_{s_i} \exp \left[K \sum_{\langle i,j \rangle} s_i s_j \right], \quad (7.1)$$

where $K = J/kT$ and the symbol $\langle i,j \rangle$ means that sites i and j are nearest neighbors, as shown in Figure 7.1. There are many options at the edges: open boundary conditions in which the spins at the edges have no neighbors in one direction, periodic boundary conditions along one direction, which makes the system a cylinder, or along both directions, which makes it a torus. For now, let us just say that M and N and the number of sites $\mathcal{N} = MN$ are huge and we are nowhere near the ends. There are $2\mathcal{N}$ bonds on a square lattice with \mathcal{N} sites because each site has four bonds emanating from it, but each bond is counted twice, once at each of its end points.

Consider the extreme limits. As $K = J/kT \rightarrow \infty$ or $T \rightarrow 0$, the spins will be all up or all down, the system will be magnetized, and $\langle M \rangle$, the average spin per site, will be at its maximum of ± 1 . Let us pick $\langle M \rangle = +1$. As $K \rightarrow 0$ or $T \rightarrow \infty$, the Boltzmann weight will be 1 for all configurations, the spins will fluctuate independently, and $\langle M \rangle$ will vanish. The graph of $\langle M(T) \rangle$ will thus start out at $+1$ and decrease as we heat the system. It should be zero at $T = \infty$. One possibility is that it does not reach zero until we reach $T = \infty$. If, however, it vanishes at some finite $T = T_c$ and remains zero thereafter, we have a phase transition. There must be a singularity at T_c , since a non-trivial analytic function cannot

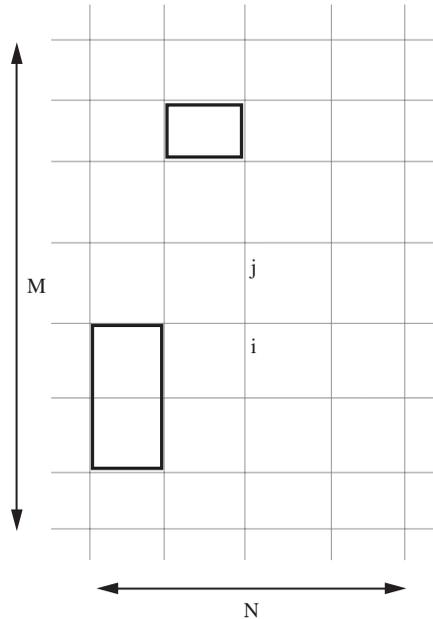


Figure 7.1 The square lattice with M rows and N columns. The site j is a nearest neighbor of i ; there are three more. The solid rectangles correspond to terms at fourth and sixth orders in the $\tanh K$ (high- T) expansion.

identically vanish beyond some point. This singularity or non-analyticity is the signature of a phase transition. The free energy and its derivatives will also be singular at this point.

If we are to see this singular behavior analytically, we must go to the thermodynamic limit. Let us understand why. For any finite system, Z is a finite sum over positive terms, and its logarithm (the free energy F) will be analytic at any real T or K . But, Z could vanish arbitrarily close to the real axis, and in the limit of infinite system size these zeros could pinch the real axis, producing a singularity. However, we cannot simply compute F in this limit since it is extensive in system size like the energy, and will not approach a limit. However, under normal conditions f , the *free energy per site* will have a limit and this is what we are after. In the $d = 1$ Ising model we were able to obtain f , but it did not exhibit a finite- T phase transition; the system was unmagnetized at all $T > 0$. The $d = 2$ case is the celebrated example with a finite- T transition, displayed rigorously in Onsager's solution. Prior to Onsager it was not universally accepted that the innocuous sum over Boltzmann weights could reproduce a phenomenon as complex as a phase transition.

Before we plunge into the exact solution, let us learn some approximate methods that work in generic situations.

7.2 High-Temperature Expansion

This is a perturbative expansion around the point $K = 0$ when the Boltzmann weight is unity for all states, the spins do not talk to each other, and $Z = 2^{\mathcal{N}}$. For any one bond we may write

$$e^{Ks_i s_j} = \cosh K + s_i s_j \sinh K = \cosh K (1 + s_i s_j \tanh K), \quad (7.2)$$

a result that follows from $(s_i s_j)^2 = 1$. So

$$Z(K) = \sum_{s_i} \prod_{\text{bond } ij} \cosh K (1 + s_i s_j \tanh K). \quad (7.3)$$

Each bond can contribute either a 1 or a $\tanh K$, and there are $2^{\mathcal{N}}$ terms in the product over bonds. The leading term in the $\tanh K$ expansion has a 1 from every bond and contributes $2^{\mathcal{N}} (\cosh K)^{2\mathcal{N}}$. The sum over 1 at each site gives us the $2^{\mathcal{N}}$, while the product of $\cosh K$ over the $2\mathcal{N}$ bonds gives the rest. The next term has a $\tanh K$ from one bond and a 1 from the others. There are $2\mathcal{N}$ such terms. These $2\mathcal{N}$ terms do not survive the sum over s_i since $\sum_{s=\pm 1} s = 0$ and we have two such free or dangling spins at each end. To order $\tanh^2 K$, we still get nothing: if the bonds share no sites, we have four spin sums that vanish, and if they do share a site we have a spin sum over the other two that vanish. The first non-zero contribution is at order $\tanh^4 K$, when we pick four bonds that form a square, as shown by the dark square in Figure 7.1. Now the spins at each corner appear twice (since there are two bonds from the square incident at each site), and we now get a contribution $2^{\mathcal{N}} (\cosh K)^{2\mathcal{N}} \cdot \mathcal{N} \tanh^4 K$ to $Z(K)$. The factor of \mathcal{N} comes from the number of squares we can have, and this equals \mathcal{N} because we can label each square by the site at its lower left-hand corner. The series so far looks as follows:

$$\frac{Z(K)}{2^{\mathcal{N}} (\cosh K)^{2\mathcal{N}}} = 1 + \mathcal{N} \tanh^4 K + 2\mathcal{N} \tanh^6 K + \dots, \quad (7.4)$$

where the derivation of the $\tanh^6 K$ term is left to Exercise 7.2.1. We expect this series to work for small K .

Exercise 7.2.1 *Derive the sixth-order term in Eq. (7.4).*

The high-temperature series takes the form

$$\frac{Z(K)}{2^{\mathcal{N}} (\cosh K)^{2\mathcal{N}}} = \sum_{\text{closed loops}} C(L) \tanh^L K, \quad (7.5)$$

where $C(L)$ is the number of closed loops of length L we can draw on the lattice without covering any bond more than once.

The free energy per site is, to this order,

$$\begin{aligned} -\frac{f}{kT} &= \frac{1}{\mathcal{N}} \ln Z = \ln \left[2 \cosh^2 K \right] + \frac{1}{\mathcal{N}} \ln(1 + \mathcal{N} \tanh^4 K + \dots) \\ &= \ln \left[2 \cosh^2 K \right] + \tanh^4 K + \dots \end{aligned} \quad (7.6)$$

using $\ln(1+x) = x + \dots$. It is significant but not obvious that as we go to higher orders, we will keep getting a limit for f that is independent of \mathcal{N} . For example, if you consider the case of two disjoint elementary squares that contribute with factor $\tanh^8 K$, there will be $\mathcal{N}(\mathcal{N}-5)/2$ of them since the two squares cannot share an edge or be on top of each other. In addition, there are single loops with perimeter 8. Upon taking the logarithm, the $\mathcal{N}^2/2$ part of this cancels against the square of the \mathcal{N} term due to the elementary square when $\ln(1+x) = x - \frac{1}{2}x^2 + \dots$ is expanded. For more practice, do Exercise 7.2.2.

Exercise 7.2.2 Show that to order $\tanh^8 K$,

$$-\frac{f}{kT} = \ln \left[2 \cosh^2 K \right] + \tanh^4 K + 2 \tanh^6 K + \frac{9}{2} \tanh^8 K + \dots \quad (7.7)$$

7.3 Low-Temperature Expansion

Consider now the regime near $T = 0$ or $K = \infty$. The spins will tend to be aligned in one direction, say up. The Boltzmann weight is e^K on each of the $2\mathcal{N}$ bonds. We say the bonds are all unbroken. If a spin is now flipped down, the four bonds linking it to its four nearest neighbors will be broken and the Boltzmann factor will be reduced by e^{-8K} . Since the spin flip can occur in any of \mathcal{N} sites, we have

$$Z = e^{2\mathcal{N}K} (1 + \mathcal{N}e^{-8K} + \dots). \quad (7.8)$$

Let us now consider flipping two spins. The energy cost depends on their relative locations. It is lowest if they are neighbors: the bond connecting them is unbroken, but the six other bonds linking them to all other neighbors will be broken. There are \mathcal{N} ways to pick the first spin and two ways to pick the second: to its north or east. (Positions to the south and west are not needed since we will then be double counting.) Thus we have

$$\frac{Z}{e^{2\mathcal{N}K}} = (1 + \mathcal{N}e^{-8K} + 2\mathcal{N}e^{-12K} + \dots). \quad (7.9)$$

We expect this series to work for small e^{-2K} or large K .

The obvious way to represent these configurations in the low-temperature expansion is to show flipped spins, as in Figure 7.2. A cleverer way due to Kramers and Wannier [2] is to surround the flipped spins by a contour that is made up of perpendicular bisectors of the broken bonds. The dotted lines in the figure correspond to the dual lattice.

If we create an island of spins pointing opposite to the majority, it costs an energy proportional to the perimeter of the island, as reflected in the Boltzmann factor e^{-2KL} .

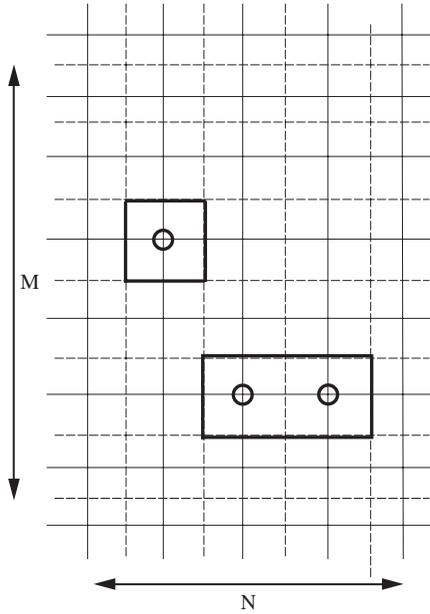


Figure 7.2 The low-temperature expansion in which only spins flipped relative to a fully aligned state are shown as tiny circles. These are then surrounded by bonds on a dual lattice of dotted lines. Note that the two such closed figures are topologically identical in shape and multiplicity to the leading ones in the high-temperature $\tanh K$ expansion.

Compare this to the $d = 1$ case where the cost of an island is just e^{-2K} regardless of its size. (The “perimeter” of this island is made of just the two end points.) This is why the $d = 1$ system loses its magnetization at any non-zero T . In $d = 2$, we can estimate the critical temperature by asking when large islands will go unsuppressed. Imagine laying out a loop of length L . At each stage we can move in three directions, since going back is not an option because each bond can be covered only once. Ignoring the condition that we end up where we began, and that we cannot run into other loops starting in other places and so on, we roughly get a factor $3^L e^{-2KL}$, so that loops of arbitrarily large size are no longer suppressed when we reach

$$e^{(-2K_c + \ln 3)L} = 1, \text{ or } K = K_c = 0.5493, \quad (7.10)$$

which you can compare to the exact result $K_c = 0.4407$ (to four places).

We can also do a similar analysis for the high-temperature series to estimate K_c :

$$(\tanh K_c)^L \cdot 3^L \simeq 1 \quad (7.11)$$

$$\tanh K_c = \frac{1}{3}, \text{ or } K = K_c = 0.3466. \quad (7.12)$$

The correct answer is seen to lie between these two estimates.

7.4 Kramer–Wannier Duality

Let us now note that the low- T expansion resembles the high- T expansion on the *dual lattice*: the lattice whose edges are the perpendicular bisectors of the original ones and whose sites are located at the center of each square element (plaquette) in the original lattice. You will agree that up to the order considered, the diagrams for the high- and low-temperature series have the same shapes (unit squares, 2×1 rectangles, etc.) and multiplicity (\mathcal{N} , $2\mathcal{N}$, etc.).

They do not, however, have the same weights. So we do the following: Since K is a dummy variable in Eq. (7.9), let us replace it by K^* to obtain

$$\frac{Z(K^*)}{e^{2\mathcal{N}K^*}} = (1 + \mathcal{N}e^{-8K^*} + 2\mathcal{N}e^{-12K^*} + \dots). \quad (7.13)$$

So far, K^* is just a dummy variable. Let us now choose, for each K , a *dual temperature* $K^*(K)$ such that

$$e^{-2K^*(K)} = \tanh K. \quad (7.14)$$

Now the two series in Eqs. (7.4) and (7.13) agree numerically to the order shown. It can be shown that the agreement is good to all orders, implying the *self-duality* relation

$$\frac{Z(K)}{2^{\mathcal{N}} (\cosh K)^{2\mathcal{N}}} = \frac{Z(K^*)}{e^{2\mathcal{N}K^*}}. \quad (7.15)$$

Using

$$\sinh 2K \cdot \sinh 2K^* = 1, \quad (7.16)$$

one can rewrite Eq. (7.15) more symmetrically as

$$\frac{Z(K)}{(\sinh 2K)^{\frac{\mathcal{N}}{2}}} = \frac{Z(K^*)}{(\sinh 2K^*)^{\frac{\mathcal{N}}{2}}}. \quad (7.17)$$

Exercise 7.4.1 Prove Eqs. (7.16) and (7.17).

If K is small, then in order to satisfy Eq. (7.14) $K^*(K)$ has to be large, which is fine, as the low-temperature expansion works for large values of its argument. What is remarkable is that the thermodynamics of the model at low and high energies are related even though the physical properties are very different: one side has magnetization and one does not. *Self-duality*, relating the model at weak coupling (at a small value of a parameter, K in our example) to the *same* model at strong coupling (large K values) is quite rare. It is more common to encounter simply *duality*, in which one model at strong coupling is related to *another* model at weak coupling.

Recall from Chapter 1 that the inverse relation of Eq.(7.14) is

$$e^{-2K} = \tanh K^*. \quad (7.18)$$

In other words, K^* as a function of K coincides with K as a function of K^* . Consequently, the dual of the dual is the original K :

$$(K^*)^* = K. \quad (7.19)$$

(A trivial example of a function that is its own inverse is $y = \frac{1}{x}$, which implies $x = \frac{1}{y}$. The relation between K and K^* is, of course, much more interesting.)

Kramers and Wannier used duality to find the critical temperature of the Ising model as follows: Equation (7.17) implies that in the thermodynamic limit any singularity at some K , such as at a phase transition, implies one at $K^*(K)$. If, however, we assume that there is just one transition, it must occur at a critical value K_c that is its own dual:

$$K_c^* = K_c, \text{ or } e^{-2K_c} = \tanh K_c, \text{ or } e^{-2K_c} = \sqrt{2} - 1, \text{ or } K_c = 0.4407\dots \quad (7.20)$$

But bear in mind that there can be, and there *are*, problems where there are two phase transitions at critical points related by duality, with nothing interesting going on at the self-dual point itself.

We conclude this section with some remarks on the *anisotropic Ising model* with couplings K_x and K_τ in the two directions. We label the second direction by τ rather than y since it will play the role of imaginary time when we use the transfer matrix. By comparing high- and low- T expansions we find that the dual couplings K_x^d and K_τ^d are given by

$$e^{-2K_x^d} = \tanh K_\tau, \quad e^{-2K_\tau^d} = \tanh K_x. \quad (7.21)$$

Now recall that we have defined the dual X^* of any real number by the symmetric relations

$$e^{-2X^*} = \tanh X, \quad e^{-2X} = \tanh X^*. \quad (7.22)$$

Let us use this to trade the tanh's in Eq. (7.21) for exponentials of the duals to obtain exponentials on both sides:

$$e^{-2K_x^d} = e^{-2K_\tau^*}, \quad e^{-2K_\tau^d} = e^{-2K_x^*}, \quad (7.23)$$

which allows us to read off the coordinates dual to any K_x and K_τ :

$$K_x^d = K_\tau^*, \quad K_\tau^d = K_x^*. \quad (7.24)$$

To summarize, duality maps points in the (K_x, K_τ) plane as follows:

$$(K_x, K_\tau) \xrightarrow{\text{duality}} (K_\tau^*, K_x^*). \quad (7.25)$$

The self-dual points obey

$$(K_x, K_\tau) = (K_\tau^*, K_x^*). \quad (7.26)$$

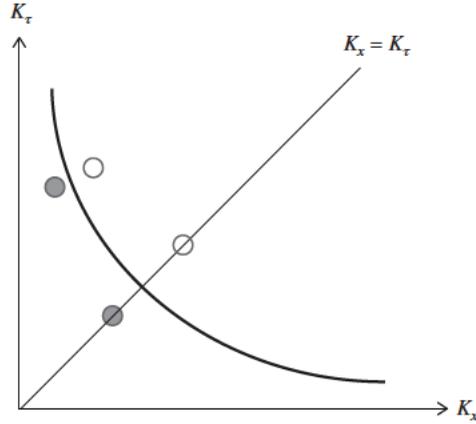


Figure 7.3 The situation in the K_x - K_τ plane. The isotropic points lie on the 45° line. The critical points lie on $\sinh 2K_x \sinh 2K_\tau = 1$. Also shown are a pair of points (solid circles) and their duals (open circles).

The condition obtained by matching the first coordinates,

$$K_x = K_\tau^*, \quad (7.27)$$

is the same as we get from equating the second coordinates, as can be seen by taking the dual of both sides. The self-dual points lie on a one-dimensional curve,

$$\sinh 2K_x \sinh 2K_\tau = 1. \quad (7.28)$$

The system is critical on this whole line and the isotropic point lies on the 45° line at $\sinh^2 2K_c = 1$ or $K_c = 0.4407\dots$

For later use, note that no matter how small or large K_τ is, there is a critical value of K_x that goes with it, and vice versa. Figure 7.3 illustrates the situation in the K_x - K_τ plane.

Exercise 7.4.2 Establish Eq. (7.21) by comparing high- and low- T expansions. Sketch the locus of self-dual points. Find the duals to a few points with (K_x, K_τ) both small, both large, and one large and one small. Show that the self-dual points obey $\sinh 2K_x \sinh 2K_\tau = 1$.

7.5 Correlation Function in the tanh Expansion

Consider the thermal average $\langle s_i s_f \rangle$ between spins at some “initial” site i and “final” site f , named that way for a good reason. Let us employ the high-temperature tanh expansion. By definition,

$$\langle s_i s_f \rangle = \frac{\sum_s s_i s_f \prod_{\text{bonds}} (1 + s_m s_n \tanh K)}{\sum_s \prod_{\text{bonds}} (1 + s_m s_n \tanh K)}, \quad (7.29)$$

where m and n are neighbors connected by a bond, and common factors have been canceled between numerator and denominator. The first non-zero term in the numerator occurs when we have bonds starting at i and ending at f , yielding a product $s_i s_{i+1} s_{i+1} s_{i+2} \cdots s_{f-1} s_f s_f$, where $i+1$ is the neighbor to i . Now there are no free Ising spins that can be summed over to give zero. In the simple case where the points lie on the same axis, this product will occur with a factor $(\tanh K)^{|i-f|}$, where $|i-f|$ is the distance between the end points. The first non-zero term in the denominator is just 1, so that

$$\langle s_i s_f \rangle = (\tanh K)^{|i-f|} (1 + \cdots), \quad (7.30)$$

where the ellipses denote higher-order corrections from longer paths joining i and f , as well as contributions from closed paths in the numerator and denominator. If the points are not along the same axis, $|i-j|$ will be simply the Manhattan distance and there will be many paths of the shortest length; the 1 in the brackets above will be replaced by this multiplicity. This most important dependence is, of course, in the exponential prefactor $\exp(\ln \tanh K |i-f|)$.

Exercise 7.5.1 Use the $\tanh K$ expansion for the N -site $d = 1$ Ising model with periodic boundary conditions to compute Z as well as $\langle s_i s_f \rangle$.

References and Further Reading

- [1] B. McCoy and T. T. Wu, *The Two-Dimensional Ising Model*, Harvard University Press (1973).
- [2] H. A. Kramers and G. H. Wannier, *Phys. Rev.* **60**, 252 (1941).