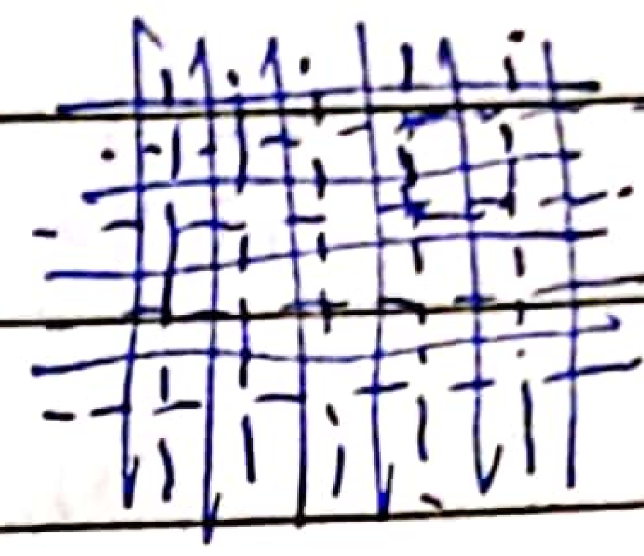
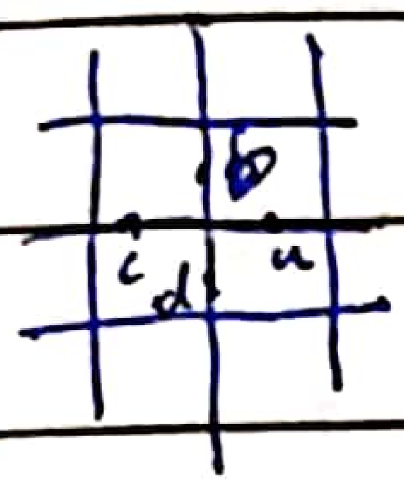
$\delta\left[\sum_\mu (l_\mu(r) - l_\mu(r-\mu))\right]$ $\quad // \; i\sum_r \theta(r) \sum_\mu (l_\mu(r) - l_\mu(r-\mu))$

$$\sum e^{-\sum_{r\mu}\frac{l_\mu^2(r)}{2\beta J}} \int D\theta(r)\, e^{\sum_{r\mu} i\, l_\mu(r)(\theta(r)-\theta(r+\mu))}$$

$$i\sum_r (l_\mu(r) - l_\mu(r-\mu))\,\theta(r)$$

$$\underline{\delta\left[\sum_\mu (l_\mu(r)-l_\mu(r-\mu))\right]}$$





$a-c+b-d=0$

~~无散条件~~. 定义 $\vec l(r) = (l_x(r), l_y(r)) = (\nabla\times\vec n)$

$l_x = \partial_y n \quad l_y = -\partial_x n \qquad \vec n = (0, 0, n) \; // \; \hat z$

$l_x = n(r) - n(r-y) \quad l_y = -n(r) + n(r-x)$

无源条件. $\quad l_x(r) - l_x(r-x) + l_y(r) - l_y(r-x) = 0$

dual lattice 上 $l_\mu(r) \in \mathbb{Z}$. 无散 $\vec l = \nabla\times\vec n$ 是无源的 ~~$\nabla\times\vec l$~~ $\Rightarrow \nabla\cdot\vec l = 0$

解: $\begin{cases} l_x(r) = n(r) - n(r-y) \\ l_y(r) = -n(r) + n(r-x) \end{cases}$ 就除去积分之后

$$Z = \int D\theta\, e^{\beta J\cos(\theta_i-\theta_j)} = \sum_{l_{ij}} e^{-\frac{1}{2\beta J} l_{ij}^2}\, \underbrace{\delta(\cdots)}_{\text{无散条件}}$$

$$= \sum \sum_{\{n(r)\}} e^{-\frac{1}{2\beta J}(n(r)-n(r-\mu))^2}$$

由泊 Poisson 求和

4) 在 Honeycomb lattice 中 的 BKT 相变

对偶, 变换 $\qquad$ ref: wegner duality in generalized Ising Model

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$$Z = \sum_{\{n(r)\}} e^{-\sum_{r\mu} \frac{1}{2\beta J}(n(r)-n(r-\mu))^2}$$

再利用一次 Villian Poission 公式.

记 $\Delta_\mu n(r) = n(r) - n(r-\mu)$ 将其看成离散的 $\Delta_\mu \phi = \phi(r) - \phi(r-\mu)$

反推得

$$\sum_{l} \int e^{-\sum_{r\mu} \frac{1}{2\beta J}(\Delta_\mu \phi(r))^2 + 2\pi i l \phi_\mu(r)} d\phi_\mu$$

$$= \sum_{\Delta_\mu n(r)} \int d\phi_\mu e^{-\sum_{\mu r} \frac{1}{2\beta J}(\Delta_\mu \phi(r))^2 + 2\pi i \Delta_\mu n(r) \phi_\mu(r)}$$

这一步[illegible]了. [illegible]

$$Z = \sum_{l_\mu(r)} \left(\frac{1}{\sqrt{2\pi\beta J}}\right)^{N_s} e^{\beta J N_s} \prod_{\{l_\mu(r)\}} e^{-\frac{1}{2\beta J} l_\mu^2(r)} \, \delta\left(\sum_\mu (l_\mu(r) - l_\mu(r-\mu))\right)$$

第一步 求和 $l_\mu(r)$ 之和. 因为δ函数而有 $\delta\left(\sum_\mu (l_\mu(r) - l_\mu(r-\mu))\right)$ 有 $l_1 - l_3 + l_2 - l_4 = 0$

定义 $\vec{l}(r) = (l_x(r), l_y(r))$ 则 $\nabla \cdot \vec{l} = 0$ $\quad \vec{l} = \nabla \times n(r)$ 完全确定

在离散化. $\quad l_x(r) = n(r) - n(r-y) \quad l_y(r) = -n(r) + n(r-x)$

则 $\sum_\mu l_\mu(r) - l_\mu(r-\mu) = 0$ 自动满足

$$Z = \mathrm{Tr}\left(e^{-\frac{1}{2\beta J}\sum_{r\mu}(n(r) - n(r-\mu))^2}\right) \quad n(r) \in \text{所有整数}$$

$\Downarrow$

离散的导数

$\Delta_\mu n(r) = n(r) - n(r-\mu)$ 连续极限下: $\Delta_\mu \to \partial_\mu$

第二步 再利用 Villian Possion 公式 Sine-Gordon field

(整体上 自旋 $\longrightarrow$ n 离散 $\longrightarrow$ φ 连续)

$$Z = \int d\phi \sum_m e^{-\frac{1}{2\beta J}(\Delta_\mu \phi(r))^2 + 2\pi i m\phi} \qquad \sum_m e^{2\pi i m\phi} = \sum_n \delta(\phi - n)$$

$$= \sum_n \int d\phi\, e^{-\frac{1}{2\beta J}(\Delta_\mu\phi)^2} \delta(\phi - n)$$

$$= \sum_n e^{-\frac{1}{2\beta J}(\Delta_\mu n)^2} = \text{原始}$$

反过来 $\sum \int d\phi\, e^{-\frac{1}{2\beta J}(\Delta_\mu\phi)^2 + 2\pi i m\phi}$ 二次型积分可算

$$\sum_{\{m\}} \int d\phi\, e^{-\frac{1}{2\beta J}(\Delta_\mu\phi)^2 + 2\pi i \phi m} \qquad \begin{array}{l} m \to \Delta_\mu n(r) \\ \phi \to \phi_\mu(r) \end{array}$$

下面来计算: $Z = e^{-\sum_{r\mu}\frac{1}{2\beta J}(\Delta_\mu n(r))^2}$

$$\int d\phi_\mu(r)\, e^{-\frac{1}{2\beta J}\sum_{r\mu}(\Delta_\mu \phi_\mu(r))^2 + 2\pi i \Delta_\mu n(r)\, \phi_\mu(r)}$$

二次型积分公式: $\int D\phi\, e^{\int -k(\partial\phi)^2 + J(x)\phi(x)} = \text{const} \cdot e^{-\int J(x) G(x-y) J(y)\, dx\, dy}$

(离散版 $\int Dx\, e^{-A_{ij}x_i x_j + J_i x_i} = \int Dx\, e^{-x^T A x + J x} = \text{const} \cdot e^{-J A^{-1} J/4}$)

$$Z = Z_{sw} \sum_{\{m(r)\}} e^{-2\pi^2 \beta J \sum_{r,r'} m(r) G(r-r') m(r')}$$

$$G(r-r') = \int_{-\pi}^{\pi} \frac{dk_x dk_y}{(2\pi)^2} \frac{e^{ik(r-r')}}{4-2\cos k_x - \cos k_y}$$

$$\frac{e^{ik(r-r')}}{4-2\cos k_x - \cos k_y} \approx \frac{e^{ik(r-r')}}{4-2-2+k_x^2+k_y^2} = \frac{e^{ik(r-r')}}{k_x^2+k_y^2}, \quad \text{即} \int \frac{dk_x dk_y}{(2\pi)^2} \frac{e^{ik(r-r')}}{k_x^2+k_y^2}$$

即求 $\frac{1}{k^2}$ 的Fourier变换，即 $\frac{1}{k^2}$ 是 $k^2 f(k) = 1$ 的解，其Fourier变换为

$\nabla^2 g(r) = \delta^2(r)$ 的解，即 $\ln|\vec{r}-\vec{r}'|$

得到 $G(r-r') = \underbrace{-\frac{1}{2\pi}\ln\left(\frac{|\vec{r}-\vec{r}'|}{a}\right) - \frac{1}{4}}_{G'(r-r')} + G(0)$

代入得 $Z = Z_{sw} \sum_{\{m(r)\}} e^{-2\pi^2 \beta J \sum_{r,r'} m(r) m(r') (G(0) + G'(r-r'))}$

$$= Z_{sw} \sum_{\{m(r)\}} e^{-2\pi^2 \beta J \left[\sum_{r,r'} G(0) m(r) m(r') + \sum_{r,r'} G'(r-r') m(r) m(r')\right]}$$

① $e^{-2\pi^2 \beta J \left|\sum_r m(r)\right|^2 G(0)}$ ∴只有 $\sum m(r) = 0$ 的才有贡献，

即电中性条件

② $\sum_{r r'} G'(r-r') m(r) m(r') = -\frac{1}{4} \sum_{rr'} m(r) m(r') - \frac{1}{2\pi} \sum_{rr'} m(r) m(r') \ln\left|\frac{r-r'}{a}\right|$

$\sum_{rr'} m(r) m(r') = \left(\sum_r m(r)\right)^2 = ?$ 　　 Z一[illegible]项[illegible]

$$Z = Z_{sw} \sum_{m(r)} \exp\left[\ln y \sum m^2(r) - \pi \beta J \sum_{r} m(r) m(r') \ln\left|\frac{r-r'}{a}\right|\right]$$

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$$Z = Z_{SW} \sum_{\{m(r)\}} e^{-2\pi^2 \beta J \sum_{rr'} m(r) G(r-r') m(r')}$$

$$G(r-r') = \int_{-\pi}^{\pi} \frac{dk_x dk_y}{(2\pi)^2} \frac{e^{ik(r-r')}}{4-2\cos k_x - 2\cos k_y}$$

$$\simeq -\frac{1}{2\pi} \ln\left(\frac{|r-r'|}{a}\right) - \frac{1}{4} + G(0) \qquad \left([illegible],\ \nabla^2 G = \delta(r) \quad [illegible] \quad \to \frac{1}{2\pi}\ln|r| \right)$$

$$G(r-r') = \underbrace{-\frac{1}{2\pi}\ln\left|\frac{r-r'}{a}\right| - \frac{1}{4}}_{G'(r-r')} + G(0)$$

$$Z = Z_{SW} \sum_{\{m(r)\}} e^{-2\pi^2\beta J\left[\underline{\sum_{rr'} G(0)\, m(r) m(r')} + \sum_{rr'} G'(r-r') m(r) m(r')\right]}$$

$$\to G(0)\underbrace{\left[\sum_r m(r)\right]^2}_{\text{total charge}}$$

$$e^{-2\pi^2\beta J\, G(0) \left|\sum_r m(r)\right|^2}$$ [illegible] $\sum_r m(r) = 0$ [illegible] $E_v \sim \ln\frac{L}{a}$

$$\sum_{rr'} G'(r-r') m(r) m(r') = -\frac{1}{4}\underline{\sum_{rr'} m(r) m(r')} - \frac{1}{2\pi}\sum_{rr'} m(r) m(r') \ln\left|\frac{r-r'}{a}\right|$$

$$\left(\sum_r m(r)\right)^2 = 0 \iff \sum_r m^2(r) + \sum_{r \neq r'} m(r) m(r') = 0$$

$$= \frac{1}{4}\sum_r m^2(r) - \frac{1}{2\pi}\sum_{rr'} m(r) m(r') \ln\left|\frac{r-r'}{a}\right|$$

$$Z = Z_{SW} \sum_{m(r)} \exp\left[\ln y \sum_r m^2(r) - \pi\beta J \sum_r m(r) m(r') \ln\left|\frac{r-r'}{a}\right|\right]$$

$$y = e^{\beta\mu} \iff \ln y = \beta\mu \qquad \mu = -\frac{\pi^2}{2} J$$

(BKT [illegible] $\beta J = \frac{2}{\pi}$, $\mu = -\frac{\pi^2}{2} J$, $\beta\mu = -\frac{\pi^2}{2}\beta J$ ...)

第七步. Sine-Gordon eq 与 [illegible]

(平均) $$Z = \int d\phi \sum_{m(r)} e^{-\frac{1}{2\beta J}(\Delta_\mu \phi)^2 + 2\pi i m \phi + \ln y\, m^2(r)}$$

$m(r) = 0, \pm 1$ [illegible]

([illegible] BKT [illegible])

YCP

$$1 + e^{\ln y + 2\pi i\phi} + e^{\ln y - 2\pi i\phi} = 1 + 2y\cos(2\pi\phi) \simeq e^{2y\cos(2\pi\phi)}$$

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$\sinh(-K) = -\sinh(K)$

$\because$: $\cosh(K) = \cosh(-K)$

$$Z = \mathrm{Tr}(e^{-\beta H}) = \sum_{\{i,j\}} \prod_{\langle i,j \rangle} e^{K S_i S_j} = \sum_{\{i,j\}} \prod_{\langle i,j \rangle} \left\{ \cosh(K) + \sinh(K) S_i S_j \right\}$$

$$= \cosh^{N_0}(K) \sum_{\{i,j\}} \prod_{\langle i,j \rangle} \left\{ 1 + \tanh(K) S_i S_j \right\}$$

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$$Z = \int d\phi\, e^{-\frac{1}{2}\beta J (\partial_\mu \phi)^2 + 2y \cos 2\pi\phi}$$

$$\Rightarrow \int d\phi\, e^{-\int \frac{1}{2}(\nabla\phi)^2 - 2y\cos(2\pi\sqrt{\beta J}\,\phi)}$$

(临界点: $1 = \frac{n^2}{8\pi} = \frac{4\pi^2 \beta J}{8\pi} = \frac{\pi}{2}\beta J$

$\Leftrightarrow$ $k_B T_c = \frac{\pi}{2} J$ X (o Wen)

Kramers - Wannier 对偶. PRB. 60, 252 / PRB 60. 263 (对偶)

Onsager , 对偶 PRB, 66 117

$H = -J\sum_{\langle i,j\rangle} S_i S_j$ $K = \beta J$ $K_c = 0.44067 = \frac{1}{2}\ln(1+\sqrt{2})$

$$Z = \mathrm{Tr}\, e^{-\beta H} = \sum_{\{i,j\}} \prod e^{K S_i S_j} = \sum_{\{i,j\}} \prod \{\cosh(K) + \sinh(K) S_i S_j\}$$

N_b bond 个数
N_s site 个数

$$= \cosh(K)^{N} \prod \{1 + \tanh K\, S_i S_j\}$$

$\tanh K$

$K = \beta J$

展开 $(1+x_1)(1+x_2)(1+x_3)\cdots$

$= 1 + x_1 + x_2 + x_3 + \cdots + x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots + x_1 x_2 x_3 + \cdots$

$\sum S_1 S_2 = 0$ $\sum S_1 S_2 S_3 S_4 = 0$ 闭合.

~~$S_1 S_2 S_{23} S_{34} S_4$~~ $S_1 S_2 S_2 S_3 S_3 S_4 S_4 S_1 = 1$

只有闭合回路 $\neq 0$

$Z = \text{const} \sum_{\text{loop}} \left(\bigcirc \right)$ ← 激发 { loop excitation / loop expansion

$Z = \sum_i e^{-\beta E_i} = e^{-\beta E_0}\left(\sum e^{-\beta \cdots}\right)$

$\Downarrow$

2^{N_s} ~~const~~ $\cosh^{N_b}(K)$

$f(x) = \sum_n c_n x^n$

$$Z = \sum_{\{S_i\}} \prod e^{K S_i S_j} = 2^{N_s} \cosh^{N_b}(K)\, f(\tanh(K))$$

关联 $\langle S_i S_j \rangle$

$\sim \sum_l c_l \tanh^l(K)$

$\tanh^l K \sim e^{-l \ln \tanh(K)}$