2020.1.6

BKT相变是一套完整的理论

- 半经典
- Wilson RG: $\theta = \theta_< + \theta_>$ (1971)
- Real space RG: Koskolitz, Thouless (1971)
- Duality: 起源是 Kramers-Wannier ~~Wilson~~ 对 Ising model 的工作 (1944) (Onsager, 1944, $\beta J = \frac{1}{2}\ln(1+\sqrt{2}) \simeq 0.44067$) Transfer matrix; Yang, $\nu = 1/8 \neq \frac{1}{2}$

高T/低T展开 可能最早是来自于 Kramers-Wilson duality.
(不具体讲)

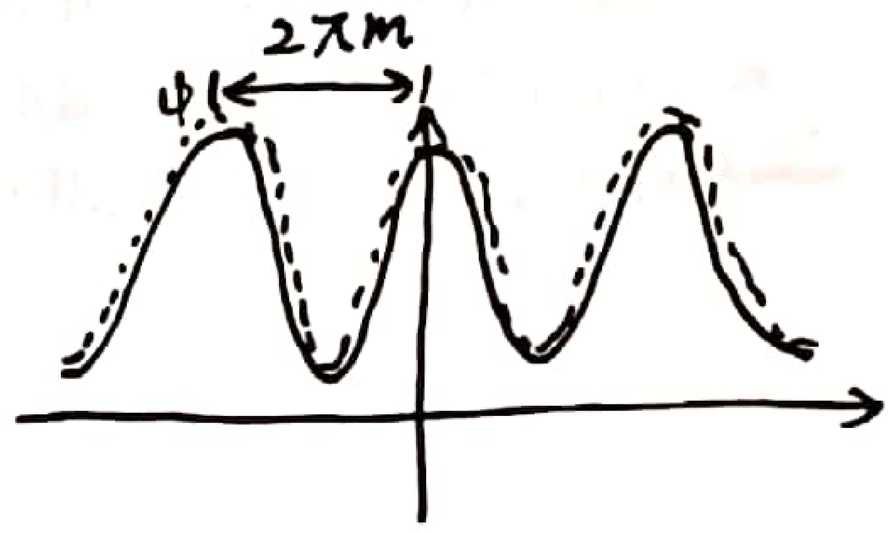
本节内容: 1. BKT相变的 duality; 2. Ising model 的 duality

B

BKT相变的 duality

最重要的是: Villiam transformation + Possion求和

$$\boxed{e^{k\cos(x)} \simeq \frac{1}{\sqrt{2\pi k}} \sum_{q=-\infty}^{+\infty} e^{k - \frac{q^2}{2k} + iqx}}$$



连续场 $\longrightarrow$ 离散场 (如 Ising model $S_i = \pm 1$)

类似

$$e^{-(A)x^2} \overset{FT}{\longleftrightarrow} e^{-\frac{q^2}{A}}$$

回顾:

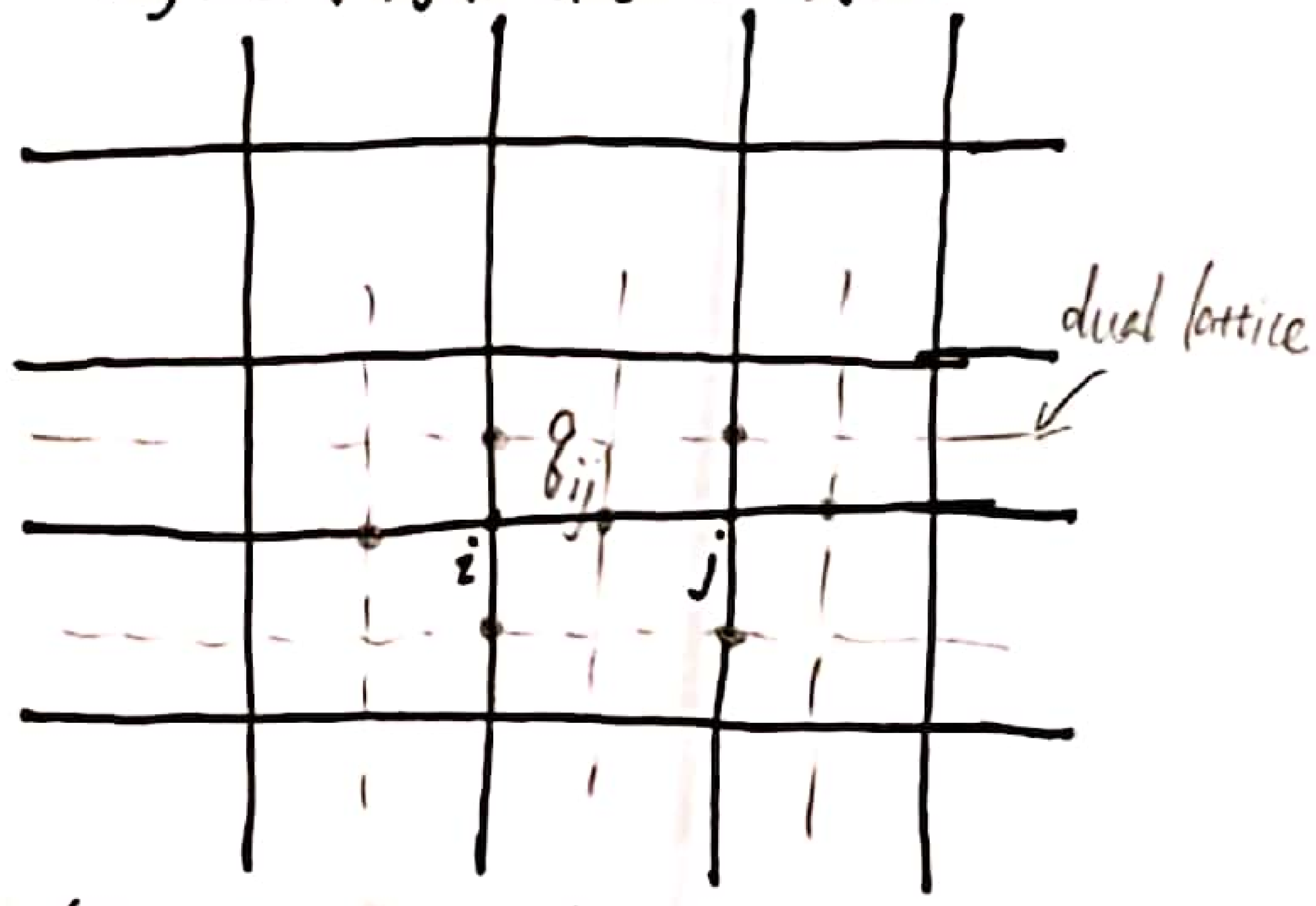
<u>第一步</u> $Z = \int D\theta \; e^{\beta J \sum_{ij} \cos(\theta_i - \theta_j)}$

*作业: 请求解 Honeycomb lattice 中的 BKT相变

强调: 画 lattice 图

Ref: Wegner, duality in generalized Ising models. (对偶图)

q_{ij} 是对偶 lattice 上的顶点



$(\theta_i - \theta_j) \longleftrightarrow q_{ij}$

l_{ij} 与 $\theta_i - \theta_j$ 之间1-1对应, 使标记明确.

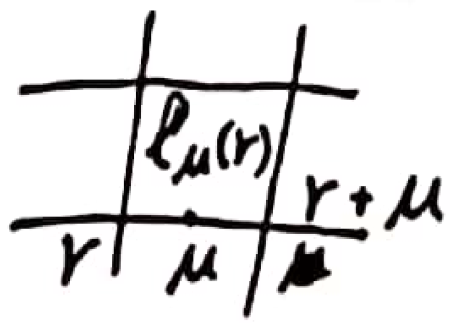
(为与书上保持一致, $q_{ij} \to l_{ij}$)

<u>第二步</u>

$$e^{\beta J \cos(\theta_i - \theta_j)} = \frac{1}{\sqrt{2\pi\beta J}} \sum_{l_{ij}} e^{\beta J - \frac{l_{ij}^2}{2\beta J} + i l_{ij}(\theta_i - \theta_j)} \qquad \left(\int d\theta \; e^{ik\theta} = \delta(k)\right)$$

<u>第三步</u> 重新标记符号

$\theta_i \longrightarrow \theta(\vec{r}) = \theta(r)$



相应

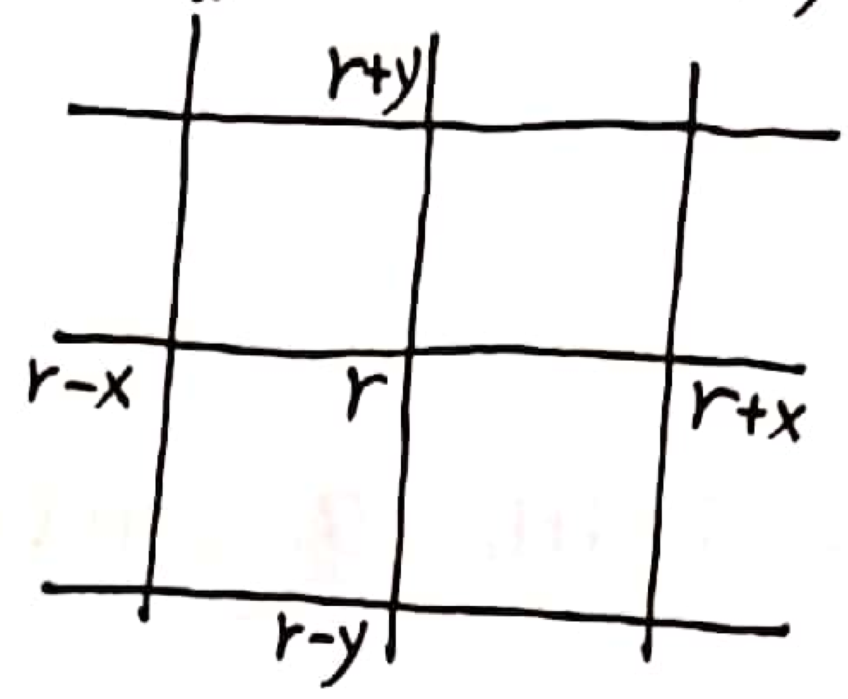
$$\begin{cases} l_{ij}^2 \to (l_\mu(r))^2 \\ l_{ij}(\theta_i - \theta_j) \to l_\mu(r) \, (\theta(r) - \theta(r+\mu)) \end{cases}$$

$$e^{-\frac{1}{2\beta J} l_\mu^2(r) - i l_\mu(r)(\theta(r)-\theta(r+\mu))}$$

$$\Rightarrow \left(e^{-\frac{1}{2\beta J}\sum_{r\mu} l_\mu^2(r)}\right) e^{\underbrace{-i\sum_{r\mu} l_\mu(r)(\theta(r)-\theta(r+\mu))}_{=\,-i\sum_{r\mu}\theta(r)(l_\mu(r)-l_\mu(r-\mu))}}$$

dual 关系

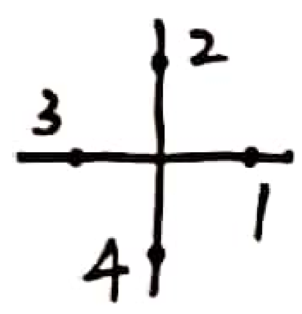
2种计数法 $\begin{cases}\text{bond}\\ \text{vertex(顶点)}\end{cases}$ $= -i\sum_r \theta(r)\sum_\mu (l_\mu(r)-l_\mu(r-\mu))$ $= l_1+l_2-l_3-l_4$



$$Z=\left(\frac{1}{\sqrt{2\pi\beta J}}\right)^{N_s} e^{\beta J N_s}\prod_{\{l_\mu(r)\}} e^{-\frac{1}{2\beta J} l_\mu^2(r)}\,\delta\Big(\sum_\mu (l_\mu(r)-l_\mu(r-\mu))\Big)$$

离散

第四步 找 $l_\mu(r)$的解，去掉 $\delta[\sum_\mu l_\mu(r)-l_\mu(r-\mu)]$ 限制



$\underbrace{l_1-l_3}_{x方向}+\underbrace{l_2-l_4}_{y方向}$ 即 $\boxed{\nabla\cdot\vec{l}=0}$ 的解

定义 $\vec{l}(r)=(l_x(r), l_y(r))$

"梯无旋，旋无散"

$$\begin{cases}\vec{l}(r)=\nabla\times\vec{n} & \Rightarrow \text{离散} =(\partial_y\theta, -\partial_x\theta)\\ l_x(r)=n(r)-n(r-y) & \vec{n}=(0,0,n)\\ l_y(r)=-n(r)+n(r-x) & \end{cases}$$

上节课到此. Nagaosa书上 eq 3.3.20 $\sum_\mu l_\mu(r)-l_\mu(r-\mu)=0$

$Z=\mathrm{Tr}\left[e^{-\frac{1}{2\beta J}\sum_{r\mu}(n(r)-n(r-\mu))^2}\right]$，注意. $n(r)$ 为整数

*连续时 $(n(r)-n(r-\mu))^2 \longrightarrow (\partial_\mu n(r))^2$ 可行

第五步 再利用Villian-Possion 公式

连续的 θ $\longrightarrow$ 离散的 n $\longrightarrow$ 连续化 ϕ (Sine-Gordon field)

$$Z=\int d\phi\sum_m e^{-\frac{1}{2\beta J}(\Delta_\mu\phi(r))^2+2\pi i m\phi}$$

$$\Delta_\mu\phi=\phi(r)-\phi(r-\mu)$$

连续化下 $\Delta_\mu\to\partial_\mu$

$$\sum_m e^{2\pi i m\phi}\equiv\sum_n\delta(\phi-n)$$

$$=\sum_n\int d\phi\, e^{-\frac{1}{2\beta J}(\Delta_\mu\phi)^2}\delta(\phi-n)$$

$$=\sum_n e^{-\frac{1}{2\beta J}(\Delta_\mu n)^2}=\text{前面}$$

特点：$\int d\phi \sum_m = $ 原式

反过来

$$\sum_n \int d\phi\, e^{-\frac{1}{2\beta J}(\Delta_\mu\phi)^2+2\pi i m\phi}$$

化为二次型积分，可解！ $\begin{cases} m\to\Delta_\mu n(r)\\ \phi\to\phi_\mu(r)\end{cases}$

真实的计算应该是：

$$Z = e^{-\sum_{r_\mu}\frac{1}{2\beta J}(\Delta_\mu n(r))^2}$$

$$\int d\phi_\mu(r)\, e^{-\frac{1}{2\beta J}\sum_{r_\mu}\underbrace{(\Delta_\mu\phi_\mu(r))^2}_{\text{类似于二次型}(\partial\phi)^2} + 2\pi i\Delta_\mu n(r)\phi_\mu(r)}$$

二次型基本结论

$$\int D\phi\, e^{\int -K(\partial\phi)^2+J(x)\phi(x)} = (\text{const})\, e^{-\int J(x)G(x-y)J(y)\,dxdy}$$

e.g. Gaussian Integral

$$\int D \quad = \int Dx\, e^{-x^TAx+J\cdot x} = \text{const}\cdot e^{-JA^{-1}J/4}$$

体现关联

中间FT过程使 $\frac{1}{2\beta J}\to 2\beta J$

$$Z = \underbrace{Z_{SW}}_{const}\sum_{\{m(r)\}} e^{-2\pi^2\beta J\sum_{r,r'} m(r)G(r-r')m(r')}$$

其中 $G(r-r') = \int_{-\pi}^{\pi}\frac{dk_x\, dk_y}{(2\pi)^2}\frac{e^{ik(r-r')}}{4-2\cos k_x-2\cos k_y}$

$\int dk\left(\frac{e^{ik(r-r')}}{k_x^2+k_y^2}\right) \longleftrightarrow \nabla^2 G=\delta(r)$ (FT) $k^2G(k)=1$

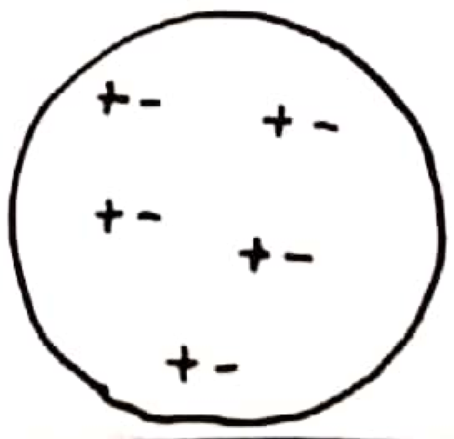
$\sim -\frac{1}{2\pi}\ln|r|$

$$\simeq \underbrace{-\frac{1}{2\pi}\ln\left(\frac{|r-r'|}{a}\right)-\frac{1}{4}}_{G'(r-r')} + G(0)$$

$$Z = Z_{SW}\sum_{\{m(r)\}} e^{-2\pi^2\beta J\left[\underbrace{\sum_{r,r'}G(0)m(r)m(r')}_{G(0)(\sum_r m(r))^2\ \text{total charge}} + \sum_{r,r'}G'(r-r')m(r)m(r')\right]}$$

① $e^{-2\pi^2\beta J\, G(0)|\sum_r m(r)|^2}$ 有贡献时，$\sum m(r)=0$ 由于

即 $\boxed{\text{中性条件}}$ $E_v\sim\ln\left(\frac{L}{a}\right)$



② $\sum_{rr'} G'(r-r')\, m(r)\, m(r')$

$= -\frac{1}{4}\sum_{rr'} m(r)\, m(r') - \frac{1}{2\pi}\sum_{rr'} m(r)\, m(r') \ln\left|\frac{r-r'}{a}\right|$

利用中性条件 $\sum_r m(r) = 0$

$$\left(\sum_r m(r)\right)^2 = 0 \iff \sum_r m^2(r) + \sum_{r,r'} m(r) m(r') = 0$$

$= \frac{1}{4}\sum_r m^2(r) - \frac{1}{2\pi}\sum_{r,r'} m(r)\, m(r') \ln\left|\frac{r-r'}{a}\right|$

其中 fugacity : $e^{\beta\mu}$

$$Z = Z_{sw}\sum_{m(r)} \exp\left[\ln y \sum_r m^2(r) - \pi\beta J \sum_r m(r)\, m'(r) \ln\left|\frac{r-r'}{a}\right|\right]$$

其中 $y = e^{\beta\mu} \iff \ln y = \beta\mu$, $\mu = -\frac{\pi^2}{2}J$ 这里细节有纰漏

BKT相变条件：$\left.\begin{aligned}\beta J &= \frac{2}{\pi^2} \\ \mu &= -\frac{\pi^2}{2}J\end{aligned}\right\}$ $\begin{aligned}\ln y &= 0 \\ y &= 1\end{aligned}$ 处发生相变.

第六步 Sine-Gordon eq 与近似

手动加

$$Z = \oint \int d\phi \sum_{m(r)} e^{-\frac{1}{2\beta J}(\Delta_\mu \phi)^2 + 2\pi i m\phi + \ln y\, m^2(r)}$$

取 $m = 0, \pm 1$，其它激发无贡献

* 注意：这一点不是推导出来的，
而是与对BKT相变的认识有关

$\oint d\theta = 2\pi n$

$E \propto n^2$

$1 + e^{\ln y + 2\pi i\phi} + e^{\ln y - 2\pi i\phi}$

$= 1 + 2\underbrace{e^{\ln y}}_{小}\cos(2\pi\phi) = 1 + 2y\cos(2\pi\phi)$

$= e^{2y\cos(2\pi\phi)}$

$$\left.\begin{aligned} Z &= \int d\phi\, e^{-\frac{1}{2\beta J}(\partial\phi)^2 + 2y\cos(2\pi\phi)} \\ &\Rightarrow \int d\phi\, e^{-\int \frac{1}{2}(\nabla\phi)^2 - 2y\cos(2\pi\sqrt{\beta J}\,\phi)}\end{aligned}\right\} \text{Sine-Gordon model.}$$

Ref. X. G. Wen

$1 = \frac{n^2}{8\pi} = \frac{4\pi^2\beta J}{8\pi} = \frac{\pi}{2}\beta J$

$\iff \boxed{k_B T_c = \frac{\pi}{2}J}$ 半经典

Kramers - Wannier 1944

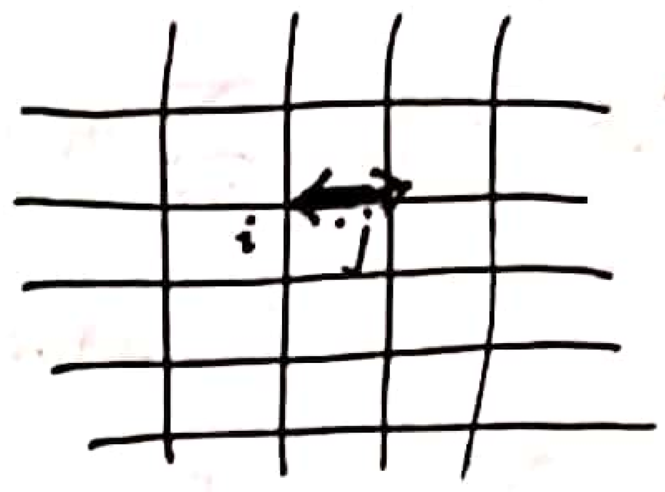
PRB, 60, 252

PRB, 60, 263

Onsager, 1944, PRB, 66, 117

$$H = -J\sum_{\langle ij\rangle} S_i S_j \quad , \quad k = \beta J \ , \quad k_c = \frac{1}{2}\ln(1+\sqrt{2}) = 0.44067$$

$$Z = Tr(e^{-\beta H}) = \sum_{\{ij\}} \prod e^{K S_i \cdot S_j}$$



1. 高温展开

$$= \sum \prod \{\cosh(K) + \sinh(K) S_i \cdot S_j\}$$

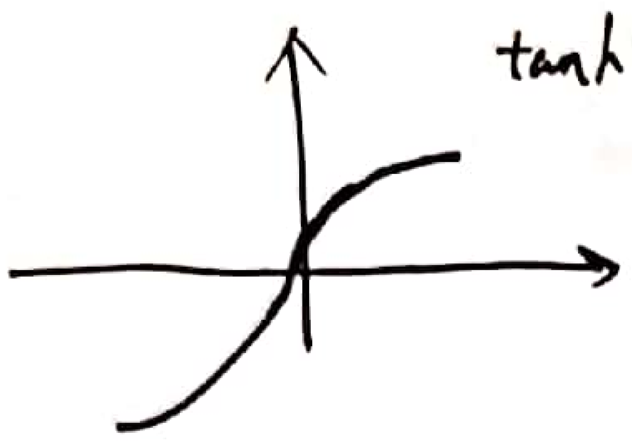
$$= \cosh^{N_b}(K) \sum \prod ((1 + \tanh(K)) S_i S_j)$$

~~N_b bg~~

N_b: bond个数

N_s: site个数

tanh(K)



展开： $(1+x_1)(1+x_2)(1+x_3)+\cdots$

$$= 1 + x_1 + x_2 + x_3 + x_1 \cdot x_2 + x_1 x_3 + x_2 \cdot x_3 + x_1 x_2 x_3 + \cdots$$

如果 $\sum S_1 S_2 = 0$

$\sum S_1 S_2 S_3 S_4 = 0$

图表示 $S_1 S_2 S_2 S_3$, $S_4 S_4 S_1 = 1$

所有回路部分才 $\neq 0$

$$Z = \sum_i e^{-\beta\varepsilon_i}$$

$$= e^{-\beta\varepsilon_0}\left(\sum e^{-\beta\varepsilon_i}\right)$$

$$Z = \frac{const}{(GS)} \cdot \sum_{loop} \left(\quad \right)$$

激发

loop excitation

loop expansion

$2^{N_s} \cosh^{N_b}(k)$

Ref: Shankar书上

$$Z = 2^{N_s} \cosh^{N_b}(K) \quad f(\tanh(K))$$

多项式 $f(x) = \sum_l C_l x^l$

$$= 2^{N_s} \cosh^{N_s}(K)$$

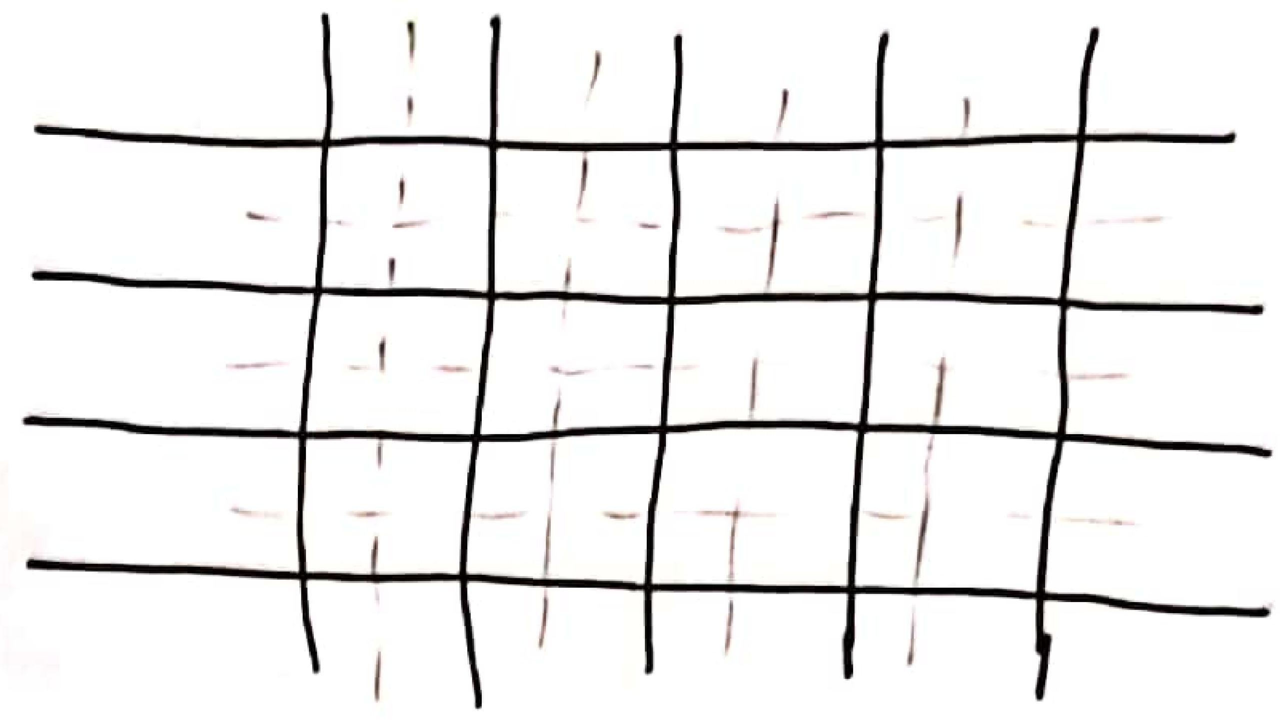
补充: Polykov书上 P6.

计算关联时

2. 低温展开

$J>0$

$H=-J\sum_{ij} S_i S_j$



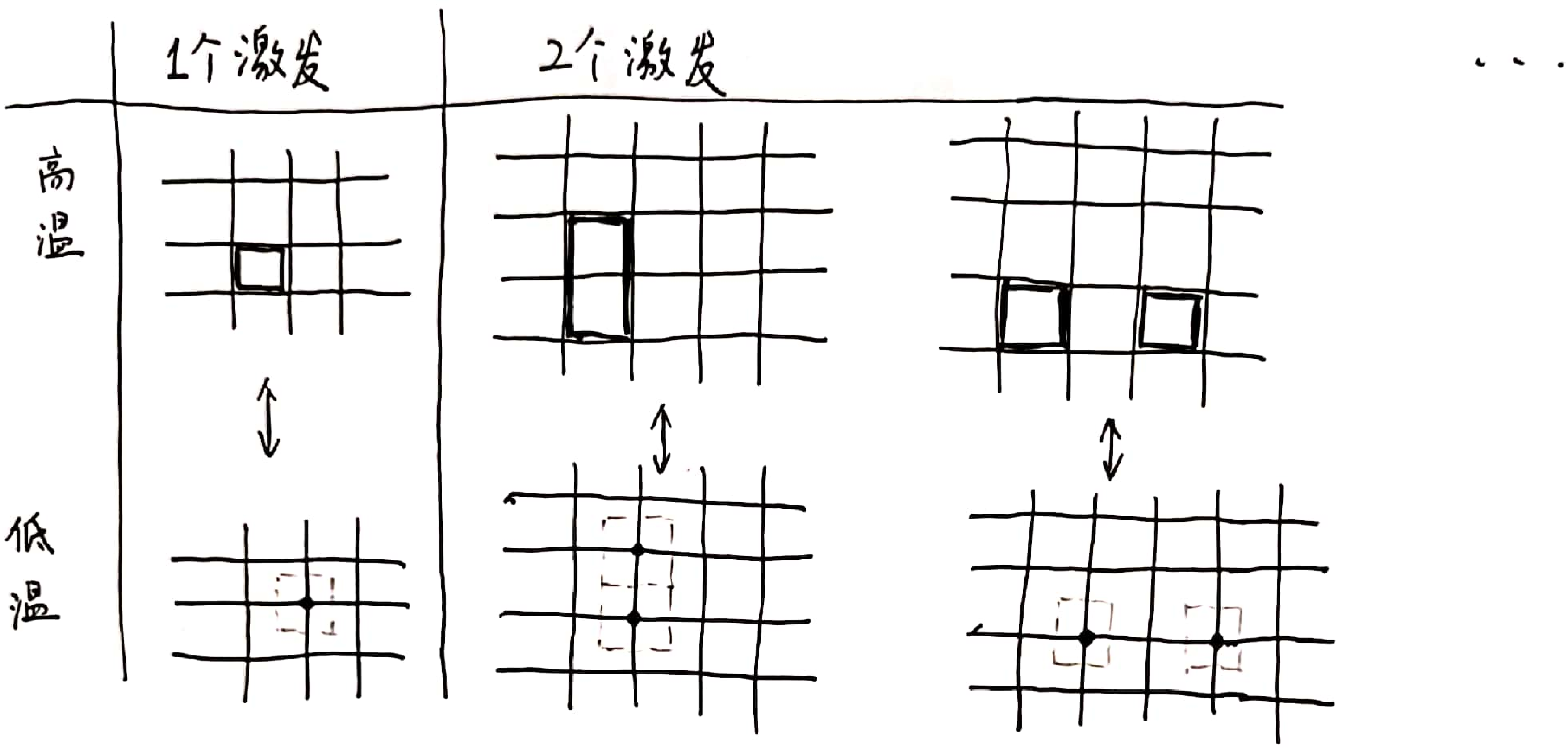
$$Z=\sum \prod e^{K S_i S_j}$$

$$=2e^{K\varepsilon_0}+e^{K\varepsilon_0}N_s e^{-4K}$$

$$+\underset{\text{loops}}{\text{两个}}+\underset{\text{loops}}{\text{三个}}+\underset{\text{loops}}{\text{四个}}+\cdots$$

$$\propto f(e^{-2K})$$

Ref. Wegner



Shanker书 P370

$$\frac{Z_H}{2^N \cosh^{2N}(K)}=1+N\tanh^4(K)+2N\tanh^6(K)+\cdots$$

$$\frac{Z_L}{e^{2NK}}=1+Ne^{-8K}+2Ne^{-12K}$$

$$\boxed{\tanh(K)=e^{-2K}}$$

$$K_c=\frac{1}{2}\ln(1+\sqrt{2})$$
$$=0.44067$$

思考：1.高温下关联是指数衰减的；低温行为肯定不同，应该是长程关联.(可以尝试计算)

2. Wilson Loop (Nagaosa)