12.30 $k_B T_c = \frac{t}{2} J$ 前面：Poisson求和.

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{k=-\infty}^{+\infty} \tilde{f}(k)$$

证明：$\tilde{f}(k) = \sum_n f(n) e^{2\pi i n k}$ 应该是$\int_n$

$$\sum_k \tilde{f}(k) = \sum_n f(n) \sum_k e^{2\pi i n k}$$ n为整数.

$$\sum_n f(n) \sum_m \delta(n-m) = \sum_m f(m)$$

e.g. $f(n) = e^{-an^2-bn}$. $\tilde{f}(k) = \sqrt{\frac{\pi}{a}}\, e^{(b-2\pi k i)^2/4a} = \sqrt{\frac{\pi}{a}}\, e^{-(2\pi k - ib)^2/4a}$

e.g. $a=1$, $b=0.53$. 则 $\sum_n f(n) = \sum_k \tilde{f}(k) = 1.70138$

a小时 $f(n)$中 n的贡献 | $\tilde{f}(k)$ k的贡献 [illegible]
[illegible] | [illegible] ⇒ 之间存在对偶关系.
$\alpha \propto \beta$ 收敛慢 | 快
[illegible]

Euler Summation

$$\sum_{n=1}^{k} f(n) \to \int_1^k f(x)\,dx + \dots$$

BKT 相变.

① 单个涡旋：$F = U - TS = J\pi \ln\frac{L}{a} - k_B T\, 2\ln\frac{L}{a} = (J\pi - 2k_B T)\ln\frac{L}{a}$

$k_B T_c = \frac{J\pi}{2}$ ⇒ [illegible]

②. Thouless Nelson-Kosterlitz (PRL)

③ Wilson. RG. ref. [illegible] Wen . Stat [illegible]

④ Jose (ref. [illegible]) Nagaosa.

BKT. RG: (why [illegible])

$\langle e^{u} \rangle = e^{\langle u \rangle + \frac{1}{2}\langle u^2 \rangle_c}$ 由于 $\langle e^{J\cos\theta} \rangle = e^{\langle J\cos\theta \rangle + \frac{1}{2}\langle J^2 \cos^2\theta \rangle_c}$

$\langle u \rangle = \langle e^{k} \rangle = e^{\langle k \rangle + \frac{1}{2}\langle k^2 \rangle_c}$ $u = \cos\theta = e^{i\theta} + e^{-i\theta}$ $u = e^{iA\cos\theta}$

$\langle u \rangle = e^{\langle iA\cos\theta \rangle} = e^{\frac{iA}{2}(\langle e^{i\theta} \rangle + \langle e^{-i\theta} \rangle)} = e^{\frac{iA}{2}(e^{\theta - \frac{1}{2}\langle\theta^2\rangle} + e^{-\frac{1}{2}\langle\theta\rangle^2})}$

$= e^{iA}\, e^{-\frac{1}{2}\langle\theta^2\rangle}$

That's Fucking Amazing

No

Date . .

$$\cos n(\theta_> + \theta_<) - \cos n\theta_<$$
$$= \cos n\theta_> \cos n\theta_< - \sin n\theta_> \sin n\theta_< - \cos n\theta_<$$
$$= \cos n\theta_< \left(1 - \tfrac{1}{2} n^2 \theta_>^2\right) - n\theta_> \sin n\theta_<$$

$$S = \int \frac{k}{2}(\partial_\mu \theta_<)^2 - g\cos n\theta_< + \int \frac{k}{2}(\partial_\mu\theta_>)^2 - g\left(\underline{\cos n(\theta_< + \theta_>) - \cos n\theta_<}\right)$$

$$= \int \frac{k}{2}(\partial_\mu\theta_<)^2 - g\cos n\theta_< + \int \frac{k}{2}(\partial_\mu\theta_>)^2 + g\frac{n^2}{2}\cos n\theta_> \theta_>^2 + ng\sin n\theta_< \theta_>$$

$$Z = \int D\theta_< \, e^{-\int \frac{k}{2}[(\partial_\mu\theta_<)^2 - g\cos n\theta_<]} \int D\theta_> \, e^{-\int \frac{k}{2}(\partial_\mu\theta_>)^2} \left\langle e^{-g\int [\frac{n^2}{2}\cos n\theta_< \theta_>^2 + n\sin n\theta_< \theta_>]} \right\rangle$$

$$\langle e^{u} \rangle = e^{\langle u \rangle + \frac{1}{2}\langle u^2 \rangle_c}$$

一阶近似 $\langle \theta_>^2 \rangle$ $\qquad e^{-g\frac{n^2}{2}\langle\theta_>^2\rangle \int \cos n\theta_<}$ [illegible]

二阶近似 $\qquad g^2 n^2 \int \sin n\theta_<(x) \sin n\theta_<(x') \, g(x - x')$

$$= g^2 n^2 \int \sin n\theta_<(x) \sin n\theta_<(x+y) \, g_>(y)$$

$$= g^2 n^2 \int \sin n\theta_<(x) \left[\sin n[\theta_<(x) + \partial\theta_< y]\right] g_>(y)$$

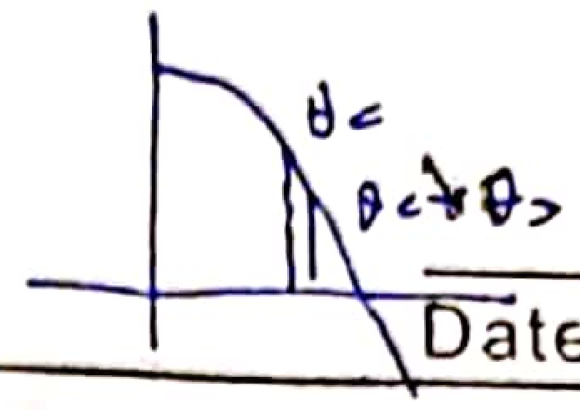
$$= g^2 n^2 \int \sin n\theta_<(x) \left\{\sin n\theta_<(x) \underline{\cos n\partial\theta_< y} + \cos n\theta_<(x) \sin n\partial\theta_< y\right\} g_>(y)$$

$$= g^2 n^2 \int \sin n\theta_<(x) \left[\sin n\theta_<(x)\left\{1 - \tfrac{1}{2} n^2 (\partial\theta_<)^2 y^2\right\} + \cos n\theta_<(x) \sin n\partial\theta_< y\right] g_>(y)$$

$$= \cdots \quad - \frac{1}{2} g^2 n^4 \int \sin^2 n\theta_<(x) (\partial_x \theta_<)^2 y^2 g_>(y)$$

$$= \cdots \quad - \frac{1}{4} g^2 n^4 \int (1 - \cos 2n\theta_<(x)) (\partial_x\theta_<)^2 y^2 g_>(y)$$

$$= \cdots \quad - \frac{1}{4} g^2 n^4 \int (\partial_x\theta_<)^2 y^2 g_>(y)$$



x. G. Wen p111 (P.)

$\theta=\theta_c+\theta_>$ 　 $S=\int \frac{K}{2}(\partial_\mu\theta_c)^2-g\cos^n\theta_c + \int \frac{K}{2}(\partial_\mu\theta_>)^2-g[\cos n(\theta_c+\theta_>)-\cos n\theta_c]$

$=\int \frac{K}{2}(\partial_\mu\theta_>)^2+\underbrace{g\frac{n^2}{2}\cos n\theta_c\,\theta_>^2-ng\sin n\theta_c\,\theta_>}_{perturbation}$

$Z=\int D\theta_c e^{-S_0}\int D\theta_> e^{-\int\frac{K}{2}(\partial_\mu\theta_>)^2}\langle e^{-g\int\frac{n^2}{2}\cos(n\theta_c)\theta_>^2-n\sin n\theta_c\,\theta_>}\rangle$

一阶 $\langle\theta_>^2\rangle$ 　 $\langle\theta(x)\theta(y)\rangle=g(x-y)$

$e^{-\frac{n^2}{2}g\underbrace{\langle\theta_>^2\rangle}_{const}\int\cos n\theta_c}$

二阶 $\int dx\langle\theta_>^2(x)\,\theta_>^2(x')\rangle=?$

$\boxed{\langle 0\rangle}$

$\langle\theta_>(x)\,\theta_>(x')\rangle\Rightarrow$ 只… $g_>(x-x')$ …

$0=\langle\theta_>^2(x)\,\theta_>(x')\rangle$

$0=\langle\theta_>(x)\,\theta_>^2(x')\rangle$

$g^2n^2\int\cos(n\theta_c(x))\cos(n\theta_c(x'))\,g_>(x-x')$ 　 $g^2n^2\int\cos(n\theta_c(x))\cos n\theta_c(x+y)\,\underbrace{g_>(y)}_{短程}$

$\cancel{\cos n\theta_c(x)}\,[\cos n\theta_c(x)$

…

$\cos n\theta_c(x)\cdot\cos(n\theta_c(x)+\partial\theta_c\,y)$

$=\cos n\theta_c(x)\,\cancel{\cos n\theta_c(x)}\cdot[1-\frac{n^2}{2}y^2(\partial\theta_c)^2]-\sin n\theta_c(n\partial_\mu\theta_c\,y)$

$\int\cos^2 n\theta_c(y)\,g_>(y)\,dy$ 　 …

$\int\cos n\theta_c\,\sin n\theta_>\,y\,g_>(y)\,\partial_\mu\theta_c$

$-\frac{n^4}{2}g^2n^2\int\cos^2 n\theta_c\,(\partial_\mu\theta_c)^2\int dy\,g_>(y)\,y^2$ 　 $\cos 2\theta=2\cos^2\theta-1$

$\int\cos^2(n\theta_c)\,g_>(y)\,dy\,dx$

$-\frac{n^2}{2}g^2n^2\int\cos 2\theta_c\,(\partial_\mu\theta_c)^2\int dy\,g_>(y)\,y^2$

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Date

$$Z = \int D\theta_< \, e^{-\int \{ (\frac{k}{2} + \frac{g^2}{8} n^2 \int y^2 g_>(y) dy)(\partial \theta_<)^2 - g(1 - \frac{n^2}{2}\langle \theta_>^2 \rangle) \cos n\theta_< \}}$$

$$S = \int d^2x \frac{k_\lambda}{2} (\partial \theta_<)^2 - g_\lambda \cos n\theta_<$$

$$k_\lambda = k + \frac{g^2}{4} n^4 \int y^2 g_>(y) dy$$

$$g_\lambda = g - \frac{n^2}{2} g \langle \theta_>^2 \rangle$$

$$\Rightarrow \begin{cases} g_\lambda = g_l - \frac{n^2}{2} g_l K(0) \\ k_\lambda = k_l + \frac{n^2}{8} g^2 k_2 \end{cases}$$

$$K(0) = \int_{2\pi/\lambda < |k| < 2\pi/l} \frac{d^2k}{(2\pi)^2} \frac{n^2}{k_l |k|^2} = \frac{n^2}{2\pi k_l} \int \frac{dk}{k} = \frac{n^2}{2\pi k_l} \ln \frac{\lambda}{l}$$

$$k_2 = \int d^2x |x|^2 K(x) = \int_{2\pi/\lambda < |k| < 2\pi/l} \frac{d^2x \, d^2k}{(2\pi)^2} \frac{n^2 x^2 e^{ikx + 0^+|x|}}{k_l k^2}$$

$$= \frac{\lambda - l}{l} \frac{3n^2 l^4}{16\pi^4 k_l} \int d\theta \frac{1}{(\cos\theta + i0^+)^4} = \frac{\lambda - l}{l} \frac{3n^2 l^4}{2\pi^4 k_l}$$

$$b = \frac{l}{\lambda} \qquad 0 < |x| < \lambda \qquad x' = bx \qquad 0 < |x'| < l$$

$$\int d^2x \frac{k_\lambda}{2} (\partial\theta_<)^2 - g_\lambda \cos(n\theta_<) = \frac{k_\lambda}{2} \int d^2x' \frac{1}{b^2} b^2 (\partial \theta_<)^2$$

$$- g_\lambda \frac{1}{b^2} \int d^2x' \cos n\theta_<(x')$$

$$\frac{k}{2}(l - \epsilon) = \frac{1}{2}(k + \frac{n^2}{8} g^2 k_2) = \frac{1}{2}(k_l + \frac{n^2}{8} g^2 (\frac{1}{b} - 1) \frac{3n^2 l^4}{2\pi^4 k_l})$$

$$-g(l - \epsilon) = -\frac{1}{b^2}(g - \frac{1}{2} g K(0)) = -\frac{1}{b^2}(g + \frac{1}{2}\frac{n^2}{2\pi k_l} \ln b)$$

$$\ell = \ln\lambda \qquad \lambda\frac{d}{d\lambda}$$

$$k = k + \frac{n^2}{2} g^2 (\frac{\lambda - l}{l}) \frac{3n^2 l^4}{8\pi^4 k}$$

$$g = \frac{\lambda^2}{l^2}(g + \frac{1}{2}\frac{n^2}{2\pi k_l} \ln\frac{l}{\lambda} g)$$

$$\Downarrow$$

$$\frac{dk}{d\ell} = b\frac{dk}{db} \to b\,\lambda[\frac{n^2}{2} g^2 \frac{1}{l} \frac{3n^2 l^4}{8\pi^4 k}] = \frac{3n^4 \lambda^4 g^2}{16\pi^4 k}$$

$$\frac{dg}{d\ell} = \lambda \{ \frac{2\lambda}{l^2} (g + \frac{1}{2}\frac{n^2}{2\pi k_l} \ln\frac{l}{\lambda} g)$$

$$+ \frac{\lambda^2}{l^2} (-\frac{1}{2}\frac{n^2}{2\pi k_l}\frac{1}{\lambda} g) \} \Big|_{l = \lambda} = l \{ \frac{2}{l}(g + \frac{1}{2}\frac{n^2}{4\pi k_l} g) - \frac{n^2}{4\pi l k_l} g \} = -\frac{n^2}{4\pi k_l} g$$

No
Date . .

$$S = \int dx \, \frac{k_\lambda}{2} (\partial\theta_<)^2 - g_\lambda \cos(n\theta_<)$$

$$k_\lambda = k + \frac{n^2}{8} g^2 k_2$$

$$g_\lambda = g - \frac{1}{2} k(0) \qquad k(0) = \left(\frac{1}{2\pi}\right)^2 \int_{\frac{2\pi}{\lambda} \le |k| \le 2\pi/l} \frac{n^2}{k|k|^2} = \frac{n^2}{2\pi k} \ln\frac{\lambda}{l}$$

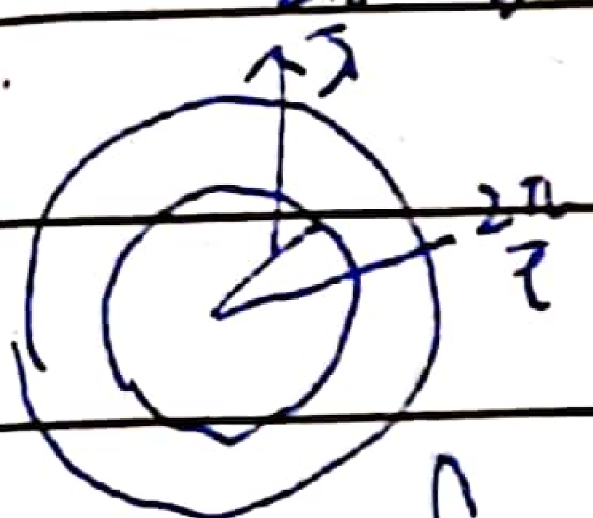
2d BKT [illegible]

$$K_2 = \int d^2x \, K(x) \sim \langle \theta_>(x)\theta_>(0) \rangle \sim \frac{1}{V}\sum_{k\,\text{shell}} \frac{n^2 e^{ikx}}{k|k|^2} \qquad \frac{\lambda-1}{l} \frac{3n^2 l^4}{2\pi^4 k}$$

rescaling :

$$\int dx \, \frac{k_\lambda}{2}(\partial\theta_<)^2 - g_\lambda \cos(n\theta_<) \qquad |k| \le \frac{2\pi}{l} \to |k| \le \frac{2\pi}{\lambda}. \quad \lambda > l.$$

$$b \cdot \frac{2\pi}{l} = \frac{2\pi}{\lambda} \qquad b = \frac{l}{\lambda}. \qquad |k| \le b\Lambda. \quad k = bk' \quad |k'| \le \Lambda$$



$$x = \frac{1}{b}x' \qquad \theta_<(x) = \theta_<\left(\frac{1}{b}x'\right) \doteq z\,\theta_<(x')$$

$$\int dx \, \underline{\frac{k_\lambda}{2}(\partial\theta_<)^2} - g_\lambda \cos(n\theta_<)$$

$$= \frac{k_\lambda}{2}\int \left(\frac{1}{b}\right)^2 dx' \left(\frac{\partial}{\partial(\frac{1}{b}x')}\theta_<(x)\right)^2 \qquad z=1$$

$$= \frac{k_\lambda}{2}\int dx' \left(\frac{\partial}{\partial x'}\theta_<\left(\frac{1}{b}x'\right)\right)^2$$

$$-g_\lambda\left(\frac{1}{b}\right)^2 \int dx' \cos(n\theta_<(x)) = -g_\lambda\left(\frac{1}{b}\right)^2 \int dx' \cos(n\theta(x'))$$

$$\text{rescaling}: = \frac{k_\lambda}{2}\int dx' (\partial'\theta(x'))^2 - g_\lambda\left(\frac{1}{b}\right)^2 \cos(n\theta(x'))$$

$$= \underbrace{\frac{1}{2}\left(k + \frac{n^2}{8}g^2 k_2\right)}_{k(b)} \int dx' (\partial'\theta(x'))^2 - \underbrace{\frac{1}{b^2}\left(g - \frac{1}{2}k(0)\right)}_{g(b)} \int dx' \cos(n\theta(x'))$$

$$\left(b = \frac{l}{\lambda}\right)$$

$$\Rightarrow b = \frac{l}{\lambda} = 1 - \epsilon$$

$$\frac{k}{2}(1-\epsilon) = \frac{1}{2}\left(k + \frac{n^2}{8}g^2 k_2\right)$$

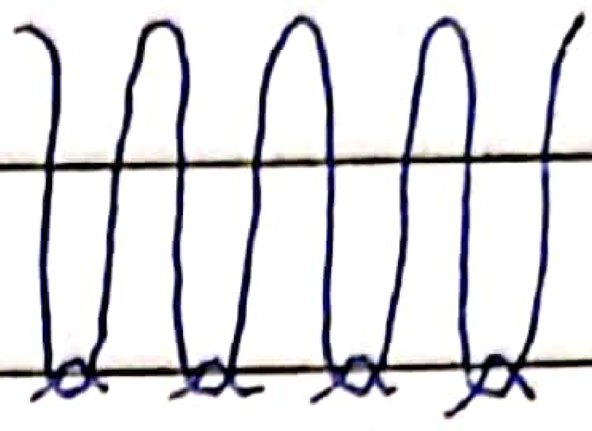
$$-g(1-\epsilon) = -\frac{1}{b^2}\left(g - \frac{1}{2}k(0)\right)$$

$$\begin{cases} \frac{\partial g_\lambda}{\partial b} = -\frac{n^2 g}{4\pi\lambda} \\ \frac{\partial k}{\partial b} = \frac{3n^4 g^2 \pi}{16\pi^4 k} \end{cases}$$

标度变换: $\begin{cases} k \to \bar{k} \\ g \to \bar{g}\lambda^2 \end{cases}$ $\begin{cases} \dfrac{d\bar{g}}{db} = \left(2 - \dfrac{n^2}{4\pi\bar{k}}\right)\bar{g} \\ \dfrac{d\bar{k}}{db} = \dfrac{3n^4\bar{g}^2}{16\pi^2\bar{k}} \end{cases}$

讨论结果. ① $2 - \frac{n^2}{4\pi\bar{k}} > 0$ $\bar{g} \to \infty$ 产生涡旋-反涡旋

$(\partial_\mu\theta)^2 - g\cos\theta$



② $2 - \frac{n^2}{4\pi\bar{k}} < 0$ $\bar{g} \to 0$ $S = \int \frac{1}{2}(\partial_\mu\theta)^2 \Rightarrow$ super fluid phase

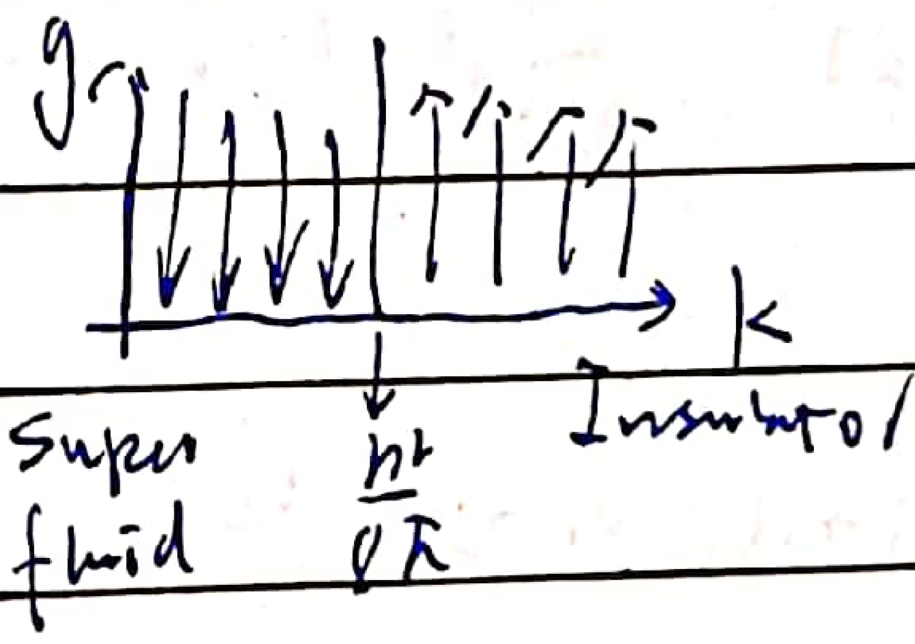
③ $2 - \frac{n^2}{4\pi\bar{k}} = 0$ 相变点

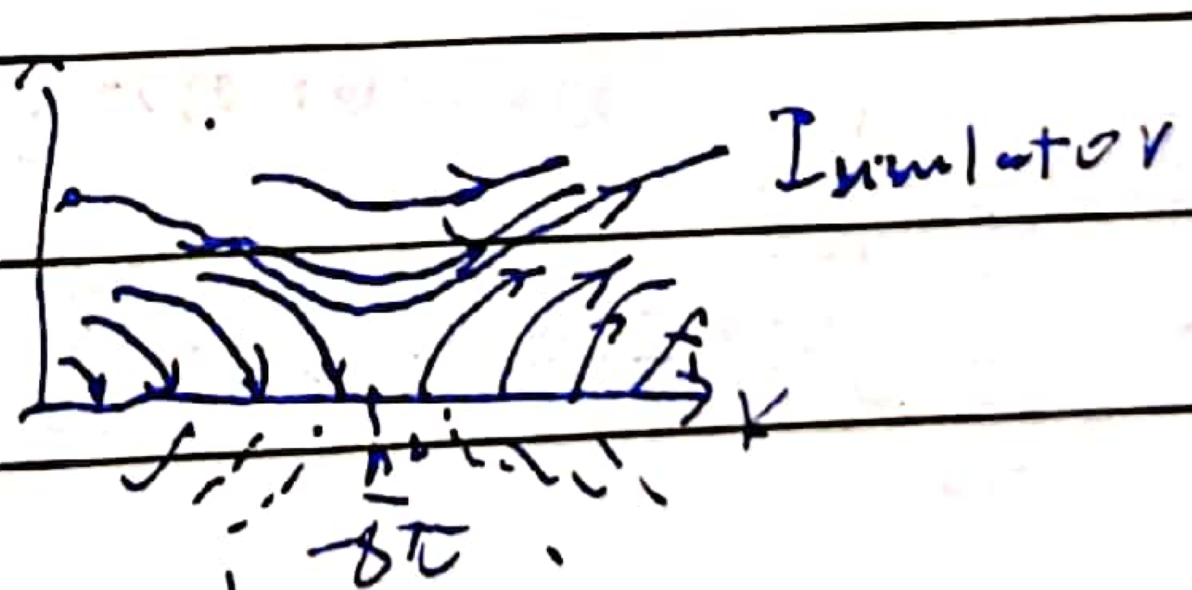
相变. J与β关系 $2 - \frac{n^2}{4\pi\bar{k}} = 0$ $K = \beta J$ 由 $J\cos(\theta_i - \theta_j) = -J\theta^2/2 + \frac{1}{2}J(\partial\theta)^2$

$e^{-\beta H} \Leftrightarrow e^{\frac{1}{2}\beta J(\partial\theta)^2} \triangleq \frac{K}{2}(\partial\theta)^2$ $\beta J = \frac{n^2}{8\pi} \overset{n=4}{=} \frac{16}{8\pi} = \frac{2}{\pi}$

βJ_2

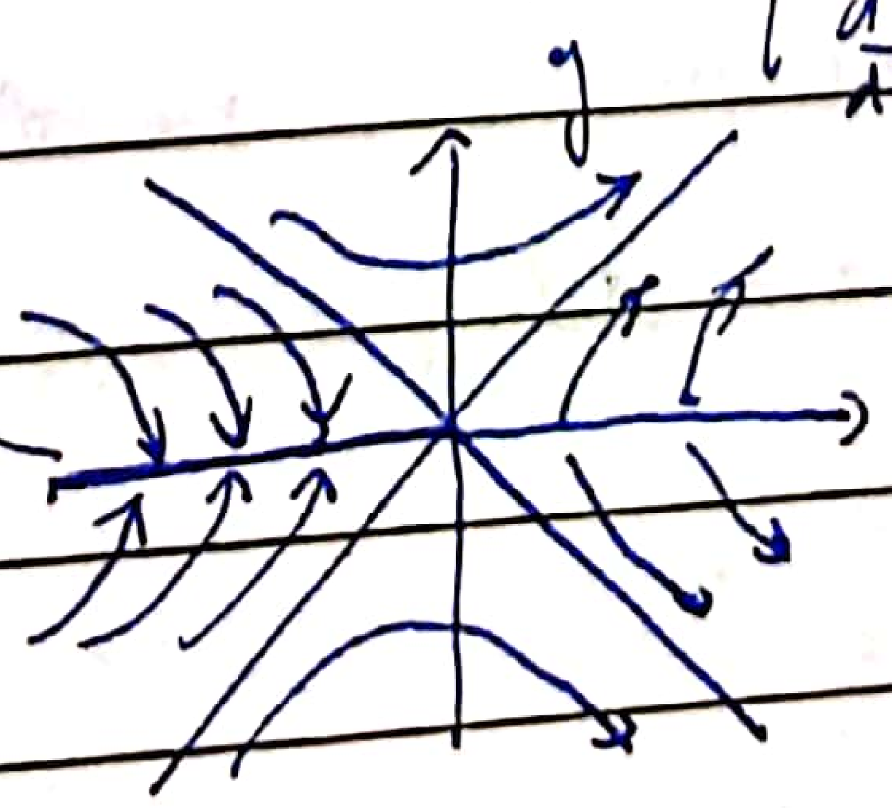
画出 running coupling const.





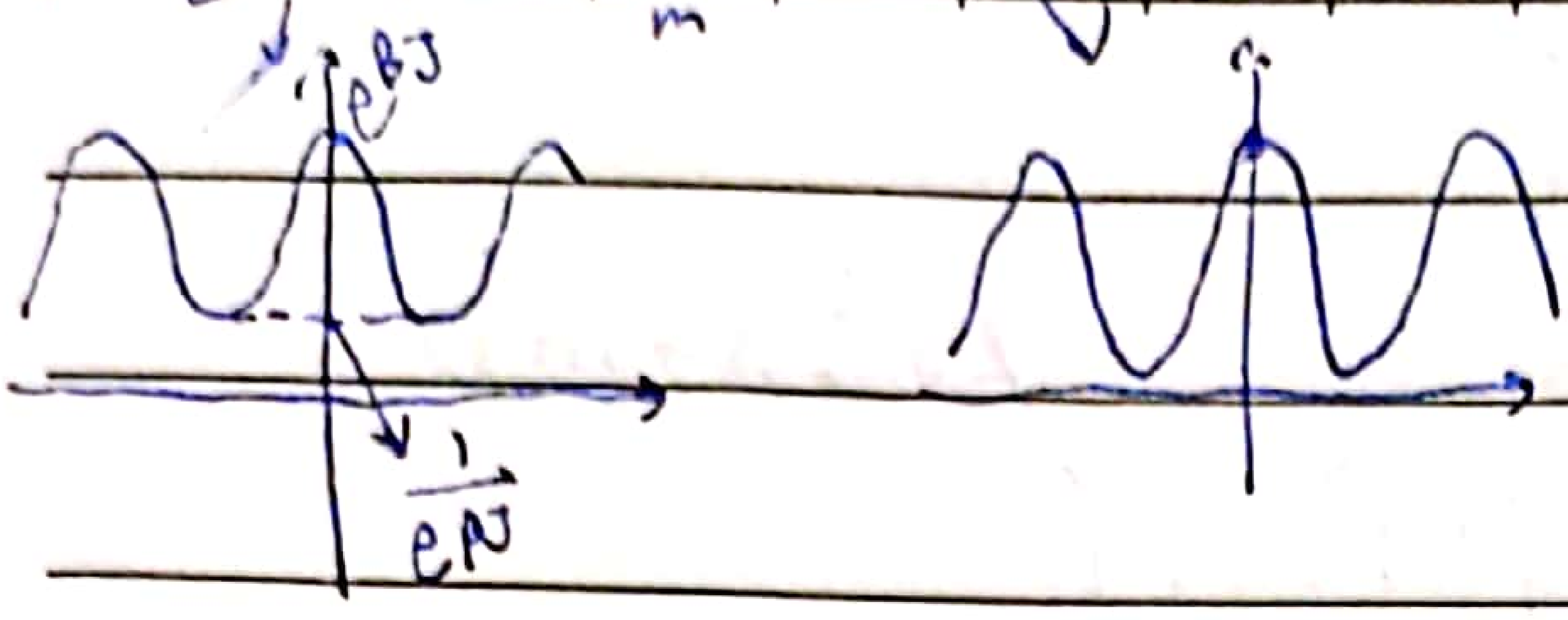
令 $2 - \frac{n^2}{4\pi\bar{k}} = \kappa$. $\begin{cases} \dfrac{dg}{db} = \kappa g \\ \dfrac{d\kappa}{db} = \dfrac{n^2}{4\pi\bar{k}^2}\left(\dfrac{d\bar{k}}{db}\right) \end{cases}$

Fixed point 形成区间



reference.
Shankar 书
Fig. 18.2.

$$e^{\beta J\cos\theta} \rightarrow \sum_m e^{\beta J - \frac{1}{2}\beta J(\theta - 2\pi m)^2}$$



$$Z = \int_{-\infty}^{+\infty} D\theta_1 \cdots D\theta_n \, e^{\beta J \sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)}$$

$$= \int_{-\infty}^{+\infty} D\theta_1 \cdots D\theta_n \prod_{\langle ij\rangle} \underline{\sum_m e^{\beta J - \frac{1}{2}\beta J(\theta_i - \theta_j - 2\pi m)^2}}$$

$$\sum_{l_{ij}} \int_\phi e^{\beta J - \frac{1}{2}\beta J(\theta_i - \theta_j - 2\pi\phi)^2 + 2\pi i \phi l_{ij}}$$

$$= \int_{-\infty}^{+\infty} D\theta_1 \cdots D\theta_m \prod_{\langle ij\rangle} \sum_{l_{ij}} \underline{\int d\phi \, e^{\beta J - \frac{1}{2}\beta J(\theta_i - \theta_j - 2\pi\phi)^2 + 2\pi i \phi l_{ij} - i l_{ij}(\theta_i - \theta_j) + i l_{ij}(\theta_i - \theta_j)}}$$

$$\parallel$$

$$e^{\beta J} \int d\phi \, e^{-\left[\sqrt{\frac{\beta J}{2}}(\theta_i - \theta_j - 2\pi\phi)\right]^2 + \frac{i l_{ij}(2\pi\phi - \theta_i + \theta_j)}{2\sqrt{\frac{\beta J}{2}}(\theta_i - \theta_j - 2\pi\phi)}} + \frac{l_{ij}^2}{2\beta J} - \frac{l_{ij}^2}{2\beta J} + i l_{ij}(\theta_i - \theta_j)$$

$$\parallel$$

$$\frac{1}{\sqrt{2\beta J\pi}} e^{\beta J + i l_{ij}(\theta_i - \theta_j) - \frac{l_{ij}^2}{2\beta J}}$$

$$= \int_{-\infty}^{+\infty} D\theta_1 \cdots D\theta_m \prod_{\langle ij\rangle} \sum_{l_{ij}} \frac{1}{\sqrt{2\beta J\pi}} e^{\beta J + i l_{ij}(\theta_i - \theta_j) - \frac{l_{ij}^2}{2\beta J}}$$

$$= \int_{-\infty}^{+\infty} D\theta_1 \cdots D\theta_m \sum_{l_{ij}} \frac{1}{\sqrt{2\pi\beta J}} e^{\beta J + \sum_{\langle ij\rangle} i l_{ij}(\theta_i - \theta_j) - \frac{l_{ij}^2}{2\beta J}}$$

$$= \sum_{l_{ij}} e^{\beta J - \frac{l_{ij}^2}{2\beta J}} \int D\theta_1 \cdots D\theta_m \, e^{\sum_{ij} i l_{ij}(\theta_i - \theta_j)}$$

[illegible] $\frac{1}{\sqrt{2\pi\beta J}}$ [illegible]

[illegible] $l_{ij} \to l_r(\mu)$ r 为格点. μ 为方向 $\underset{i \quad l_{ij} \quad j}{\longrightarrow}$

$$-\frac{l_{ij}^2}{2\beta J} + \sum_{\langle ij\rangle} i l_{ij}(\theta_i - \theta_j) = -\frac{l_\mu^2(r)}{2\beta J} + i l_\mu(r)(\theta(r) - \theta(r+\mu))$$

$$Z = \sum e^{-\sum_{r\mu} \frac{l_\mu^2(r)}{2\beta J}} \int D\theta_1 \cdots D\theta_m \, e^{\underline{\sum_{r\mu} i l_\mu(r)(\theta(r) - \theta(r+\mu))}}$$

$$\Downarrow$$

$$\sum_{r\mu} i(l_\mu(r) - l_\mu(r-\mu))\theta(r)$$

$$\int D\theta_1 \cdots D\theta_m \, e^{\sum_{r\mu} i(l_\mu(r) - l_\mu(r-\mu))\theta(r)}$$

$$= \prod_r \delta(l_\mu(r) - l_\mu(r-\mu)) \, \delta_{\Sigma}$$

Date . .

下节：Wannier – Krammer duality

证明. $Z=\int_{-\infty}^{+\infty} D\theta_1\cdots D\theta_N\, e^{\beta J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)}$

[illegible]

$e^{\beta J\cos\theta}\to\sum_m e^{\beta J-\frac{1}{2}\beta J(\theta-2\pi m)^2}$ [illegible]

低温. $\beta J=\frac{2}{\pi}\approx 1$.

Poisson 求和. $\sum_n h(m)=\sum_l\int d\phi\,\tilde{h}(\phi)e^{2\pi i l\phi}=\sum_m\int d\phi\,\tilde{h}(\phi)\delta(\phi-m)$

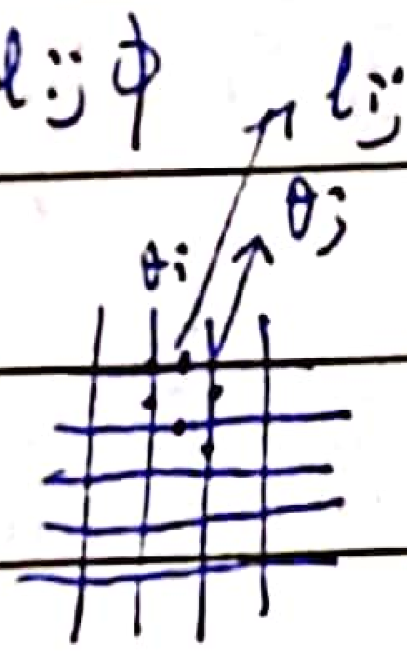
[illegible]

$Z=\int_{-\infty}^{+\infty} D\theta_1\cdots D\theta_N\, e^{\beta J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)}$

$\left(=\int D\theta\,\exp\left[-\frac{\beta J}{2}(\theta_i-\theta_j-2\pi\phi)^2+2\pi i l_{ij}\phi\right]d\phi\right)$

$=\int D\theta\sum_m e^{\beta J-\frac{\beta J}{2}(\theta_i-\theta_j-2\pi m)^2}$

$=\int D\theta\sum_{l_{ij}=-\infty}^{+\infty}\int d\phi\, e^{\beta J-\frac{1}{2}\beta J(\theta_i-\theta_j-2\pi\phi)^2+2\pi i l_{ij}\phi}$



$l_{ij}\to$ l_{ij} 定义在 lattice 上的 ... l_{ij} ⇔ $\theta_i-\theta_j$ [illegible]

[illegible] 上定义一个 l_{ij}

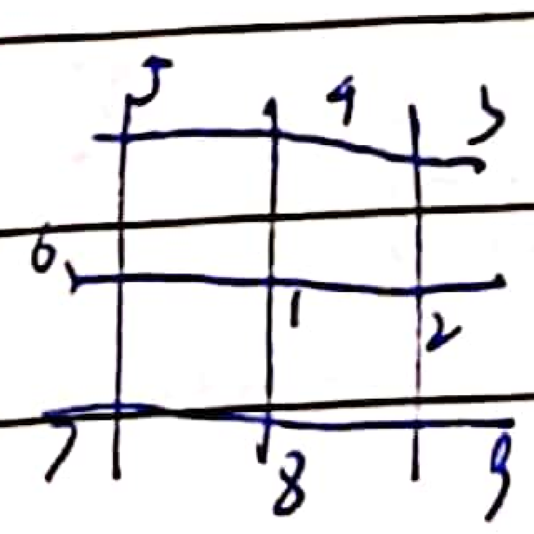
所有 bond 上定义一个 l_{ij}

$=\int D\theta\frac{1}{\sqrt{2\pi\beta J}}\sum_{l_{ij}} e^{\beta J+i\underline{l_{ij}}(\theta_i-\theta_j)-\frac{1}{2\beta J}l_{ij}^2}$

$\underline{\text{整数}}$

$\cancel{\sum_{l_{ij}} e^{-\frac{1}{2\beta J}l_{ij}^2}}$

$Z=\sum e^{-\sum_{ij}\frac{l_{ij}^2}{2\beta J}}\int D\theta\, e^{\sum_{ij} i l_{ij}(\theta_i-\theta_j)}$



$l_{12}(\theta_1-\theta_2)$

$+l_{14}(\theta_1-\theta_4)$

$+l_{16}(\theta_1-\theta_6)$

$+l_{18}(\theta_1-\theta_8)$

[illegible] $l_{ij}\to l_\mu(r)$ r 为 index，μ 为方向

$l_{ij}=-l_{ji}$

$\frac{l_{ij}^2}{2\beta J}-i\,l_{ij}(\theta_i-\theta_j)$

$=\frac{l_\mu^2(r)}{2\beta J}-i\,l_\mu(r)\left(\theta(r)-\theta(r+\mu)\right)$

$Z=\sum e^{-\sum_{r\mu}\frac{l_\mu^2(r)}{2\beta J}}\int D\theta(r)\, e^{\sum_{r\mu} i\, l_\mu(r)(\theta(r)-\theta(r+\mu))}$

$\cancel{\sum_r i\theta(r)\sum}$ $\quad \overset{\shortparallel}{i}\sum_{\mu r}\left(l_\mu(r)-l_\mu(r-\mu)\right)\theta(r)$

No

Date . .

定义 $\vec{l}(r) = (l_x(\vec r), l_y(\vec r)) = \nabla\times\vec n$ 设 $\vec n = (0, 0, n)$

则 $l_x(r) = \partial_y n_z - \partial_z n_y = \partial_y n$ $l_y(r) = \partial_z n_x - \partial_x n_z = -\partial_x n$

$l_x = n(r) - n(r-y)$ $l_y(r) = n(r-x) - n(r)$

此时 $l_x(r) - l_x(r-x) + l_y(r) - l_y(r-y) = \cancel{n(r)} - n(r-y) - n(r-x) + n(r-x-y)$

$- \cancel{n(r)} + n(r-x) + n(r-y) - n(r-x-y)$

$= 0 = \nabla\cdot\vec l$

$$Z = \int D\theta_1 \cdots D\theta_m \, e^{\beta J \sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)}$$

$$= \sum_{l(r)} e^{-\sum_{r\mu} \frac{l_\mu^2(r)}{2\beta J}} \prod_r \delta(l_\mu(r) - l_\mu(r-\mu))$$

$$= \sum_{l_\mu(r)} e^{-\sum_{r\mu} \frac{l_\mu^2(r)}{2\beta J}} \prod_r \delta\Big(\underbrace{\sum_\mu (l_\mu(r) - l_\mu(r-\mu))}_{\text{在此基础设计自由的场}}\Big)$$

$$= \sum_{\{n(r)\}} e^{-\sum_{r\mu} \frac{1}{2\beta J} (n(r) - n(r-\mu))^2}$$

BKT: $e^{K\cos(x)} = \frac{1}{\sqrt{2\pi K}} \sum_q e^{K - \frac{q^2}{2K} + iqx} = \frac{1}{\sqrt{2\pi K}} \sum_q e^{K - \frac{q^2}{2K} + iqx}$

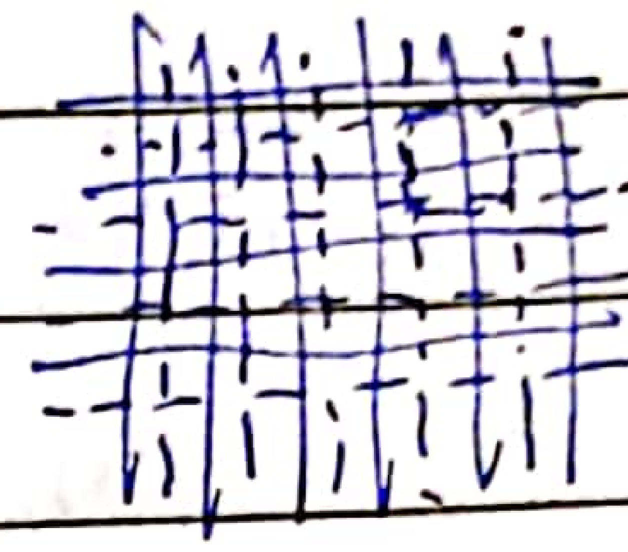
[illegible] Villiam transformation + Poission 求和

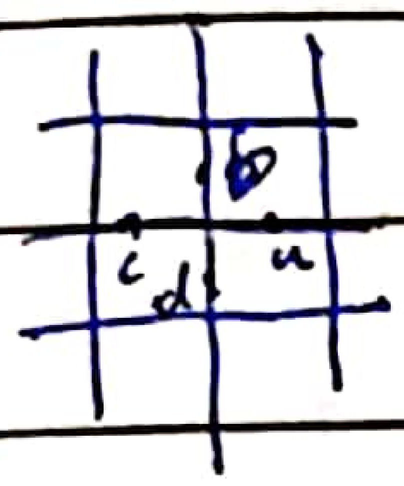
$$\delta\Big[\sum_\mu (l_\mu(r) - l_\mu(r-\mu))\Big] \qquad {}^{\prime\prime} \, i\sum_r \theta(r) \sum_\mu (l_\mu(r) - l_\mu(r-\mu))$$

$$\sum e^{-\sum_{r\mu} \frac{l_\mu^2(r)}{2\beta J}} \int D\theta(r)\, e^{\sum_{r\mu} i\, l_\mu(r)(\theta(r) - \theta(r+\mu))}$$

$$i\sum_{\mu r} (l_\mu(r) - l_\mu(r-\mu))\,\theta(r)$$

$$\underline{\delta\Big[\sum_\mu (l_\mu(r) - l_\mu(r-\mu))\Big]}$$





$a - c + b - d = 0$

~~[illegible]~~ : 定义 $\vec{l}(r) = (l_x(r), l_y(r)) = (\nabla\times\vec{n})$

$l_x = \partial_y n \qquad l_y = -\partial_x n \qquad \vec{n} = (0, 0, n)$ // $\hat{z}$

$l_x = n(r) - n(r-y) \qquad l_y = -n(r) + n(r-x)$

[illegible]: $l_x(r) - l_x(r-x) + l_y(r) - l_y(r-x) = 0$

dual lattice 上 $l_\mu(r) \in \mathbb{Z}$. [illegible] $\vec{l} = \nabla\times\vec{n}$ 定义[illegible] $\nabla\times\vec{l} \Rightarrow \nabla\cdot\vec{l} = 0$

则: $\begin{cases} l_x(r) = n(r) - n(r-y) \\ l_y(r) = -n(r) + n(r-x) \end{cases}$ 消除 δ [illegible]

$$Z = \int D\theta\, e^{\beta J \cos(\theta_i - \theta_j)} = \sum_{l_{ij}} e^{-\frac{1}{2\beta J} l_{ij}^2 - \delta(\cdots)}$$

[illegible]

$$= \sum_{[illegible]} \sum_{\{n(r)\}} e^{-\frac{1}{2\beta J}(n(r) - n(r-\mu))^2}$$

[illegible] Poisson 求和

[illegible] Honeycomb lattice 中的 BKT 相变

[illegible] ref: wegner duality in generalized Ising Model