泊松求和.

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \tilde{f}(k)$$

證明. $\tilde{f}(k) = \sum_n f(n) e^{2\pi i k}$ $\left(\int dx\, e^{ikx} \sim \delta(k). \right)$

$$\sum_k \tilde{f}(k) = \sum_n f(n) \underbrace{\sum_k e^{2\pi i k}}_{= \sum_m \delta(n-m)} = \sum_n f(n) \sum_m \delta(n-m) = \sum_m f(m).$$

例 $f(n) = e^{-an^2-bn}$ $\tilde{f}(k) = \sqrt{\frac{\pi}{a}} e^{(b-2\pi k i)^2/(4a)} = \sqrt{\frac{\pi}{a}} e^{-(2\pi k - ib)^2/(4a)}.$

對偶

a小	4a大.
收斂慢	收斂快
interaction weak	interactions strong.
高 $T (a \propto \beta)$.	低T

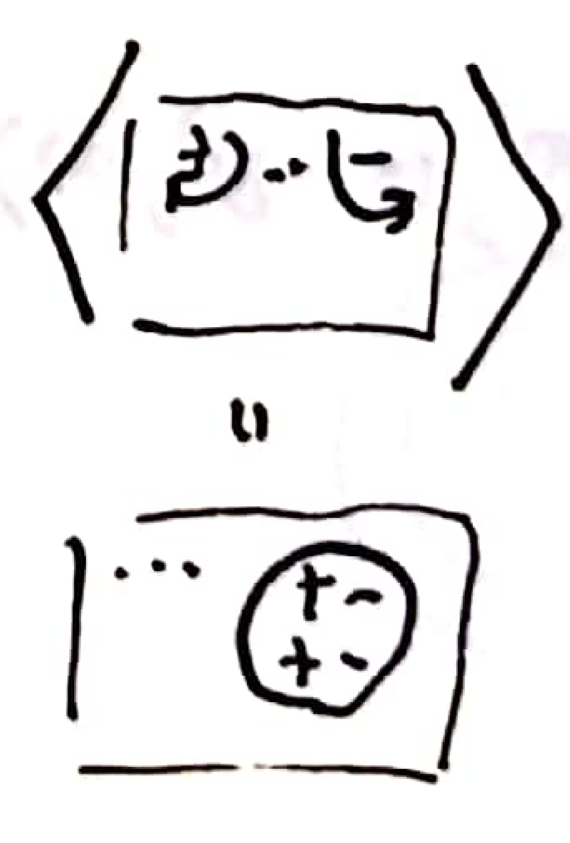
BKT 相變.

1. 半經典處理. $\Rightarrow k_B T_c = \frac{\pi}{2} J$. 方法不具有普適性. 如對1d Ising model的處理.

$$F = \textcircled{U} - TS = J\pi \ln\left(\frac{L}{a}\right) - k_B T \cdot 2\ln\left(\frac{L}{a}\right) = (J\pi - 2k_B T)\ln\left(\frac{L}{a}\right).$$

2. (In 凝聚態場論 simons) 實空間的 RG. $U \to U_{effective}$. (Thaless. Nelson-Kosterlitz.)
3. ~~Wilson~~ Wilson RG. (Ref Wen X.G., shankar. Sine-Gordon)
4. Jose. dual (Nagaosa).

殘差回歸



BKT RG:

$\langle e^{U}\rangle = e^{\langle U\rangle + \frac{1}{2}\langle U^2\rangle_c}$

$\langle e^{J\cos\theta}\rangle = e^{\langle J\cos\theta\rangle}$

$\langle U\rangle = \langle e^{k}\rangle = e^{\langle k\rangle}$.

指数理函数特点.

$U = \cos\theta = e^{i\theta} + e^{-i\theta}$

$U = e^{iA\cos\theta}$ $\quad \langle U\rangle = e^{\langle iA\cos\theta\rangle} = e^{\frac{iA}{2}(\langle e^{i\theta}\rangle + \langle e^{-i\theta}\rangle)} = e^{\frac{iA}{2}(\langle e^{-\frac{\theta^2}{2}}\rangle + \langle e^{-\frac{\theta^2}{2}}\rangle)}$

~~$= e^{\frac{iA}{2}2 e^{-\frac{\langle\theta^2\rangle}{2}}}$~~ $= e^{iA e^{-\frac{1}{2}\langle\theta^2\rangle}}$.

英 P111. X.G. Wen.

$\theta = \theta_< + \theta_>$.

$S = \underbrace{\int \frac{1}{2}K(\partial_\mu\theta_<)^2 - g\cos\theta_< n}_{S_0} + \overbrace{\int \frac{1}{2}K(\partial_\mu\theta_>)^2 - g(\cos(\theta_< n + \theta_> n) - \cos\theta_< n)}^{\text{抵消}}$

懂.

$= \underbrace{\int \frac{1}{2}K(\partial_\mu\theta_>)^2}_{S_0} + \underbrace{\frac{1}{2}g n^2\cos(n\theta_<)\theta_>^2 - ng(\sin\theta_<)\theta_>}_{\text{微扰}}$.

~~$Z = \int D\theta_K e^{-S_0}\int D\theta_>$~~

$Z = \int D\theta_K e^{-S_0} \int D\theta_> e^{-\int\frac{1}{2}K(\partial_\mu\theta_>)^2}\left(e^{-g\underbrace{\int \frac{n^2}{2}(\cos n\theta_K)\theta_>^2 - n(\sin\theta_<)\theta_>}_{U}}\right)$

一階. $\langle\theta_>^2\rangle$. $\quad \langle\theta(x)\theta(y)\rangle = g(x-y)$ $\quad e^{-\frac{1}{2}n^2 g\langle\theta_>^2\rangle}\int\cos(n\theta_<)$

新相互作用 $\quad \int g^2 n^2 \underbrace{(\cos n\theta_<(x))(\cos n\theta_<(x'))}_{\langle\theta_>(x)\theta_>(x')\rangle.}$

$\underline{\qquad}$ $g_>(x-x')$.

二階. $\langle\theta_>^2(x)\,\theta_>^2(x)\rangle$. 無

令 $x' = x+y$.

$g^2 n^2 \int (\cos n\theta_<(x))(\cos n\theta_<(x'))\, g_>(x-x') = g^2 n^2 \int (\cos n\theta_<(x))(\cos n\theta_<(x+y))\, g_>(y)$.

$\cos(n\theta_<(x) + n(\partial\theta_<) y)$ （衰減. $g \to \infty$ 才有貢獻.）

$= \cos(n\theta_<(x))\left(1 - \frac{n^2}{2} y^2 (\partial\theta_<)^2\right) - \sin(n\theta_<)\,(n(\partial_\mu\theta_<) y)$

其中項 $\int \cos^2(n\theta_<(x))\, g_>(y)\, dy$.

$\int \cos(n\theta_<)\sin(n\theta_<)\, y\, g_>(y)\,(\partial_\mu\theta_<)$

奇函數、積分後無貢獻。

$\therefore \Rightarrow -\frac{n^2}{2} g^2 n^2 \int \cos^2(n\theta_<)(\partial_\mu\theta_<)^2 \int y^2 g_>(y)\, dy$

⇓ 動能項。 常數

$\frac{1}{2}(1+\cos(2n\theta_<))$

irrelevant.

$\frac{1}{2}$ 很重要.

一階微擾 相互作用修正、 二階微擾 動能修正.

$S = \int dx \left(\frac{1}{2} K_\lambda (\partial\theta_<)^2 - g_\lambda \cos(n\theta_<)\right)$ ← 二階過程.

$K_\lambda = K + \frac{n^2}{8} g^2 K_2$.

$g_\lambda = g - \frac{1}{2} K(0)$.

跑動耦合常數.

$K(0) = \left(\frac{1}{2\pi}\right)^2 \int \frac{n^2}{K|k|^2} = \frac{n^2}{2\pi K}\ln\frac{\lambda}{\ell}$.

$\left|\frac{2\pi}{\lambda}\right| \le |k| \le \left|\frac{2\pi}{\ell}\right|$

2d BKT. 紫外 紅外 發散
4d. 紫外發散.
需引入 $\ln(\frac{\lambda}{\ell})$ 做截斷
1d Phonon. 紅外發散

$K_2 = \int d^2x\, K(x)|x|^2$

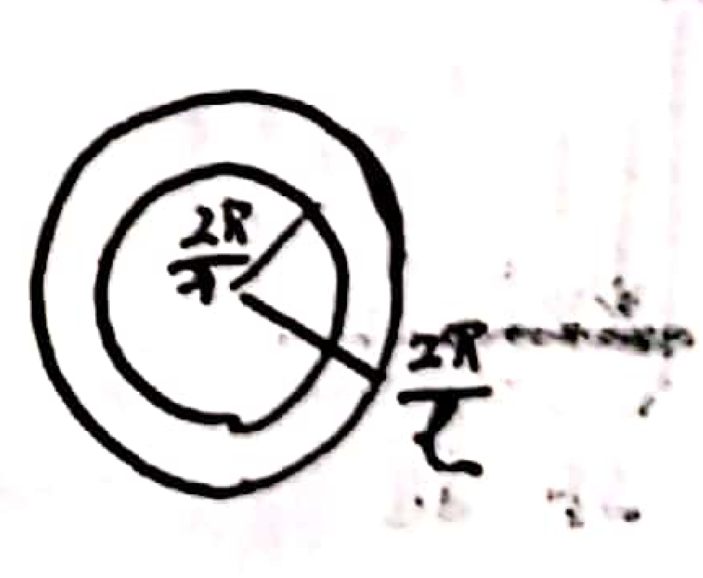
$\langle\theta_>(x)\theta_>(0)\rangle \sim \frac{1}{V}\sum_{k\ \text{shell}} \frac{n^2 e^{i\vec{k}\cdot\vec{x}}}{K|k|^2} = \frac{\lambda-\ell}{\ell}\left(\frac{3n^2\ell^4}{2\pi^4 K}\right)$.

Rescaling

$\int dx\, \frac{K_\lambda}{2}(\partial\theta_<)^2 - g_\lambda\cos(n\theta_<)$

$|k| \le \frac{2\pi}{\ell} \to |k| \le \left(\frac{2\pi}{\lambda}\right)$

$b\cdot\frac{2\pi}{\ell} = \frac{2\pi}{\lambda} \Rightarrow b = \frac{\ell}{\lambda}$.



$\therefore |k| \le b\lambda$. $|k'| \le \lambda$.

$k = bk'$ $\quad x = \frac{1}{b}x'$

$\theta_<(x) = \theta_<\left(\frac{1}{b}x'\right) = Z_b\,\theta_<(x')$.

$\int dx \frac{K_\lambda}{2}(\partial\theta_<)^2 - g_\lambda \cos(n\theta_<)$；前 $= \frac{K_\lambda}{2}\int \left(\frac{1}{b}\right)^2 dx' \left(\frac{\partial}{\partial \left(\frac{1}{b}\right) x'}\theta_<(x)\right)^2$ (cancel.)

$= \frac{K_\lambda}{2}\int dx' \left(\partial'_\mu (\theta_<(x'))\right)^2 = \frac{K_\lambda}{2}\int dx' \left(\partial'_\mu \theta(x')\right)^2$ 後 =

Rescaling $= \frac{K_\lambda}{2}\int dx' \left(\partial'\theta(x')\right)^2 - g_\lambda \left(\frac{1}{b}\right)^2 \int dx' \cos(n\theta(x'))$

$= \underbrace{\frac{1}{2}\left(K + \frac{n^2}{8} g^2 K_2\right)}_{\frac{1}{2}K(b)} \int dx' \left(\partial' \cos(\theta(x'))\right)^2 - \underbrace{\frac{1}{b^2}\left(g - \frac{1}{2}K(0)\right)}_{-g(b).\ b=\frac{1}{\lambda}.} \int dx' \cos(n\theta(x'))$.

變化後相互作用.

$\therefore \begin{cases} \frac{K(1-\epsilon)}{2} = \frac{1}{2}\left(K + \frac{n^2}{8} g^2 K_2\right) \\ -g(1-\epsilon) = -\frac{1}{b^2}\left(g - \frac{1}{2}K(0)\right) \end{cases} \Rightarrow \begin{cases} \frac{dg_\lambda}{db} = -\frac{n^2 g}{4\pi K}. \\ \frac{dK}{db} = \frac{3n^4 g^2 \lambda^4}{16\pi^4 K}. \end{cases}$ 重新定義. $\Rightarrow \begin{cases} K \to \bar{K} \\ g \to \bar{g}\lambda^2. \end{cases}$

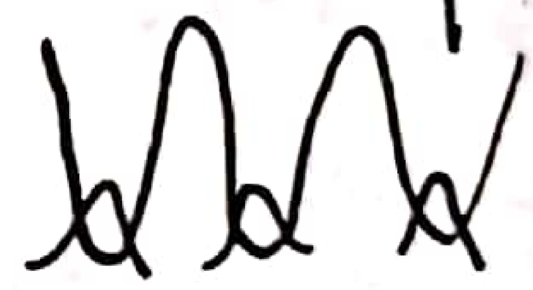
$\Downarrow$

$\Rightarrow$ Thouless $\begin{cases} \frac{d\bar{g}}{db} = \left(2 - \frac{n^2}{4\pi\bar{K}}\right)\bar{g} \\ \frac{d\bar{K}}{db} = \frac{3n^4 \bar{g}^2}{16\pi^4 \bar{K}}. \end{cases}$

兩種結果.

①. $2 - \frac{n^2}{4\pi\bar{K}} > 0 \Rightarrow g \to \infty$. 重整化意義：高能→低能→平均場/經典解.

勢能增強.



②. $2 - \frac{n^2}{4\pi K} < 0 \Rightarrow g \to 0$. $S = \int \frac{1}{2}K(\partial_\mu\theta)^2 \to$ Super F phase

相變點. $2 - \frac{n^2}{4\pi K} = 0$. $K = \beta J$. $\Rightarrow \beta J = \frac{n^2}{8\pi}$. 正方形晶格 $n=4$. $\Rightarrow \frac{2}{\pi}$

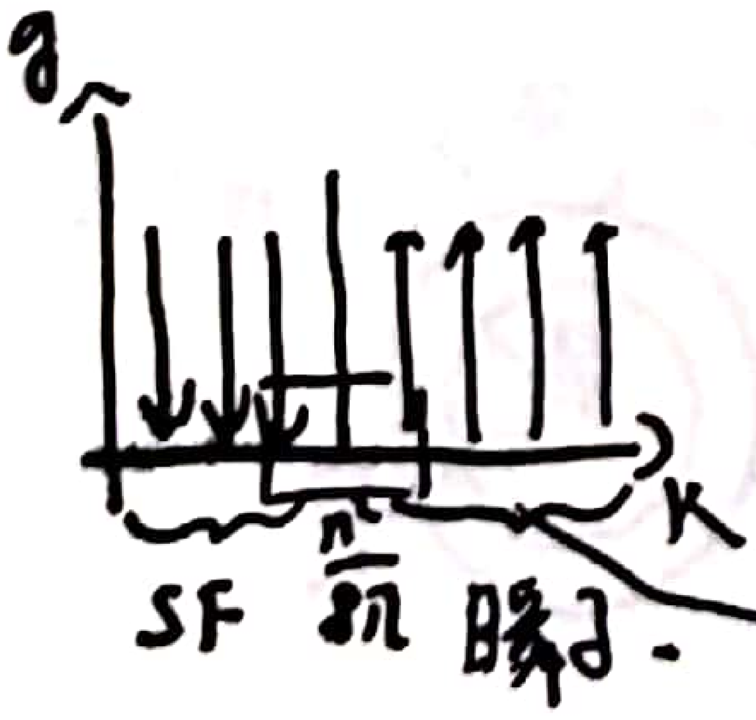
$-J\cos(\theta_i - \theta_j) = \text{constant} + \frac{1}{2}J(\partial\theta)^2$.

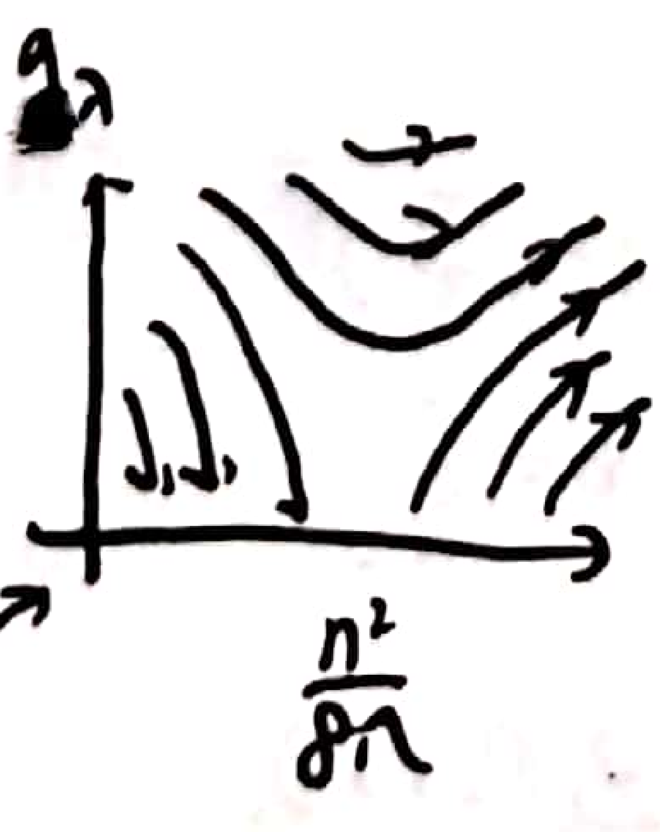
$e^{\beta H} \Leftrightarrow \frac{1}{2}\beta J(\partial\theta)^2 \triangleq \frac{K}{2}$

$\frac{J}{k_B T} = \frac{2}{\pi} \Rightarrow k_B T_c = \frac{\pi}{2} J$.

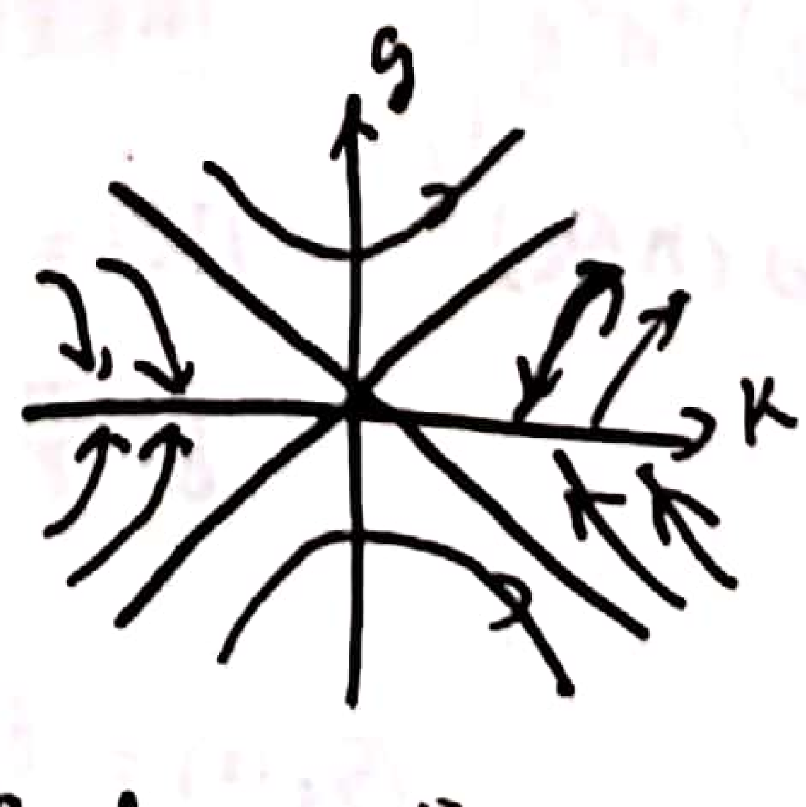
令 $K = 2 - \frac{n^2}{4\pi\bar{K}}$.

$\begin{cases} \frac{dg}{db} = Kg \\ \frac{dK}{db} = \frac{n^2}{4\pi\bar{K}^2}\left(\frac{d\bar{K}}{db}\right). \end{cases}$





瞬子



Shanker P358.

求解

$$Z=\int D\theta_1\cdots D\theta_N\, e^{\beta J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)}$$

Villian Transformation.

$$e^{\beta J\cos(\theta)} \longrightarrow \sum_m e^{\beta J-\frac{1}{2}\beta J(\theta-2\pi m)^2}$$

相變點附近，$\beta J\approx\frac{2}{\pi}\sim 1$. 在其附近這種近似表示很好

Poisson求和

$$\sum_n h(n) = \sum_l \int d\phi\, \tilde{h}(\phi)\, e^{2\pi i l\phi}.$$

交換順序。

$$\sum_l e^{2\pi i l\phi} = \sum_m \delta(\phi-m).$$

不易認識，建議畫圖。

~~$Z=\int D\Theta \exp\left(\frac{\beta J}{2}((\theta_i-\theta_j)-2\pi\phi)^2+2\pi i l_{ij}\phi\right)$~~

$$Z=\int D\theta_1\cdots D\theta_N \exp\left(\beta J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\right)$$

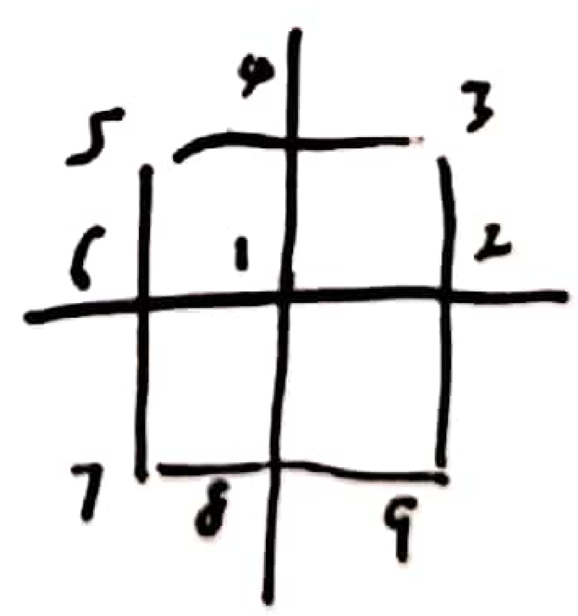
$$=\int D\Theta\left(\sum_m \underbrace{\exp\left(\beta J-\frac{\beta J}{2}(\theta_i-\theta_j-2\pi m)^2\right)}_{h(\phi)}\right)$$

$$=\int D\Theta \sum_{l_{ij}=-\infty}^{\infty}\int d\phi \exp\left(\beta J-\frac{1}{2}\beta J(\theta_i-\theta_j-2\pi\phi)^2+2\pi i l_{ij}\phi\right).$$

$$\int D\Theta \frac{1}{\sqrt{2\pi\beta J}}\sum_{l_{ij}}\exp\left(\beta J+i l_{ij}(\theta_i-\theta_j)-\frac{1}{2\beta J}l_{ij}^2\right).$$

$$\sim\sum_{\{l_{ij}\}} e^{-\frac{1}{2\beta J}l_{ij}^2}\int D\theta \underbrace{\sum_{ij} e^{i l_{ij}(\theta_i-\theta_j)}}$$

$$=\int D\theta\, e^{i l_{ij}\theta_i\cdots}$$



$l_{12}(\theta_1-\theta_2)$

$l_{14}(\theta_1-\theta_4)$

$l_{16}(\theta_1-\theta_6)$

$l_{18}(\theta_1-\theta_8)$、

重新定義符號、

$l_{ij}\longrightarrow$ (i, j) $\quad\therefore l_{ij}=-l_{ji}.$

$l_{ij}(\theta_i-\theta_j)=l_{ji}(\theta_j-\theta_i).$

$\therefore l_{ij}\equiv l_\mu(r).$ $\begin{cases} r\text{、位置} \\ \mu\text{、方向、}\end{cases}$

$$\therefore \frac{l_{ij}^2}{2\beta J}=\frac{l_\mu^2(r)}{2\beta J}.$$

$$i l_{ij}(\theta_i-\theta_j)=i\, l_\mu(r)(\theta(r)-\theta(r+\mu))$$

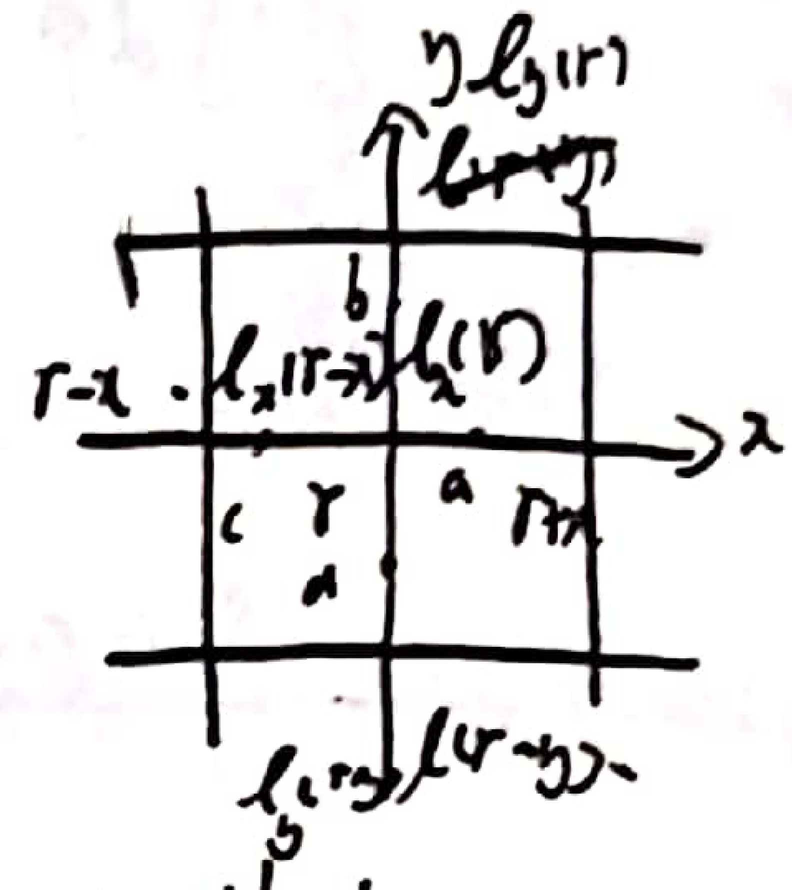
∴

$$= \sum \exp\left(-\sum_{r,\mu} \frac{l_\mu^2(r)}{2\beta J}\right) \int D\theta(r) \exp\left(\sum_{r,\mu} i\, l_\mu(r)\left(\theta(r) - \theta(r+\mu)\right)\right)$$

$\left\{ l_\mu(r)\theta(r+\mu) \Downarrow l_\mu(r-\mu)\theta(r) \right.$

$$= i\sum_{l,r} \left(l_\mu(r) - l_\mu(r-\mu)\right)\theta(r).$$

$$= \qquad \delta\left(\sum_\mu \left(l_\mu(r) - l_\mu(r-\mu)\right)\right)$$



$a - c + b - d = 0.$

$$\Rightarrow Z = \int D\theta\, e^{\beta J \cos(\theta_i - \theta_j)}$$

$$= \sum_{ij} e^{-\frac{1}{2\beta J} l_{ij}} \quad \delta(\cdots)$$

有约束 這裡顾 剛除掉约束。

不易於計算

$$= \sum_{\{n(r)\}} e^{-\frac{1}{2\beta J}(n(r) - n(r-\mu))^2}$$

無约束。

$\vec{l}(r) = (l_x(\vec{r}), l_y(\vec{r}))$

$= (\nabla \times \vec{n}),\ \vec{n} = (0, 0, n).$

$$\begin{cases} l_x = \partial_y n = n(r) - n(r-y). \\ l_y = -\partial_x n = -n(r) + n(r-x). \end{cases}$$

$\Rightarrow l_x(r) - l_x(r-x) + l_y(r) - l_y(r-x) = 0$

對偶晶格上的整數 $l_\mu(r) \in \mathbb{Z}$

必須要求 $\vec{l} = \nabla \times \vec{n}$ 是無散的.

$\nabla \cdot \vec{l} = 0.$