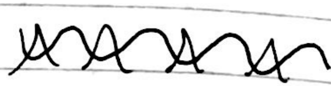
Date Page 笑死老爸

☆ 关于sign 一阶微扰，正负之差，二阶微扰 一定会出现正的

下节课：RG必修，拔爸们（1d经典 → 1+1d量子）

$(\nabla\theta)^2 + g\cos(\theta)$ $g=0$ 自由场

若 $g\to+\infty$  局域模式

⇒ Insulator phase

期末小论文 → !!! 周末完成 !!!

2020年 12月30日 签

①. 结论 $k_B T_c = \frac{\pi}{2} J$

② 方法 possion summation $\sum_{n=-\infty}^{+\infty} f(n) = \sum_{k=-\infty}^{+\infty} \tilde{f}(k)$

证明：$\tilde{f}(k) = \sum_n f(n) e^{2\pi i n k} \Rightarrow \sum_k \tilde{f}(k) = \sum_n f(n) \sum_k e^{2\pi i n k}$ （n整数）

$= \sum_n f(n) \sum_m \delta(n-m) = \sum_m f(m)$

$\therefore f(n) = e^{-an^2-bn}$, $\tilde{f}(k) = \sqrt{\frac{\pi}{a}} e^{(b-2\pi k i)^2/4a} = \sqrt{\frac{\pi}{a}} e^{-(2\pi k - ib)^2/4a}$

BKT 相变 ⟶

有用 a小 收敛慢	a大 快	
a → interaction weak	strong	
若 $a \propto \beta$ 高T	低T	

(dual 对偶)

eg. $a=1.0$, $b=0.53$

$\sum f(n) = 1.90138$

$\sum \tilde{f}(k) = 1.90138$

补：欧拉求积 $\sum_{n=1}^{k} f(n) \to \int_1^k f(x) dx$

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BKT：① 半经典处理 $F = U - TS$ 不一定可靠 ⇝ $k_B T_c = \frac{\pi}{2} J$

② Thouless (1973) 还有 K( ) PRL Simon 书有

⟨日⟩ $U \to U_{eff}$ (real-space RG) 的方法

③ Wilson RG ref：X.G.Wen，shankar

③ Jose → dual Nagaosa 20世纪 CMP 里最重要的篇之一

指数：$\langle e^{U} \rangle = e^{\langle U \rangle + \frac{1}{2}\langle U^2 \rangle_c}$

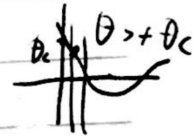
$\langle e^{J\cos\theta} \rangle = e^{\langle J\cos\theta \rangle}$ $\cos\theta = 1 - \frac{\theta^2}{2} + \cdots$

$\langle U \rangle = \langle e^{k} \rangle = e^{\langle k \rangle}$ 一层一层 套

$\therefore U = 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$U = e^{\lambda\cos\theta}$

$\langle U \rangle = e^{\langle \lambda\cos\theta \rangle} = e^{\frac{\lambda}{2}\langle e^{i\theta} \rangle + \langle e^{-i\theta} \rangle}$

$= e^{\frac{\lambda}{2}\left(e^{-\frac{\langle\theta^2\rangle}{2}} + e^{-\frac{\langle\theta^2\rangle}{2}}\right)} = e^{\lambda e^{-\frac{\langle\theta^2\rangle}{2}}}$ 

X.G.Wen $\theta = \theta_< + \theta_>$

$S = \int \frac{k}{2}(\partial_\mu\theta_<)^2 - g\cos(n\theta_<) + \int \frac{k}{2}(\partial_\mu\theta_>)^2 - g[\cos(n(\theta_< + \theta_>))$

$-\cos(n\theta_<)] = \int \frac{k}{2}(\partial_\mu\theta_>)^2 + g\frac{n^2}{2}\cos(n\theta_<)\theta_>^2 - ng\sin(\theta_<)\theta_>$

$\therefore Z = \int D\theta_< e^{-S_0} \int D\theta_> e^{-\int\frac{k}{2}(\partial_\mu\theta_>)^2} \left(e^{-g\int\frac{n^2}{2}\cos(n\theta_<)\theta_>^2 - n\sin(\theta_<)\theta_>}\right.$

U

一阶：$\langle\theta_>^2\rangle$ $\langle\theta(x)\theta(y)\rangle = g(x-y)$

$e^{-\frac{n^2}{2} g \langle\theta_>^2\rangle} \int \cos(n\theta_<)$ const.

Ruitai paper Ruitai paper

$\left\{\begin{array}{l} 4d^{(3+1)} \text{ 出外发散} \\ 2d \text{ 12外1日叶发久} \end{array}\right.$ (区别)

二阶: $\langle \theta^2_>(x), \theta^2_>(x') \rangle$ $\langle \theta_>(x), \theta_>(x') \rangle = g_>(x-x')$ 已有

排掉 $\langle \theta^2_>(x), \theta_>(x') \rangle^{\times}$, $\langle \theta_>(x), \theta^2_>(x') \rangle^{\times}$

$\int g^2 n^2 \cos(n\theta_<(x))\cos(n\theta_<(x')) \rangle$ 非局域相互作用

$\downarrow$ Taylor 展开

令 $x' = (x+y)$

$g n^2 \int \cos(n\theta_<(x))\cos(n\theta_<(x')) \, g_>(x-x')$

$= g^2 n^2 \int \cos(n\theta_<(x))\cos(n\theta_<(x+y)) \, g_>(y)$ 衰减, 只有 $y \to 0$ (小) 才有贡献

$\downarrow$

$\cos(n\theta_<(x) + n(\partial\theta_<)y)$

$= \cos(n\theta_<) \left[1 - \frac{n^2}{2} y^2 (\partial\theta_<)^2\right] - \sin(n\theta_<)(n(\partial\theta_<))$

$\Downarrow \left\{\begin{array}{l} \cos^2(n\theta_<) \, g_>(y) \, dx\, dy \\ \cos^2(n\theta_<)(\partial_\mu\theta_<)^2 \underline{\int y^2 g_>(y)\, dy} \text{ 常数} \end{array}\right.$

$\downarrow \frac{1}{2}(1 + \underbrace{\cos(2n\theta_<)}_{\text{irrelevant}})$

一阶对相互作用修正, 二阶对动能修正 (以后 θ, 动能修正意味着对波函数修正)

$\therefore$ 最后 $S = \int dx \left[\frac{k_\lambda}{2}(\partial\theta_<)^2 - g_\lambda \cos(n\theta_<)\right]$

$k_\lambda = k + \underbrace{\frac{n^2}{8} g^2 k_2}_{\text{二阶主导}}$ $\quad g_\lambda = g - \frac{1}{2}k(0)$, $k(0) = (\frac{1}{2\pi})^2$

$\rightarrow \int_{\frac{2\pi}{\lambda} \le |k| \le 2\pi}$ $\frac{n^2}{k|k|^2} = \frac{n^2}{2\pi k} \ln\left(\frac{\lambda}{2}\right)$

$k_2 = \int d^2x \, k(x) |x|^2 = \int d^2x \, \langle \theta_>(x)\theta_>(0) \rangle |x|^2 \sum_{k\,\text{shell}} \frac{1}{V} \frac{n^2 e^{ik\cdot x}}{k|k|^2}$

$= \frac{\lambda - 2}{2}\left(\frac{3n^2 z^4}{2\pi^4 k}\right)$

但完全重整化之后 还要做 rescaling

$\int dx \frac{k_\lambda}{2}(\partial\theta_<)^2 - g_\lambda \cos(n\theta_<)$ $\quad |k| \le \frac{2\pi}{2} \rightarrow |k| \le \left(\frac{2\pi}{\lambda}\right)$ $\lambda > 2$

$b\frac{2\pi}{2} = \frac{2\pi}{\lambda} \Rightarrow b = \frac{2}{\lambda} \Rightarrow |k| \le b\Lambda$, $k = bk'$ $|k'| \le \Lambda$

$x = \frac{1}{b}x'$ $\quad \theta_<(x) = \theta_<(\frac{1}{b}x') = z\theta_<(x')$

$\therefore = \frac{k_\lambda}{2}\int (\frac{1}{b})^2 dx' \left(\frac{\partial}{\partial(\frac{1}{b}x')}\theta_<(x)\right)^2$

$= \frac{k_\lambda}{2}\int dx' (\partial_\mu(\theta_<(x')))^2$ (不会改变, 是相同, 与 b 无关)

第二项 $= -g_\lambda (\frac{1}{b})^2 \int dx' \cos(n\theta_<(x'))$ $\quad z = 1$

$= \boxed{-g_\lambda(\frac{1}{b})^2} \int dx' \cos(n\theta(x'))$

$\Rightarrow$ rescaling $= \frac{k_\lambda}{2}\int dx' (\partial\theta(x'))^2 - g_\lambda(\frac{1}{b})^2 \int dx' \cos(n\theta(x'))$

$= \frac{1}{2}(k + \frac{n^2}{8}g^2 k_2) \int dx' (\partial'\theta(x'))^2 - (g - \frac{1}{2}k(0)) \left(\frac{1}{b}\right)^2 \int dx' \cos(n\theta(x'))$

令 $b = \left(\frac{2}{\lambda}\right)$, 由变化后相互作用 $\frac{1}{2}k(b) = \frac{1}{2}(k + \frac{n^2}{8}g^2 k_2)$

$g(b) = -(g - \frac{1}{2}k(0))(\frac{1}{b})^2$

$\therefore b \approx \frac{2}{\lambda} = 1 - \varepsilon \Rightarrow \left\{\begin{array}{l} \frac{k(1-\varepsilon)}{2} = \frac{1}{2}(k + \frac{n^2}{8}g^2 k_2) \\ g(1-\varepsilon) = -\frac{1}{b^2}(g - \frac{1}{2}k(0)) \end{array}\right.$

$\Rightarrow$ 最后 $\left\{\begin{array}{l} \frac{dg_\lambda}{db} = -\frac{n^2 g}{4\pi k} \\ \frac{dk}{db} = \frac{3n^4 g^2 \lambda^4}{16\pi^4 k} \end{array}\right.$ $\Rightarrow \frac{dg}{dk} = -(\,) \frac{g}{k} \frac{k}{g^2\lambda^4} = -(\,) \frac{1}{g\lambda^4}$

和 λ 有关系

Ruitai paper Ruitai paper

重定义 $\begin{cases} k \to \bar{k} \\ g \to \bar{g}\lambda^2 \end{cases}$ → K 有些人这样定义

$$\begin{cases} \dfrac{d\bar{g}}{db} = \boxed{\left(2 - \dfrac{n^2}{4\pi\bar{k}}\right)} \bar{g} \\ \dfrac{d\bar{k}}{db} = \dfrac{3n^4\bar{g}^2}{16\pi^4\bar{k}} \end{cases}$$

Thouless 给出(最早)

$\dfrac{dy}{db} = Ay \Rightarrow y = e^{Ab}$ $\begin{cases} A>0 \\ A<0 \end{cases}$

① $2 - \frac{n^2}{4\pi k} > 0 \Rightarrow g \to \infty$ 忽略 $(\partial\theta)^2 - g\cos(\theta)$

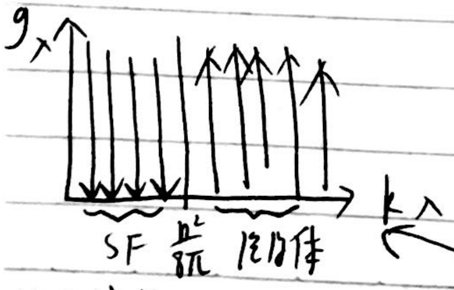
RG → 高 → 低 → 平均场 而 $g\cos(\theta)$ 非常大 ⇒ 回到 lattice

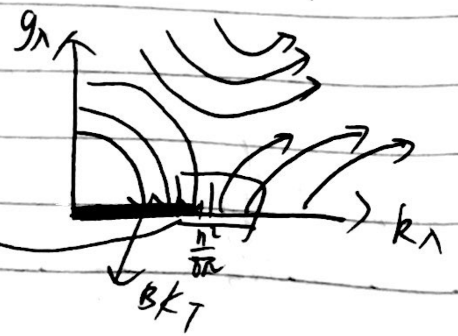
② $2 - \frac{n^2}{4\pi k} < 0 \Rightarrow g \to 0$

只剩 $S = \int (\partial\theta)^2 \Rightarrow$ SF phase

相变在 $\boxed{2 - \frac{n^2}{4\pi k}} = 0$ $k = \frac{n^2}{8\pi}$ $k = \beta J$

$\Rightarrow \beta J = \frac{n^2}{8\pi}$ 一般 $n = 4$, $\Rightarrow \boxed{k_B T_c = \frac{\pi}{2} J}$



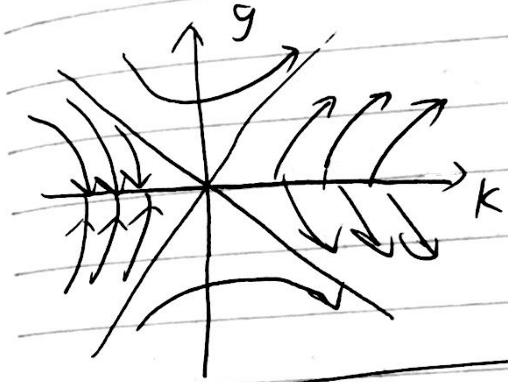


无穷!

shankar P358 Fig. [illegible].2

$$\begin{cases} \dfrac{dg}{db} = kg \\ \dfrac{dk}{db} = \dfrac{n^2}{4\pi k^2}\left(\dfrac{dk}{db}\right) \end{cases}$$

fixed point ⇒ 形成 node line



RG 很复杂式, 但是要会写

下节课 $\boxed{\text{Wannier-Kramers duality}}$ 难 → 技巧性很强, 一套物理名称 也可以写

求解 $Z = \int_{-\infty}^{+\infty} D\theta_1 \cdots D\theta_N \, e^{\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$ (周期性函数)

Villain 变换 $e^{\beta J\cos(\theta)} \to \sum_{m=-\infty}^{m=+\infty} e^{\beta J - \frac{1}{2}\beta J(\theta - 2\pi m)^2}$ (非常好的近似)

, 在相变点附近 $\beta J = \frac{2}{\pi} \sim 1$

posson 求和 $\sum_m h(m) = \sum_l \int d\phi \, \tilde{h}(\phi) e^{2\pi i l\phi}$

$= \int d\phi \sum_l \tilde{h}(\phi) e^{2\pi i l \phi}$

而 $\sum_l e^{2\pi i l\phi} = \sum_m \delta(\phi - m)$ 建议画图, 数值检查

$\therefore Z = \int D\theta \exp\left[-\frac{\beta J}{2}(\theta_i - \theta_j - 2\pi\phi)^2 + 2\pi i l_{ij}\phi\right]$ $h(\theta)$

$= \int D\theta \left(\sum_m e^{\beta J - \frac{\beta J}{2}(\theta_i - \theta_j - 2\pi m)^2}\right)$ 只是求和

$= \int D\theta \sum_{l_{ij}}^{+\infty} \int d\phi \, e^{\beta J - \frac{1}{2}\beta J(\theta_i - \theta_j - 2\pi\phi)^2 + 2\pi i l_{ij}\phi}$

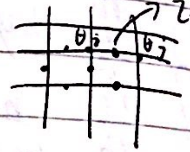
$(\theta_i - \theta_j)^2 - 2(\theta_i - \theta_j)2\pi\phi + 4\pi^2\phi^2$

$$\int d\theta\, e^{2\pi i \theta k} = \delta(k)$$

Date: ___ Page ___ → 用了 $\sum_{l_{ij}} e^{2\pi i \theta_{ij} \phi} = \sum_m \delta(\phi - m)$

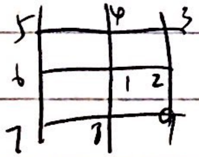
$$= \int D\theta \frac{1}{\sqrt{2\pi\beta J}} \sum_{l_{ij}} e^{\beta J + i l_{ij}(\theta_i - \theta_j) - \frac{1}{2\beta J}\theta_{ij}^2}$$

↓ 整数



在每个 link 上有一个 $l_{ij} \in \mathbb{Z}$

$$= \sum e^{-\sum \frac{l_{ij}^2}{2\beta J}} \int D\theta\, e^{i l_{ij}(\theta_i - \theta_j)}$$



$$l_{12}(\theta_1 - \theta_2) + l_{16}(\theta_1 - \theta_6)$$

$$+ l_{14}(\theta_1 - \theta_4) + l_{18}(\theta_1 - \theta_8)$$

$$\therefore e^{i\theta_1(l_{12} + l_{14} + l_{16} + l_{18})}$$

重新定义符号 $l_{ij} = -l_{ji}$ $\quad l_{ij}(\theta_i - \theta_j) = l_{ji}(\theta_j - \theta_i)$

令 $l_{ij} \to l_\mu(r)$ $\begin{cases} r \text{ site } m \text{ do } x \\ \mu \text{ 方向} \end{cases}$

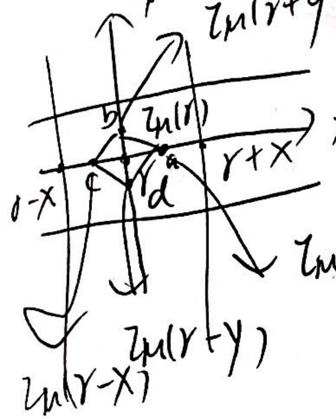
$$\therefore \frac{l_{ij}^2}{2\beta J} + i l_{ij}(\theta_i - \theta_j) = \frac{l_\mu^2(r)}{2\beta J} - i l_\mu(r)(\theta(r) - \theta(r+\mu))$$

$$\therefore = \sum e^{-\sum_\mu \frac{l_\mu^2(r)}{2\beta J}} \int D\theta(r)\, e^{\sum_{r\mu} i l_\mu(r)(\theta(r) - \theta(r+\mu))}$$

$$i \sum_r (l_\mu(r) - l_\mu(r-\mu))\theta(r)$$

积分 $\Rightarrow \delta[\sum_\mu (l_\mu(r) - l_\mu(r-\mu))] \to$ 一堆（完全独立 ↓ 不完全独立，限制了）

$\begin{cases} \text{参数} \\ \text{生活在 dual Lattice 上} \end{cases}$



$l_\mu(r+y)$, $a - c + b - d = 0$

$l_\mu(r)$, $r+x$, $r-x$, $l_\mu(r+y)$, $l_\mu(r-y)$, $l_\mu(r-x)$

在 x $\vec{l}(r) = (l_x(\vec{r}), l_y(\vec{r}))$ vector

$$= (\nabla \times \vec{n})$$

$$\Rightarrow \begin{cases} l_x = d_y n & (n \text{ 场的旋度}) \\ l_y = d_x n \end{cases}$$

$$= \begin{cases} n(r) - n(r-y) \\ n(r) - n(r-x) \end{cases}$$ 如果是在 x

$$\vec{n} = (0, 0, n)$$

则 $l_x(r) - l_x(r-x) + l_y(r) - l_y(r-x) = 0$ 自动满足

在 dual Lattice 上定义另一个整数，但这个整数不需要满足 $\boxed{\nabla \cdot \vec{l} = 0} \Rightarrow$ 无散场

$l_\mu(r) \in \mathbb{Z}$

→ 很强约束

$$\Rightarrow \sum e^{-\sum_\mu \frac{l_\mu^2(r)}{2\beta J}} \delta[\sum_\mu (l_\mu(r) - l_\mu(r-\mu))]$$

$\Rightarrow$ 现在可以找到相对应的 无散 $\Rightarrow$ 找到一组旋度场

这时约束天然消失 $\Rightarrow$ 前面 δ 函数约束

$$\therefore Z = \int D\theta \, e^{\beta J \cos(\theta_i - \theta_j)}$$

$$= \sum_{Z_{ij}} e^{Z_{ij}^{2}} \delta(1 \ldots)$$

有约束!

不好算

史诗级工作 $= \sum_{\{n(r)\}} e^{-\frac{1}{2\beta J}\left(n(r) - n(r-\mu)\right)^2}$

无约束

...

然... 还是有关联

$\Rightarrow$ 对偶 possion 求和