Problem 3.4.1.
Note that, near T_c, the action S_{eff} in eqn (3.4.1) can be approximated by

$$S_{\text{eff}} = \beta \int \mathrm{d}^d \boldsymbol{x} \; \left[\frac{1}{2m^*} |\partial_{\boldsymbol{x}} \varphi_c|^2 + a(T - T_c)|\varphi_c|^2 + b|\varphi_c|^4 \right] + O(|\varphi_c|^6)$$

because the order parameter φ_c is small near the critical point (or the phase transition point) at $T = T_c$. In the mean-field (or semiclassical) approach to the phase transition and the critical point, we first find the mean-field solution that minimizes the action. We then assume that fluctuations around the mean-field solution are small and expand the action to quadratic order in the fluctuations. The quadratic approximation of S_{eff} can be used to calculate various correlations.

1. Use the mean-field approach to calculate the decay exponent γ in $\langle \varphi_c(\boldsymbol{x}) \varphi_c^*(0) \rangle \sim 1/|\boldsymbol{x}|^\gamma$ at the critical point.
2. The above result is not always valid because the classical theory may break down. Repeat the discussions at the end of Section 3.3.8 (i.e. write S_{eff} in the form $g^{-1}\tilde{S}$ with dimensionless $\tilde{S}$) to see when the mean-field approach can correctly describe the critical point and when the critical point is controlled by strong fluctuations; that is, to find the upper critical point d_c.
3. Here we would like to introduce the concept of relevant versus irrelevant perturbations. We know that above the upper critical dimension the classical theory correctly describes the critical point at the phase transition. Now we add a perturbation $\beta \int \mathrm{d}^d \boldsymbol{x} \; c|\varphi|^\sigma$ to the effective action S_{eff}. If the perturbation is important and modifies the critical exponents, then we say it is a relevant perturbation. If the perturbation becomes vanishingly small near the critical point, then we say it is an irrelevant perturbation. Use the same scalings that you found above to see how the perturbation $\beta \int \mathrm{d}^d \boldsymbol{x} \; c|\varphi|^\sigma$ modifies the scaled action $\tilde{S}$. Determine for what range of σ the perturbation is relevant, and for what range of σ the perturbation is irrelevant.

3.5 Renormalization group

3.5.1 Relevant and irrelevant perturbations

- **Relevant perturbations change the long-distance (or low-energy) behavior of a system, while irrelevant perturbations do not.**
- **We can use the scaling dimension of a perturbation to determine if the perturbation is relevant or irrelevant.**

In the above discussion of the KT transition, we note that, when $e^{-S_c} \ll 1$, the vortex fluctuations are just 'small perturbations'. However, if $h < 2$, then no matter how small e^{-S_c} is, the vortex fluctuations always destroy the algebraic long-range correlation of $\left\langle e^{\,i\theta(x)} e^{-i\theta(0)} \right\rangle$. Thus, when $h < 2$, the perturbation of including the vortex fluctuations is called a relevant perturbation. When $h >$

2, the perturbation is called an irrelevant perturbation, and, when $h = 2$, the perturbation is called a marginal perturbation. In the following we would like to discuss relevant/irrelevant/marginal perturbations in a more general set-up.

Consider a theory described by the action

$$S = S_0 + \int \mathrm{d}^d x a O(x)$$

where aO is a perturbation. We assume that S_0 has a Z_2 symmetry and $O \to -O$ under the Z_2 transformation. As a result, $\langle O \rangle = 0$ when $a = 0$. We also assume that, for large x,

$$\langle O(x) O(0) \rangle = \frac{1}{|x|^{2h}} \tag{3.5.1}$$

when $a = 0$. Here h is called the scaling dimension of the operator O (the scaling dimension of $1/x$ is defined as 1). Equation (3.5.1) also defines the normalization of the operator O.

At the second-order perturbation, the partition function is given by

$$Z = Z_0 \int \mathrm{d}^d x \mathrm{d}^d y \, a^2 \langle O(x) O(y) \rangle$$

where Z_0 is the zeroth-order partition function. We see that the second-order perturbation changes the effective action by

$$\Delta S_{\text{eff}} = -\ln Z + \ln Z_0 = -2 \ln g + 2h \ln L - 2d \ln L$$

Note that, when $h < d$ and $L > \xi = a^{-1/(d-h)}$, we have $\Delta S_{\text{eff}} < 0$. Thus, the system prefers to have two $O(x)$ insertions. When $L \gg \xi$, the system wants to have two $O(x)$ insertions for each ξ^d volume. We see that, if we are interested in correlation functions at length scales beyond ξ, then the perturbation is always important. We conclude that the perturbation $\int \mathrm{d}^d x \, O(x)$ is relevant if the scaling dimension of $O(x)$ is less than d. In this case, $O(x)$ is called a relevant operator. If the scaling dimension of $O(x)$ is greater than (or equal to) d, then $O(x)$ is called an irrelevant (marginal) operator. An easy way to remember this result is to note that the perturbation $\int \mathrm{d}^d x \, O(x)$ is relevant if $\int \mathrm{d}^d x \, O(x)$ has a dimension less than zero.

The concept of scaling dimension also allows us to use dimensional analysis to estimate the induced $\langle O \rangle$ by a finite perturbation aO. As the scaling dimension of $\delta S = \int \mathrm{d}^d x \, aO$ is zero by definition, the coefficient a has a scaling dimension $[a] = d - [O] = d - h$. When aO is an irrelevant perturbation (i.e. when $h > d$), the induced $\langle O \rangle$ is proportional to a. We have

$$\langle O \rangle \sim a l^{d-2h}$$

where l is the short-distance cut-off. When aO is a relevant perturbation (i.e. when $h < d$), the induced $\langle O \rangle$ is more than $a l^{d-2h}$. By matching the scaling dimensions, we find that

$$\langle O \rangle \sim a^{h/(d-h)} = a\xi^{d-2h}. \tag{3.5.2}$$

Problem 3.5.1.
The effective action

$$S_{\text{eff}} = \beta \int \mathrm{d}^d \boldsymbol{x} \ \frac{1}{2m^*} |\partial_{\boldsymbol{x}} \varphi|^2$$

describes a critical point. Calculate the scaling dimensions of φ, φ^2, $|\varphi|^2$, and $|\varphi|^4$. Show that, below a spatial dimension d_0, the perturbation $\int \mathrm{d}^d \boldsymbol{x}\ b|\varphi|^4$ becomes a relevant perturbation. Find the value of d_0 and explain why d_0 is equal to the upper critical dimension d_c of

$$S_{\text{eff}} = \beta \int \mathrm{d}^d \boldsymbol{x} \ [\frac{1}{2m^*} |\partial_{\boldsymbol{x}} \varphi|^2 + a(T - T_c)|\varphi|^2 + b|\varphi|^4]$$

3.5.2 The duality between the two-dimensional XY-model and the two-dimensional clock model

- The vortices in the two-dimensional XY-model can be viewed as particles. The field theory that describes those particles is the two-dimensional clock model.

In order to study the vortex fluctuations of the XY-model in more detail, we would like to map the two-dimensional XY-model to the Z_1 two-dimensional clock model. A generic Z_n clock model is defined by

$$S = \int \mathrm{d}^2 \boldsymbol{x} \ \left(\frac{\kappa}{2} (\partial_{\boldsymbol{x}} \theta)^2 - g \cos(n\theta) \right) \tag{3.5.3}$$

When $g = 0$ the clock model is the XY-model at finite temperatures. The action is the energy divided by the temperature: $S = \beta E$. The $g\cos(n\theta)$ term (explicitly) breaks the $U(1)$ rotational symmetry. If we view $(\cos(\theta), \sin(\theta)) = (S_x, S_y)$ as the two components of a spin, then, for $n = 1$, the $g\cos(\theta)$ term is a term induced by a magnetic field in the S_x direction. For general n, the clock model has a Z_n symmetry: $\theta \to \theta + \frac{2\pi}{n}$.

To show the duality relation, we consider the following partition function of eqn (3.5.3) with $n = 1$:

$$\begin{aligned} Z &= \int \mathcal{D}\theta \sum_k \frac{1}{k!} \left(g \int \mathrm{d}^2 \boldsymbol{x} \ \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2} \right)^k \mathrm{e}^{-\int \mathrm{d}^2 \boldsymbol{x} \ \frac{\kappa}{2} (\partial_{\boldsymbol{x}} \theta)^2} \\ &= Z_0 \sum_k \frac{1}{k!k!} \int \prod_{i=1}^{2k} \mathrm{d}^2 \boldsymbol{r}_i \ \mathrm{e}^{-2kS_c} \mathrm{e}^{\sum_{i<j}^{2k} 2h q_i q_j \ln \frac{r_{ij}}{l}} \end{aligned} \tag{3.5.4}$$

where Z_0 is the partition function of $S = \int \mathrm{d}^2 \boldsymbol{x} \ \frac{\kappa}{2} (\partial_{\boldsymbol{x}} \theta)^2$. Each term in the summation arises from the correlation $\langle \mathrm{e}^{\mathrm{i}\theta(\boldsymbol{r}_1)} \ldots \mathrm{e}^{\mathrm{i}\theta(\boldsymbol{r}_k)} \mathrm{e}^{-\mathrm{i}\theta(\boldsymbol{r}_{k+1})} \ldots \mathrm{e}^{-\mathrm{i}\theta(\boldsymbol{r}_{2k})} \rangle$. Also,

$q_i = 1$ for $1 \leqslant i \leqslant k$, $q_i = -1$ for $k+1 \leqslant i \leqslant 2n$, $e^{-S_c} = g/2$, and

$$h = \frac{1}{4\pi\kappa}$$

Equation (3.5.4) is identical to the partition function (3.4.5) of the XY-model (3.4.2) if $\frac{1}{4\pi\kappa} = \pi\eta$. So, the Z_1 clock model (3.5.3) is equivalent to the XY-model (3.4.2) (with vortices) if $\frac{1}{2\pi\kappa} = 2\pi\eta$. The vortex in the XY-model is mapped to $e^{i\theta(x)}$ in the clock model. Similarly, the vortex in the clock model is mapped to $e^{i\theta(x)}$ in the XY-model. The vortex in the clock model has a scaling dimension $\pi\kappa$. The $e^{i\theta(x)}$ operator in the XY-model has a scaling dimension $1/4\pi\eta$. The relation $\frac{1}{2\pi\kappa} = 2\pi\eta$ ensures that the two scaling dimensions agree with each other.

We know that the $U(1)$ symmetry in the XY-model does not allow the $e^{i\theta}$ term to appear in the action. Using the above mapping, we see that the corresponding clock model must not allow vortex fluctuations. Allowing the vortex fluctuations in the clock model corresponds to explicitly breaking the $U(1)$ symmetry in the dual XY-model. We see that there are two different types of clock model, the one with vortex fluctuations and the one without vortex fluctuations. As the corresponding dual models have different symmetries, the two types of clock model have very different properties. We will call the clock model with vortex fluctuations the compact clock model, and the one without vortex fluctuations the non-compact clock model. The XY-model with vortices is mapped to an Z_1 non-compact clock model. Such a mapping allows us to study the KT transition in the XY-model by studying the transition in the corresponding non-compact clock model.

3.5.3 Physical properties of the clock model

- A field theory model is not well defined unless we specify the short-distance cut-off.
- Ginzburg–Landau theory, containing strong vortex fluctuations, cannot describe phase transitions in the non-compact clock model.

In this section, we will discuss possible phase transitions in a generic Z_n clock model. When g is large, the field θ is trapped by one of the minima of the potential $-g\cos(n\theta)$. We believe that, in this case, the model is in a phase that spontaneously breaks the Z_n symmetry. When both κ and g are small, the fluctuation of θ is strong. We expect that the model will be in a Z_n-symmetric phase.

Despite sounding so reasonable, the above statements do not really make sense. This is because g has a dimension. It is meaningless to talk about how large g is. What is worse is that g is the only parameter in the model that has a non-trivial dimension. So, we cannot make a dimensionless combination to determine how large g is.

To understand the importance of the $g\cos(n\theta)$ term in a physical way, we would like to ask how big the $e^{in\theta}$ operator is. One physical way to answer this question is to examine the correlation of $e^{in\theta}$ for the XY-model $S = \int d^2\boldsymbol{x}\, \frac{\kappa}{2}(\partial_{\boldsymbol{x}}\theta)^2$. The correlation is given by (see eqn (3.3.25))

$$\left\langle e^{in\theta(\boldsymbol{x})} e^{-in\theta(0)} \right\rangle = (l/|\boldsymbol{x}|)^{n^2/2\pi\kappa}. \tag{3.5.5}$$

One big surprise is that the correlation depends on the short-distance cut-off l. Thus, the magnitude (or the importance) of the operator $e^{in\theta}$ is not even well defined unless we specify the cut-off l. This illustrates the point that *to have a well-defined field theory we must specify a short-distance cut-off* l. To stress this point, we would like to make the l dependence explicit and write the action as

$$S = \int d^2\boldsymbol{x}\ \left(\frac{\kappa_l}{2}(\partial_{\boldsymbol{x}}\theta_l)^2 - g_l\cos(n\theta_l)\right) \tag{3.5.6}$$

The short-distance cut-off is introduced by requiring that the θ_l field does not contain any fluctuations with wavelengths shorter than l:

$$\theta_l(\boldsymbol{x}) = \int_{|\boldsymbol{k}|<2\pi/l} d^2\boldsymbol{k}\ \theta_{\boldsymbol{k}} e^{i\boldsymbol{x}\cdot\boldsymbol{k}}$$

We see that a well-defined clock model (3.5.6) contains three parameters κ_l, g_l, and l. So, the clock model really contains two dimensionless parameters κ_l and $\bar{g}_l = g_l l^2$.

We can now make sensible statements. When $\bar{g}_l \gg 1$, we believe that the model is in a phase that spontaneously breaks the Z_n symmetry. When both κ_l and $\bar{g}_l$ are much less than 1, we expect the model to be in a Z_n-symmetric phase.

A non-trivial limit is when $\kappa_l \gg 1$ and $\bar{g}_l \ll 1$. Is the model in the Z_n-symmetric phase or in the Z_n-symmetry-breaking phase? The concept of relevant/irrelevant perturbation is very helpful in answering this question. If we treat the $g_l\cos(n\theta)$ term as a perturbation to the XY-model, then, from eqn (3.5.5), we see that the scaling dimension of $e^{in\theta}$ in the XY-model is $[e^{in\theta}] = n^2/4\pi\kappa_l$. Thus, the $g_l\cos(n\theta)$ term is relevant when $n^2/4\pi\kappa_l < 2$ and irrelevant when $n^2/4\pi\kappa_l > 2$.

This result is reasonable. When κ_l is small, the fluctuations of θ are strong. This makes the $g_l\cos(n\theta)$ term average to zero and be less effective. Hence the perturbation $g_l\cos(n\theta)$ is irrelevant. When $g_l\cos(n\theta)$ is irrelevant and $\bar{g}_l$ is small, we can drop the $g_l\cos(n\theta)$ term when we calculate long-range correlations. This suggests that, at long distances, we not only have the Z_n symmetry, but we also have the full $U(1)$ symmetry when both κ_l and $\bar{g}_l$ are small.

When κ is large, the fluctuations of θ are weak. This makes the $g_l\cos(n\theta)$ term a relevant perturbation. The effect of the $g_l\cos(n\theta)$ term becomes important at long distances, no matter how small $\bar{g}_l$ is. Thus, we expect the system to be trapped in one of the n minima of the potential term and the Z_n symmetry is spontaneously broken, even for small $\bar{g}_l$.

After realizing that the clock model can have a Z_n-symmetry-breaking phase and a Z_n-symmetric phase, the next natural question is how do the two phases transform into each other? One way to understand the transition is to introduce a complex order parameter $\varphi(\boldsymbol{x}) = \langle e^{i\theta(\boldsymbol{x})}\rangle$ and write down a Ginzburg–Landau effective theory for the transition

$$S_{GL} = \int d^2\boldsymbol{x}[\frac{\gamma}{2}(\partial_{\boldsymbol{x}}\varphi)^2 + a|\varphi|^2 + b|\varphi|^4 - c\mathrm{Re}\varphi^n] \tag{3.5.7}$$

Note that the $c\mathrm{Re}\varphi^n$ term (explicitly) breaks the $U(1)$ symmetry down to Z_n. When $n > 1$, the Ginzburg–Landau theory describes a symmetry-breaking transition as a changes from a positive value to a negative value.

When $n = 1$, the Ginzburg–Landau theory contains no phase transition because there is no symmetry breaking. This seems to suggest that the Z_1 clock model contains no phase transition and the corresponding XY-model contains no TK transition.

So what is wrong? In the Ginzburg–Landau theory, the order parameter has strong amplitude fluctuations near the transition point. A typical configuration of φ contains many points where $\varphi = 0$. So, there are strong vortex fluctuations. The Ginzburg–Landau theory describes the phase transitions in the compact clock model. The Ginzburg–Landau theory does not apply to a non-compact clock model.

3.5.4 Renormalization group approach to the non-compact clock model

- Through the concept of running coupling constants, the renormalization group (RG) approach allows us to see how a theory evolves as we go to long distances or low energies. It is very useful because it tells us the dynamical properties that emerge at long distances or low energies.
- As an effective theory only evolves into a similar effective theory, we cannot use the renormalization group approach to obtain the emergence of qualitatively new phenomena, such as the emergence of light and fermions from a bosonic model.

In this section, to understand the physical properties of the non-compact clock model, we will work directly with the θ field in the clock model.

We note that, if the fluctuations $O(\boldsymbol{x})$ and $O(\boldsymbol{y})$ at different locations fluctuate independently, then the so-called *connected correlation*

$$G_{conn} = \langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle - \langle O(\boldsymbol{x})\rangle\langle O(\boldsymbol{y})\rangle$$

vanishes. So, the connected correlation measures the correlation between the fluctuations of $O(\boldsymbol{x})$ and $O(\boldsymbol{y})$.

When $g_l = 0$, the non-compact clock model always has an algebraic long-range correlation: $\langle e^{in\theta(\boldsymbol{x})} e^{-in\theta(0)}\rangle - \langle e^{in\theta(\boldsymbol{x})}\rangle\langle e^{-in\theta(0)}\rangle \sim \frac{1}{|\boldsymbol{x}|^{n^2/2\pi\kappa_l}}$, regardless of the value of κ_l. The issue here is how the $g_l \cos(n\theta)$ term affects the algebraic long-range correlation.

As discussed in the last section, when $\kappa_l < n^2/8\pi$, $e^{in\theta(\boldsymbol{x})}$ is irrelevant and a small $g_l \cos(n\theta)$ term will not affect the algebraic long-range correlation. When $\kappa_l > n^2/8\pi$, $e^{in\theta(\boldsymbol{x})}$ is relevant. We expect that a $g_l \cos(n\theta)$ term will change the algebraic long-range correlation into a short-ranged one, no matter how small $\bar{g}_l$ is. We see that, for small $\bar{g}_l$, the non-compact clock model has a phase transition at $k_l = n^2/8\pi$. In the following, we will use the RG approach to understand the above phase transition.

We note that the clock model is well defined only after we specify a short-distance cut-off l. The key step in the RG approach is to integrate out the fluctuations between the wavelengths l and λ ($\lambda > l$). This results in a model with a new cut-off λ. To integrate out the short-wavelength θ fluctuations, we first write

$$\theta_l = \theta_\lambda + \delta\theta$$

where $\delta\theta$ only contains fluctuations with wavelengths between l and λ. As the short-wavelength fluctuations $\delta\theta$ are suppressed by the $\kappa(\partial_{\boldsymbol{x}}\delta\theta)^2$ term, we expect $\delta\theta$ to be small and expand the action to second order in $\delta\theta$ as follows:

$$\begin{aligned} S = &\int d^2\boldsymbol{x}\ \left(\frac{\kappa_l}{2}(\partial_{\boldsymbol{x}}\theta_\lambda)^2 - g_l\cos(n\theta_\lambda) + \frac{\kappa_l}{2}(\partial_{\boldsymbol{x}}\delta\theta)^2\right) \\ &+ \int d^2\boldsymbol{x}\ \left(-ng_l\sin(\theta_\lambda)\delta\theta + \frac{n^2}{2}g_l\cos(n\theta_\lambda)(\delta\theta)^2\right) \end{aligned}$$

We treat θ_λ as a smooth background field, and integrate out $\delta\theta$ (this approach is called the background-field RG approach). We obtain the effective action

$$\begin{aligned} S = &\int d^2\boldsymbol{x}\ \left(\frac{\kappa_l}{2}(\partial_{\boldsymbol{x}}\theta_\lambda)^2 - g_l\cos(n\theta_\lambda) + \frac{1}{2}g_l\cos(n\theta_\lambda)K(0)\right) \\ &- \int d^2\boldsymbol{x}d^2\boldsymbol{y}\ \frac{1}{2}(g_l)^2\sin(n\theta_\lambda(\boldsymbol{x}))K(\boldsymbol{x}-\boldsymbol{y})\sin(n\theta_\lambda(\boldsymbol{y})) \end{aligned}$$

where $K(\boldsymbol{x}) = n^2 \langle \delta\theta(\boldsymbol{x})\delta\theta(0)\rangle$. We note that the last term can be rewritten as

$$\int \mathrm{d}^2\boldsymbol{x}\mathrm{d}^2\boldsymbol{y}\, \frac{g_l^2}{4}[\sin(n\theta_\lambda(\boldsymbol{x})) - \sin(n\theta_\lambda(\boldsymbol{y}))]^2 K(\boldsymbol{x}-\boldsymbol{y}) - \int \mathrm{d}^2\boldsymbol{x}\, \frac{g_l^2}{2}\sin^2(n\theta_\lambda)\bar{K}$$
$$= \int \mathrm{d}^2\boldsymbol{x}\mathrm{d}^2\boldsymbol{y}\, \frac{n^2}{8}(g_l)^2 \cos^2(n\theta_\lambda(\boldsymbol{x}))(\partial_{\boldsymbol{x}}\theta_\lambda(\boldsymbol{x}))^2(\boldsymbol{x}-\boldsymbol{y})^2 K(\boldsymbol{x}-\boldsymbol{y})$$
$$- \int \mathrm{d}^2\boldsymbol{x}\, \frac{1}{2}(g_l)^2(\sin(n\theta_\lambda(\boldsymbol{x})))^2\bar{K}$$

where $\bar{K} = \int \mathrm{d}^2\boldsymbol{x}\; K(\boldsymbol{x})$. We see that terms like $\cos(2n\theta_\lambda)$, $(\partial_{\boldsymbol{x}}\theta_\lambda)^2$, $\cos(2n\theta_\lambda)(\partial_{\boldsymbol{x}}\theta_\lambda)^2$, $(\partial_{\boldsymbol{x}}\theta_\lambda)^4$, etc. are generated. RG flow can generate many new terms that are not in the starting action. In fact, any local terms that do not break the Z_n symmetry can be generated. However, the term $\cos(\theta_\lambda)$ is not generated when $n > 1$ because it breaks the Z_n symmetry. For the time being, let us only keep the terms $(\partial_{\boldsymbol{x}}\theta_\lambda)^2$ and $\cos(\theta_\lambda)$ that are already in our starting action.[23] We find that the action of our model becomes

$$S = \int \mathrm{d}^2\boldsymbol{x}\, \left(\frac{\kappa_\lambda}{2}(\partial_{\boldsymbol{x}}\theta_\lambda)^2 - g_\lambda\cos(n\theta_\lambda)\right) \tag{3.5.8}$$

where λ is the new cut-off. The effective coupling constants depend on the cut-off λ and are called *running coupling constants*. They are given by (assuming that $\frac{\lambda-l}{l} \ll 1$)

$$g_\lambda = g_l - \frac{1}{2}K(0), \qquad \kappa_\lambda = \kappa_l + \frac{n^2}{8}g_l^2 K_2$$

$$K(0) = \int_{2\pi/\lambda<|\boldsymbol{k}|<2\pi/l} \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\frac{n^2}{\kappa_l|\boldsymbol{k}|^2} = \frac{n^2}{2\pi\kappa_l}\ln\frac{\lambda}{l}$$

$$K_2 \equiv \int \mathrm{d}^2\boldsymbol{x}\, |\boldsymbol{x}|^2 K(\boldsymbol{x}) = \int_{2\pi/\lambda<|\boldsymbol{k}|<2\pi/l} \mathrm{d}^2\boldsymbol{x}\frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\frac{n^2\boldsymbol{x}^2\,\mathrm{e}^{\,\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-0^+|\boldsymbol{x}|}}{\kappa_l\boldsymbol{k}^2}$$
$$= \frac{\lambda-l}{l}\frac{3n^2l^4}{16\pi^5\kappa_l}\int \mathrm{d}\theta\frac{1}{(\cos\theta+\mathrm{i}0^+)^4} = \frac{\lambda-l}{l}\frac{3n^2l^4}{2\pi^4\kappa_l}$$

[23] It turns out that all of the other terms are irrelevant. If those terms are small at the start of the RG flow, then they will become even smaller after a long flow. This is the reason why we can ignore those terms. Certainly, if those terms are large at the beginning, then they can change everything.

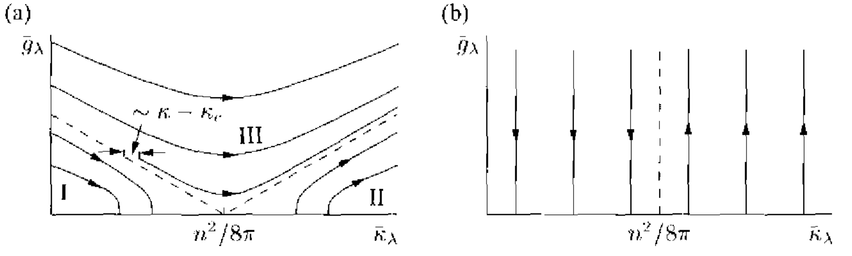

FIG. 3.10. (a) The RG flow of g_λ and κ_λ as determined by eqn (3.5.9). (b) The RG flow of g_λ and κ_λ as determined by eqn (3.5.10).

Let $b = \ln \lambda$; then the changes of the coupling constants are described by the following differential equations:

$$\frac{\mathrm{d}g_\lambda}{\mathrm{d}b} = -\frac{n^2}{4\pi\kappa_l} g_\lambda$$
$$\frac{\mathrm{d}\kappa_\lambda}{\mathrm{d}b} = \frac{3n^4 g_\lambda^2 \lambda^4}{16\pi^4 \kappa_\lambda}$$

In terms of the dimensionless couplings $\bar{\kappa}_\lambda = \kappa_\lambda \lambda^0$ and $\bar{g}_\lambda = g_\lambda \lambda^2$, these differential equations can be rewritten as follows:

$$\frac{\mathrm{d}\bar{g}_\lambda}{\mathrm{d}b} = \left(2 - \frac{n^2}{4\pi\bar{\kappa}_\lambda}\right)\bar{g}_\lambda$$
$$\frac{\mathrm{d}\bar{\kappa}_\lambda}{\mathrm{d}b} = \frac{3n^4\bar{g}_\lambda^2}{16\pi^4\bar{\kappa}_\lambda} \tag{3.5.9}$$

which are called the RG equations. The flow of $(\bar{\kappa}, \bar{g})$ is illustrated in Fig. 3.10(a).

3.5.5 Renormalization group theory and phase transition

• The concept of a fixed point and effective theory for a fixed point.

To understand the physical implications of the RG flow, let us first ignore the flow of $\bar{\kappa}_\lambda$ and study, instead, the following RG equations:

$$\frac{\mathrm{d}\bar{g}_\lambda}{\mathrm{d}b} = \left(2 - \frac{n^2}{4\pi\bar{\kappa}_\lambda}\right)\bar{g}_\lambda$$
$$\frac{\mathrm{d}\bar{\kappa}_\lambda}{\mathrm{d}b} = 0 \tag{3.5.10}$$

The flow of $(\bar{g}_\lambda, \bar{\kappa}_\lambda)$ is illustrated in Fig. 3.10(b). We find that

$$\bar{g}_\lambda = \bar{g}_l \mathrm{e}^{(2-\frac{n^2}{4\pi\kappa_l})\ln(\lambda/l)} = \bar{g}_l(\lambda/l)^{2-h}$$

from the RG equations, where $h = \frac{n^2}{4\pi\kappa_l}$ is the scaling dimension of $\cos(n\theta)$. When $\cos(n\theta)$ is relevant, a very small $\bar{g}_l$ can become as large as one wants for a long enough flow. In particular, $\bar{g}_\lambda = 1$ when $\lambda = l(\bar{g}_l)^{-1/(2-h)}$. At this point, the coupling constants stop flowing because the RG equations (3.5.9) become invalid due to the higher-order $\bar{g}_\lambda$ terms that were ignored in the RG equations. The resulting effective theory has the same form as eqn (3.5.8) and is called fixed-point theory. We can use the fixed-point theory to obtain the long-distance correlations and other long-distance physical properties of the original model.

When $\bar{g}_\lambda = 1$, everything in the renormalized fixed-point theory is of order 1 when measured in units of λ. Thus, if we believe that a large $g_\lambda \cos(n\theta_\lambda)$ will make θ_λ have short-range correlation, then the correlation length ξ must be of order 1 when measured by λ. This way, we find that

$$\xi = l\left(\frac{1}{\bar{g}_l}\right)^{1/(2-h)}$$

which agrees with the general result $\xi = a^{-1/(d-h)}$ obtained in the last section, after realizing that the perturbation O in the last section corresponds to $O = l^{-h}\cos(n\theta)$. (Equation (3.5.1) determines the normalization of O.) Thus $a = gl^h$. In terms of κ, the above result leads to

$$\xi = l\bar{g}_l^{-4\pi\kappa/(8\pi\kappa - n^2)}. \tag{3.5.11}$$

Also, a large $g_\xi \cos(n\theta_\xi)$ potential term at the length scale ξ traps θ_ξ in one of the potential minima. Thus, a relevant perturbation $g_l \cos(n\theta_l)$ always causes a spontaneous Z_n symmetry breaking, no matter how small g_l is at the cut-off scale.

When κ approaches $n^2/8\pi$, the correlation length $\xi \to \infty$. Thus, there is a phase transition at $n^2/8\pi$. When $\kappa < n^2/8\pi$, the perturbation $g_l \cos(n\theta_l)$ is irrelevant. After a long RG flow, we obtain a different fixed-point theory $\frac{\bar{\kappa}_\lambda}{2}(\partial_x\theta_\lambda)^2$ because the $\bar{g}_\lambda$ flow goes to zero. This fixed-point theory has full $U(1)$ symmetry! This is a very striking and very important phenomenon called dynamical symmetry restoration. Sometimes a term may explicitly break a certain symmetry (such as the $g_l \cos(n\theta_l)$ term breaks the $U(1)$ symmetry down to the Z_n symmetry). If the term is irrelevant, then, at long distances and/or low energies, the term flows to zero and the symmetry is restored.

To summarize, the non-compact clock model (3.5.3) has Z_n symmetry. When κ is less than a critical value $\kappa_c = n^2/8\pi$, the model is in a phase that does not break the Z_n symmetry. Furthermore, the phase has $U(1)$ symmetry at long distances. The correlation length is infinite. When κ is above the critical value κ_c, the model is in a phase that breaks the Z_n symmetry. The correlation length is finite.

The above discussion is correct and general if there is no marginal operator in the model. In that case, h can be treated as a constant. However, for the XY-model, the operator $(\partial_x\theta)^2$ has a dimension exactly equal to 2 and is an exact marginal operator. As a result, κ is a marginal coupling constant. The constant κ, and hence h, can shift their values in an RG flow. This results in the RG flow described by eqn (3.5.9) and shown in Fig. 3.10(a). We note that the RG flow described in Fig. 3.10(a) is quite different from that in Fig. 3.10(b) near the transition point $\bar{\kappa}_\infty = n^2/8\pi$. The result (3.5.11) only applies to the RG flow in Fig. 3.10(b), and is not valid for the RG flow in Fig. 3.10(a) near the transition point $\kappa_c = n^2/8\pi$. In the next section, we will calculate ξ for the RG flow in Fig. 3.10(a).

In the above, we have used the RG approach to discuss the phases and the phase transitions in the non-compact clock model. We can also use the same RG result to discuss the phases and the phase transitions in the compact clock model with vortex fluctuations.

At first sight, one may say that vortices and anti-vortices are always confined due to the potential term $g_l \cos(n\theta_l)$. This is indeed true if $\kappa > \kappa_c$ and $\kappa \cos(n\theta)$ is relevant. When $\kappa < \kappa_c$, $\kappa \cos(n\theta)$ is irrelevant. In this case, the properties of the vortices are just like those in the XY-model. The vortex fluctuations are relevant if $\kappa < 2/\pi$ and irrelevant if $\kappa > 2/\pi$. When vortex fluctuations are relevant, they modify the phase structure of the clock model.

The compact two-dimensional clock model can have several different behaviors depending on the value of n.

1. $n > 4$: The model is in the Z_n-symmetry-breaking phase when $\kappa > n^2/8\pi$. The Z_n order parameter $e^{i\theta}$ has a long-range order: $\langle e^{i\theta(\boldsymbol{x})} e^{-i\theta(0)} \rangle = \text{constant} + C e^{-|\boldsymbol{x}|/\xi}$. Near the transition point $n^2/8\pi$, we have $\kappa > 2/\pi$ and the vortex fluctuations are irrelevant. Thus, when $n^2/8\pi > \kappa > 2/\pi$, the system is in a Z_n-symmetric phase with emergent $U(1)$ symmetry at long distances. The Z_n order parameter $e^{i\theta}$ has an algebraic long-range order: $\langle e^{i\theta(\boldsymbol{x})} e^{-i\theta(0)} \rangle \sim 1/|\boldsymbol{x}|^{1/2\pi\kappa}$. When $\kappa < 2/\pi$, the vortex fluctuations are relevant, which destroys the algebraic long-range order. The system is in a Z_n-symmetric phase. The Z_n order parameter $e^{i\theta}$ has a short-ranged correlation: $\langle e^{i\theta(\boldsymbol{x})} e^{-i\theta(0)} \rangle \sim e^{-|\boldsymbol{x}|/\xi}$. As there is no long-range correlation, we cannot even talk about the emergent $U(1)$ symmetry at long distances.

2. $n = 4$: The model is in the Z_4-symmetry-breaking phase when $\kappa > n^2/8\pi = 2/\pi$, and in a Z_4-symmetric phase when $\kappa < 2/\pi$. In the symmetry-breaking phase, $g_l \cos(4\theta)$ is relevant and the vortex is irrelevant. In the Z_4-symmetric phase, $g_l \cos(4\theta)$ is irrelevant and the vortex is relevant. Thus, the Z_4-symmetric phase has no algebraic long-range order and no emergent $U(1)$ symmetry. At the transition point, both $g_l \cos(4\theta)$ and the vortex are marginal.

3. $n < 4$: The model is in the Z_n-symmetry-breaking phase when $\kappa \gg n^2/8\pi$, and in the Z_n-symmetric phase when $\kappa \ll n^2/8\pi$. Near the transition point, both $e^{in\theta}$ and the vortices are relevant and fluctuate strongly. The phase transition is described by Ginzburg–Landau theory, see eqn (3.5.7). When $n = 1$, there is no symmetry breaking and no phase transition.

Problem 3.5.2.

Running 'coupling function' Consider a model $S = \int d^2\boldsymbol{x}\ \left(\frac{\kappa}{2}(\partial_{\boldsymbol{x}}\theta)^2 + V(\theta)\right)$, where $V(\theta)$ is a small periodic function: $V(\theta + 2\pi) = V(\theta)$. Find the RG equations for the flow of the 'coupling function' V. You may ignore the flow of κ because we have assumed that V is small. Discuss the form of V after a long flow if we have started with a very small V.

Problem 3.5.3.

The $n = 1$ clock model (3.5.3) describes a two-dimensional XY-spin system in a magnetic field B_x, where $S_x = \cos(\theta)$, $S_y = \sin(\theta)$, $S_z = 0$, and $B_x = g$. Assume that $\cos(\theta)$ is relevant. Use the RG argument to find the value of S_x induced by a small magnetic field. Now assume that $\cos(\theta)$ is irrelevant. What is the S_x induced by a small magnetic field? Compare your result with eqn (3.5.2). (Hint: You may write the renormalized action in terms of the original coupling constant B_x and use the renormalized action to calculate the induced S_x. You only need to calculate the induced S_x up to an $O(1)$ coefficient.)

3.5.6 The correlation length near the transition point

To understand the behavior of ξ near the transition point for the RG flow in Fig. 3.10(a), let us expand the RG equations (3.5.9) for small $\delta\bar{\kappa}_\lambda \equiv \bar{\kappa}_\lambda - \frac{n^2}{8\pi}$ as follows:

$$\begin{aligned}\frac{\mathrm{d}\bar{g}_\lambda}{\mathrm{d}b} &= \frac{16\pi}{n^2}\delta\bar{\kappa}_\lambda \bar{g}_\lambda \\ \frac{\mathrm{d}\delta\bar{\kappa}_\lambda}{\mathrm{d}b} &= \frac{3n^2\bar{g}_\lambda^2}{2\pi^3}\end{aligned} \tag{3.5.12}$$

We find that

$$\frac{\mathrm{d}\delta\bar{\kappa}_\lambda}{\mathrm{d}\bar{g}_\lambda} = \frac{3n^4}{32\pi^4}\frac{\bar{g}_\lambda}{\delta\bar{\kappa}_\lambda}$$

The differential equation leads to $(\delta\bar{\kappa}_\lambda)^2 = \frac{3n^4}{32\pi^4}\bar{g}_\lambda^2 + C$. Depending on the sign of the constant term C, there are three classes of solutions (see Fig. 3.10(a)). Class I and class II solutions are given by

$$\delta\bar{\kappa}_\lambda = \mathrm{sgn}(\delta\bar{\kappa}_\infty)\sqrt{\frac{3n^4}{32\pi^4}\bar{g}_\lambda^2 + \delta\bar{\kappa}_\infty^2}$$

where $C = \delta\bar{\kappa}_\infty^2 > 0$. Class III solutions have the form

$$\bar{g}_\lambda = \sqrt{\frac{32\pi^4}{3n^4}\delta\bar{\kappa}_\lambda^2 + \bar{g}_{min}^2} \tag{3.5.13}$$

which is for $C < 0$.

Substituting eqn (3.5.13) into the second equation in eqn (3.5.12), we get

$$\frac{\mathrm{d}\delta\bar{\kappa}_\lambda}{\frac{32\pi^4}{3n^4}\delta\bar{\kappa}_\lambda^2 + \bar{g}_{min}^2} = n^2\,\mathrm{d}\ln(\lambda)$$

We can integrate both sides of the above equation from $\lambda = l$ to $\lambda = \xi$ to obtain

$$\int_{\delta\bar{\kappa}_l}^{\delta\bar{\kappa}_\xi} \frac{\mathrm{d}\delta\bar{\kappa}_\lambda}{\frac{32\pi^4}{3n^4}\delta\bar{\kappa}_\lambda^2 + \bar{g}_{min}^2} = n^2\ln(\frac{\xi}{l}) \tag{3.5.14}$$

We know that, at the correlation length ξ, we have $\bar{g}_\xi \sim 1$. Equation (3.5.13) tells us that $\delta\bar{\kappa}_\xi$ is also of order 1. Equations (3.5.13) and (3.5.14) relate $\delta\bar{\kappa}_l$ and $\bar{g}_l$ to ξ and allow us to determine how the correlation length ξ depends on $\kappa_l - \kappa_c$.

Let us first fix g_l and adjust κ_l to make $\delta\bar{\kappa}_l = \kappa_l - \frac{n^2}{8\pi}$ equal to $-\sqrt{\frac{3n^4}{32\pi^4}}\bar{g}_l$. From eqn (3.5.13), we see that $\bar{g}_{min} = 0$. The integral on the left-hand side of eqn (3.5.14) diverges, which implies that $\xi = \infty$. We see that the Z_n-symmetry-breaking transition really happens when $\kappa_l = \kappa_c$, where

$$\kappa_c = \frac{n^2}{8\pi} - \sqrt{\frac{3n^4}{32\pi^4}} g_l$$

If κ_l is slightly above κ_c, then we find that

$$g_{min}^2 = 2\left(\frac{32\pi^4}{3n^4}\right)^{1/2} g_l(\kappa_l - \kappa_c)$$

As $\bar{g}_{min}$ is much less than $|\delta\bar{\kappa}_l|$ and $\delta\bar{\kappa}_\xi$, eqn (3.5.14) becomes

$$\sqrt{\frac{3}{32\pi^2}}\frac{1}{g_{min}} = \ln(\frac{\xi}{l})$$

We find that

$$\xi = l e^{\left(C/(\kappa_l - \kappa_c)\right)^{1/2}}, \qquad C = \frac{n^2}{2\pi\bar{g}_l}\left(\frac{3}{32\pi^2}\right)^{3/2}$$

3.5.7 Fixed points and phase transitions

- Fixed points and universal properties.
- A fixed point with no relevant perturbations corresponds to a stable phase. A fixed point with one relevant perturbation corresponds to the transition point between two stable phases.

Running coupling constants and fixed points (or universality classes) are probably the two most important concepts in RG theory. In this section, we are going to discuss them in a general setting. Let us consider a theory with two coupling constants g_1 and g_2. When combined with the cut-off scale l, we can define the dimensionless coupling constants $\tilde{g}_a = g_a l^{\lambda_a}$, $a = 1, 2$. As we integrate out short-distance fluctuations, the dimensionless coupling constants may flow. One of the possible flow diagrams is given in Fig. 3.11(a).

What can we learn from such a flow diagram? First, we note that the flow has two attractive fixed points A and B. If $(\tilde{g}_1, \tilde{g}_2)$ is anywhere below the DCD′ line, then, after a long flow, the system will be described by $(\tilde{g}_1(A), \tilde{g}_2(A))$. So the system is described by the fixed point A at long distances. This picture demonstrates the principle of universality. The long-distance behavior of a system does not depend on the short-distance details of the system. All of the systems below the DCD′ line share a common long-distance behavior described by the fixed-point theory at A. One of the common long-distance properties is the algebraic decay exponent in the correlation function. All of the systems below the DCD′ line have

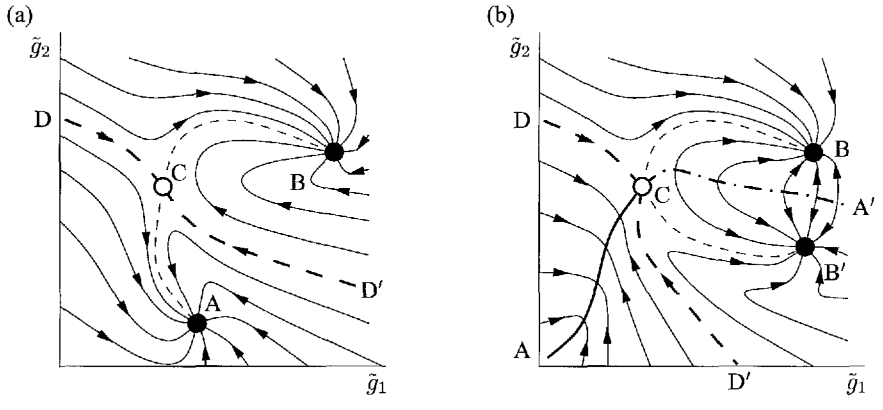

FIG. 3.11. (a) A and B are two stable fixed points representing two phases. C is an unstable fixed point with one relevant operator/direction. The transition between phase A and phase B is continuous. The critical point is described by the unstable fixed point C. (b) The fixed point/line structure of the model (3.5.3). CA is a stable fixed line. B and B′ are two stable fixed points. CA, B, and B' represent three phases. C is the critical point representing the transition between the A phase (with algebraic long-range correlations) and the B/B′ phase (with no long-range correlations). The transition is the KT transition. CA′ is an unstable fixed line, describing the transition between the B phase and the B′ phase.

the same decay exponent in the corresponding correlations. Those common properties are called universal properties. All of the systems that flow to the same fixed point form a universality class.

The systems above the DCD′ line flow to a different fixed point and form a different universality class. Those systems have different universal properties (at long distances). In particular, the decay exponents are different.

The universality classes and phases are closely related. We see that, as $(\tilde{g}_1, \tilde{g}_2)$ moves across the DCD′ line, the long-distance behavior and the long-wavelength fluctuations of the system change suddenly. As a result, the free energy of the system has a singularity at the DCD′ line. Thus, the DCD′ line is a phase transition line that separates two phases. Under this picture, we can say that the systems below the DCD′ line form one phase and the systems above the DCD′ line form the other phase. Phase and universality class mean the same thing here.

Let us start with a system exactly on the fixed point A. We add some perturbations to move the coupling constant $(\tilde{g}_1, \tilde{g}_2)$ away from $(\tilde{g}_1(A), \tilde{g}_2(A))$. As $(\tilde{g}_1, \tilde{g}_2)$ flows back to $(\tilde{g}_1(A), \tilde{g}_2(A))$, the perturbations flow to zero at long distances. Thus, the perturbations are irrelevant perturbations. As all perturbations around a stable fixed point flow to zero, *the effective theory at a stable fixed point contains no relevant or marginal perturbations.*

Now let us consider the long-distance properties of the transition point (or the critical point). If we start anywhere on the DCD′ line, then we can see that the system flows to the fixed point C. Thus, the long-distance behavior of the critical point is described by the unstable fixed point C. Here again, we see universality. No matter where we cross the transition line, the long-distance behavior of the transition point is always the same.

The fixed point C has one (and only one) unstable direction. A perturbation in that direction will flow away from the fixed point. Therefore, the fixed-point theory for C has one, and only one, relevant perturbation. In general, *a critical point describing a transition between two phases has one, and only one, relevant perturbation.* If an unstable fixed point has two relevant perturbations, then the fixed point will describe a tri-critical point.

The model (3.5.3) contains a marginal perturbation. Its flow diagram is more complicated. (See Fig. 3.11(b), where $(\tilde{g}_1, \tilde{g}_2)$ corresponds to $(\tilde{\eta}, \tilde{g})$.) The system has three phases. The phase below the DCD′ line is controlled by the stable fixed line AC. This phase has algebraic long-range correlations. The exponent of the algebraic long-range correlations depends on the position on the AC line. The phase above the DCA′ line is controlled by the stable fixed point B. It has no long-range correlation and is characterized by, say, $\langle \cos(\theta) \rangle < 0$. The phase to the right of the D′CA′ line is controlled by the stable fixed point B′. It has no long-range correlation either, and is characterized by $\langle \cos(\theta) \rangle > 0$. The transition between phase AC and phase B (or phase B′) is controlled by the unstable fixed point C, and the transition between phase B and phase B′ is controlled by the unstable fixed line CA′. The critical exponents depend on the position on the CA′ line.

From the above two simple examples, we see that we can learn a lot about the phases and phase transitions from the RG flow diagram of a system. In Section 3.3.2, we discussed phases and phase transitions from the point of view of symmetry breaking. In this section, we see that phases and phase transitions can also be understood based on an RG picture. Here, I would like to point out that the RG picture (although less concrete) is more fundamental than the symmetry-breaking picture. The symmetry-breaking picture assumes that the two stable fixed points in Fig. 3.11(a) have different symmetries and the phase transition line DCD′ is a symmetry-breaking transition line. This symmetry-breaking picture is not always true. We can construct explicit examples where the fixed points A and B have the same symmetry and the phase transition line DCD′ does not change any symmetry (Coleman and Weinberg, 1973; Halperin *et al.*, 1974; Fradkin and Shenker, 1979; Wen and Wu, 1993; Senthil *et al.*, 1999; Read and Green, 2000; Wen, 2000).