In the Luttinger model analysis,

$$\beta^2 = 16\pi K. \tag{18.108}$$

We will also use a related parameter (unfortunately also called x),

$$x = 2 - 4K = 2\left(1 - \frac{\beta^2}{8\pi}\right), \tag{18.109}$$

because the physics changes dramatically with the sign of x. It is most natural to envisage the physics in the (x, y) plane.

18.4.2 Renormalization Group Analysis of the Sine-Gordon Model

We see that the model describes a massless scalar field plus the cosine interaction due to the umklapp process ($RR \leftrightarrow LL$). It is parametrized by K and y. We need to know what the umklapp term does to the massless boson.

The answer depends on K, which determines whether or not the umklapp term is relevant. For the RG analysis it is convenient to go from the Hamiltonian in Eq. (18.106) to the Euclidean action

$$S = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{y\Lambda^2}{2}\cos\beta\phi\right) d^2x \tag{18.110}$$

and the path integral over $e^{-S(\phi)}$. Notice that we use the Lorentz-invariant bosonization formulas of Section 17.5. The replacement

$$\frac{1}{\pi\alpha} = \Lambda \tag{18.111}$$

trades the spatial momentum cut-off $1/\alpha$ for Λ, the cut-off on k, the magnitude of the two-dimensional Euclidean momentum $\boldsymbol{k}$. The evolution of y will be found by integrating out a thin shell of momenta near the cut-off $k = \Lambda$.

Let us write ϕ as a sum of slow and fast modes,

$$\phi = \phi_{\mathrm{s}} + \phi_{\mathrm{f}} \equiv \phi(0 \le k \le \Lambda(1-dt)) + \phi(\Lambda(1-dt) < k \le \Lambda). \tag{18.112}$$

The free-field action separates as well:

$$S_0 = \int \left[\frac{1}{2}(\nabla\phi_{\mathrm{s}})^2 + \frac{1}{2}(\nabla\phi_{\mathrm{f}})^2\right] d^2x. \tag{18.113}$$

The RG that leaves S_0 invariant involves integrating out ϕ_{f}, followed by the rescaling of spacetime coordinates:

$$d^2x = s^2 d^2x', \tag{18.114}$$

$$\frac{d}{dx} = \frac{1}{s}\frac{d}{dx'}, \tag{18.115}$$

$$\phi(x) = \phi'(x'). \tag{18.116}$$

Now we introduce the interaction, integrate out ϕ_f as usual, and see happens to the coupling y of the slow modes that remain. Here is the abridged analysis:

$$Z = \int d\phi_\mathrm{s} \int d\phi_\mathrm{f} \exp\left[-\int \left[\frac{1}{2}(\nabla\phi_\mathrm{s})^2 + \frac{1}{2}(\nabla\phi_\mathrm{f})^2\right] d^2x - \frac{y\Lambda^2}{2}\int d^2x \cos\beta(\phi_\mathrm{s}+\phi_\mathrm{f})\right]$$

$$= \int d\phi_\mathrm{s} \exp\left[-\int \frac{1}{2}(\nabla\phi_\mathrm{s})^2 d^2x\right]\left\langle \exp\left[-\frac{y\Lambda^2}{2}\int d^2x \cos\beta(\phi_\mathrm{s}+\phi_\mathrm{f})\right]\right\rangle_\mathrm{f} \qquad (18.117)$$

$$\simeq \int d\phi_\mathrm{s} \exp\left[-\int \left(\frac{1}{2}(\nabla\phi_\mathrm{s})^2 + \frac{y\Lambda^2}{2}\cos\beta\phi_\mathrm{s}\langle\cos\beta\phi_\mathrm{f}\rangle_\mathrm{f}\right) d^2x\right], \qquad (18.118)$$

where $\langle\cdots\rangle_\mathrm{f}$ is the average over fast modes and we are using the leading term in the cumulant expansion ($\langle e^A\rangle \simeq e^{\langle A\rangle}$); the $\sin\beta\phi_\mathrm{s}\sin\beta\phi_\mathrm{f}$ term is ignored because it has zero average over fast modes. The average $\langle\cdots\rangle_\mathrm{f}$ above is *only over the sliver of width* Λdt.

To perform the average we first set $A = i\beta\phi$, $B = 0$ in Eq. (17.57) to deduce that

$$\langle e^{i\beta\phi}\rangle = e^{-\frac{1}{2}\beta^2\langle\phi^2\rangle}. \qquad (18.119)$$

Using this result, we find that

$$\langle\cos(\beta\phi_\mathrm{f})\rangle = e^{-\frac{1}{2}\beta^2\langle\phi_\mathrm{f}^2\rangle} \qquad (18.120)$$

$$= \exp\left[-\frac{\beta^2}{2}\int_{\Lambda(1-dt)}^{\Lambda} \frac{kdkd\theta}{4\pi^2}\frac{1}{k^2}\right] \qquad (18.121)$$

$$= 1 - \frac{\beta^2}{4\pi}dt. \qquad (18.122)$$

Now we rescale the coordinates as per Eq. (18.114),

$$d^2x = s^2d^2x' = (1+2dt)d^2x', \qquad (18.123)$$

to obtain (on dropping primes)

$$\frac{y\Lambda^2}{2}\int d^2x\cos\beta\phi \to \frac{y\Lambda^2}{2}\left(1+\left(2-\frac{\beta^2}{4\pi}\right)dt\right)\int d^2x\cos\beta\phi,$$

$$\frac{dy}{dt} = \left[2-\frac{\beta^2}{4\pi}\right]y \qquad (18.124)$$

$$= (2-4K)y \ \text{ because} \qquad (18.125)$$

$$\beta^2 = 16\pi K \quad \text{in the Luttinger model.} \qquad (18.126)$$

Thus, we find that the umklapp term is

$$\text{irrelevant for } K > \tfrac{1}{2} \text{ or } \beta^2 > 8\pi, \qquad (18.127)$$

$$\text{relevant for } K < \tfrac{1}{2} \text{ or } \beta^2 < 8\pi. \qquad (18.128)$$

We rescaled x but not Λ, which just stood there. Are we not supposed to rescale all dimensionful quantities when we change units? The short answer is that in the Wilson approach the cut-off remains fixed because we use the cut-off as the unit of measurement. We could call it Λ or we could call it 1. If we begin with the ball of radius 10^{10} GeV and keep integrating away, in *laboratory units* then of course Λ_{lab} is being steadily reduced, but in rescaled units it will be fixed. It is this fixed value we are denoting by Λ above.

As a check, consider a Gaussian theory with action

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_\Lambda)^2 + \frac{1}{2}m^2\phi_\Lambda^2 \right], \tag{18.129}$$

where m is the mass in lab units and Λ is the cut-off on the momentum content of ϕ_Λ. Suppose we integrate out modes between Λ/s and Λ. We are left with

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_{\Lambda/s})^2 + \frac{1}{2}m^2\phi_{\Lambda/s}^2 \right], \tag{18.130}$$

which tells us that in lab units the theory with the reduced cut-off Λ/s continues to describe a particle of the same mass m, and asymptotic correlations will fall as e^{-mx}. There has been no change of units.

Let us now repeat this, but starting with the mass term expressed in terms of some initial cut-off Λ and a dimensionless parameter r_0:

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_\Lambda)^2 + \frac{1}{2}r_0\Lambda^2\phi_\Lambda^2 \right]. \tag{18.131}$$

Upon mode elimination this becomes

$$S = \int d^2x \left[\frac{1}{2}(\nabla\phi_{\Lambda/s})^2 + \frac{1}{2}r_0\Lambda^2\phi_{\Lambda/s}^2 \right]. \tag{18.132}$$

We now change units:

$$k = \frac{k'}{s}, \tag{18.133}$$

$$x = sx', \tag{18.134}$$

$$\frac{d}{dx} = \frac{1}{s}\frac{d}{dx'}. \tag{18.135}$$

In these new units the momentum now goes all the way to Λ and we end up with

$$S = \int d^2x' \left[\frac{1}{2}(\nabla'\phi_\Lambda)^2 + \frac{1}{2}r_0 s^2\Lambda^2\phi_\Lambda^2 \right] \tag{18.136}$$

$$\stackrel{\text{def}}{=} \int d^2x' \left[\frac{1}{2}(\nabla'\phi_\Lambda)^2 + \frac{1}{2}r_{0s}\Lambda^2\phi_\Lambda^2 \right]. \tag{18.137}$$

We see that, under the RG,

$$r_0 \to r_{0s} = r_0 s^2. \tag{18.138}$$

(We could also lump the s^2 with Λ^2 in Eq. (18.136) and identify s^2 times Λ^2 in the new units with the Λ^2_{lab} original laboratory units, thereby showing that the m^2 in laboratory units is fixed at $r_0\Lambda^2_{\text{lab}}$.)

18.4.3 Tomonaga–Luttinger Liquid: $(K > \frac{1}{2}, y = 0)$

We consider the line of fixed points $y = 0$ and focus on the sector $K > \frac{1}{2}$ where the perturbation $y\cos\sqrt{16\pi K}\phi$ is irrelevant. In terms of a variable

$$x = 2 - 4K, \tag{18.139}$$

the region where the cosine is irrelevant is

$$x = 2 - 4K < 0. \tag{18.140}$$

Not only does this line $y = 0$ for $x < 0$ describe the models with $y = 0$, it also describes models which flow to $y = 0$ under the RG. Later we will see what range of y will flow into this line under RG. In studying this line we are studying all systems in the basin of attraction of this line. Remember, however, that if you begin at some (K, y) in this basin, you will end up at $(K^*, 0)$, where $K^* \neq K$ in general. (Equivalently, $(x, y) \to (x^*, 0)$ after the RG.) So the K in what follows is in general the final K^* of a system that started away from the fixed line and got sucked into it.

For $x > 0$, the line is unstable to perturbations and the system must be tuned to stay on it. Also bear in mind that we have assumed exactly half-filling; otherwise, the umklapp term is not allowed: $e^{4iK_{\text{F}}n}$ oscillates and averages to zero unless $K_{\text{F}} = \frac{\pi}{2}$. What if we are just a little off $K_{\text{F}} = \frac{\pi}{2}$? Then the oscillations will be very slow in space to begin with, but after a lot of RG iterations, the oscillations will become rapid in the new lattice units and the seemingly relevant growth will fizzle away.

The line of fixed points $(K > \frac{1}{2}, y = 0) \equiv (x < 0, y = 0)$ is ubiquitous and appears in many guises and with different interpretations. Here it describes a fermionic liquid state called the Tomonaga–Luttinger (TL) liquid. The name was coined by Haldane [19, 20], who explored its properties and exposed the generality of the notion. It is the $d = 1$ version of Landau theory. Recall that Landau's Fermi liquid is parametrized by the F function, or its harmonics $u_m \equiv F_m$. Even if we cannot calculate the u_m from some underlying theory, we can measure them in some experiments and use them to describe others in terms of these measured values. The main point is that many low-energy quantities can be described by a few Landau parameters. Likewise, K and a velocity parameter, which I have suppressed, fully define all aspects of the fermionic system – response functions, thermodynamics, correlation functions – in the infrared.

The line of fixed points has one striking property: exponents that vary continuously with K. (This is not so for the Landau Fermi liquid, which has canonical power laws as F varies.) I will show this now, and as a by-product, establish the claim made earlier that the fermion pole at $\omega = k$ (in Minkowski space) is immediately destroyed by the smallest interaction, i.e., the smallest departure from $K = 1$.