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LETTER TO THE EDITOR

Long range order and metastability in two dimensional solids and superfluids

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Abstract. Dislocation theory is used to define long range order for two dimensional solids. An ordered state exists at low temperatures, and the rigidity modulus is nonzero at the transition temperature. Similar arguments show that the superfluid density is nonzero at the transition temperature of a two dimensional superfluid.

Peierls (1934, 1935) has argued that no long range order exists in two dimensional solids because thermal motion of low energy phonons results in a mean square deviation of atoms from their equilibrium positions which increases logarithmically with the size of the system. The absence of long range order of this simple form has been shown rigorously by Mermin (1968). Similar arguments can be used to show that there is no spontaneous magnetization in a two dimensional Heisenberg magnet (Mermin and Wagner 1966) and that the expectation value of the superfluid order parameter in a two dimensional Bose liquid is zero (Hohenberg 1967).

Numerical work on a two dimensional system of hard discs by Alder and Wainwright (1962) indicated a phase transition between a gaseous and a solid state. Stanley and Kaplan (1966) found that high temperature series expansions for two dimensional spin models indicated a phase transition at which the magnetic susceptibility becomes infinite. The evidence for such a transition is much stronger for the *xy* model (spins confined to a plane) than for the Heisenberg model, as can be seen in the papers of Stanley (1968) and Moore (1969). Low temperature expansions obtained by Wegner (1967) and Berezinskii (1970) give a magnetization proportional to some power of the field between zero and unity, and there may be a sharp transition between such behaviour, with infinite magnetic susceptibility, and the high temperature regime.

In this paper we argue in favour of a different definition of long range order based on the overall properties of the system rather than on the behaviour of a two-point correlation function. This type of long range order, which we refer to as topological long range order, may exist for the two dimensional solid, neutral superfluid, and for the *xy* model, but not for a superconductor nor for the isotropic Heisenberg model. In the case of a solid the disappearance of topological long range order is associated with a transition from a rigid to a fluid response to a small external stress, while for a neutral superfluid it is associated with the instability of persistent currents. We have recently learnt that Berezinskii (1971) has put forward similar arguments, but there are some important differences in our results.

The definition of topological long range order which we adopt arises naturally in the case of a solid from the dislocation theory of melting (Nabarro 1967). In this theory it is supposed that a liquid close to its freezing point has a local structure similar to that of a solid, but that in its equilibrium configurations there is some concentration of dislocations, which can move to the surface under the influence of an arbitrarily small shear stress, and so produce viscous flow. In the solid state there are no dislocations running across the system in equilibrium, and so the system is rigid. This theory is much easier to apply in two dimensions than in three, since a dislocation is associated with a point rather than with a curve.

The energy of a single dislocation in a two dimensional system with lattice spacing a can be found from the theory of edge dislocations (Friedel 1964), and it is given by

$$E = \left(\frac{na^2(1+\tau)}{4\pi}\right) \ln\left(\frac{A}{A_0}\right) \tag{1}$$

Here n and τ are the two dimensional rigidity modulus and Poisson's ratio, A is the area of the system, and A_0 is an area of the order of a^2. The entropy of a dislocation is

$$S = k_B \ln(A/a^2) \tag{2}$$

At temperatures which satisfy the inequality

$$k_B T < k_B T_c = na^2(1+\tau)/4\pi \tag{3}$$

the logarithmically large energy dominates, and no isolated dislocation can be formed, so the system is rigid, but once this inequality is violated there are free dislocations in the equilibrium state, and viscous flow can occur.

Although isolated dislocations cannot occur at low temperatures in a large system (except near the boundary), pairs of dislocations of equal and opposite Burgers vector have finite energy and must occur. Such pairs can respond to an applied stress and so reduce the rigidity modulus. When the inequality (3) is violated the largest pairs become unstable under an applied shearing stress, and produce a viscous response to the shear. We have worked out the behaviour of these pairs in some detail, and the results will be described in a subsequent paper.

The presence or absence of free dislocations can be determined in the following manner. We suppose that in any small local region the system is crystalline—to be definite we assume the lattice is square. We attempt to trace a rectangular path from atom to atom taking M_1 steps in the $+x$ direction, M_2 in the $+y$ direction, M_1 in the $-x$ direction, and M_2 in the $-y$ direction. Local defects can be avoided by small deformations of the path. The amount by which the path fails to close is the sum of the Burgers vectors of all the dislocations enclosed by the path. If there are isolated dislocations, the total number enclosed is proportional to the area $M_1M_2a^2$, but they are of arbitrary sign, so the expected length of the sum of the Burgers vectors is proportional to $(M_1M_2)^{1/2}a$. If there are only pairs of dislocations, only those pairs cut by the path contribute. Their number is proportional to $(M_1 + M_2)$, and so the expected length of the sum of the Burgers vectors is proportional to $(M_1 + M_2)^{1/2}\,a$. In the solid state such paths fail to close by an amount proportional to the square root of the length of the path, while in the liquid state they fail to close by an amount proportional to the length of the path. This allows us to determine whether or not topological long range order exists in a particular configuration of the system.

Similar arguments can be made for a two dimensional neutral superfluid, with vortices instead of dislocations, since the energy of a vortex also depends logarithmically on the size of the system. In this case the critical temperature is given by

$$k_B T_c = \pi \rho_s \hbar^2/2m^2 \tag{4}$$

where ρ_s is the two dimensional superfluid density and m is the atomic mass (effective mass for a thin film). Above this temperature free vortices can destroy superfluid flow. This case has been examined in some detail by Berezinskii (1971).

There should not be such a phase transition for a superconductor, since flux lines have a finite energy which depends on the penetration depth. For the isotropic Heisenberg model there is no such transition, since the topologically distinct arrangements of Heisenberg spins are separated from one another by a finite energy barrier that can be overcome by thermal fluctuations.

An important consequence of these considerations is that a solid or superfluid system with periodic boundary conditions has metastable states separated from one another by an energy barrier which cannot be overcome by thermal fluctuations in the limit of an infinite system. These metastable states cease to be distinct at T_c given by equations 3 or 4. In the case of a superfluid these states are current-carrying states; the circulation can only change by one unit if a vortex moves right round the system. For a solid the lines of atoms form a spiral whose pitch is different in different metastable states.

The argument leading to equations 3 and 4 is very similar to the argument used by Thouless (1969) for a one dimensional system with $1/r^2$ interaction. In that case it was argued that the magnetization must be nonzero at the transition, and Dyson (1971) has obtained a similar result for a soluble model. The inequality (3) shows that the rigidity cannot be zero at the transition temperature, and, from equation 4, the superfluid density cannot be zero.

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