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低维体系的涨落

1. $\langle x_i^2\rangle = \frac{1}{L}\sum_k n_k \sim \int_0^{\Lambda} dk \frac{1}{e^{\beta \gamma k}-1}$ → 发散 (d≤2)

2. 磁体. $\langle M_i^2\rangle \sim \frac{1}{V}\sum_k \overset{\text{magnon}}{n_k} \sim \int \frac{1}{e^{\beta v k^2}-1} k\,dk$ 二维 发散

$\int \frac{1}{e^{\beta v k}-1} k\,dk$ 可以收敛

3. BEC. $N_{excited} \gg N$. 粒子数 大 涨落

1.3.5. Superfluid as a toy universe   对应 vertex

(光子 → 声子 → 电磁)
交换的粒子
声子 对 roton → $\frac{1}{r^4}$ 相互作用 (交换roton/声子)

BKT 相变. $(\partial_t\theta)^2 - v^2(\nabla\theta)^2$ or $(\nabla\theta)^2 + h\cos\theta$

superfluid   super conductor   XY model.

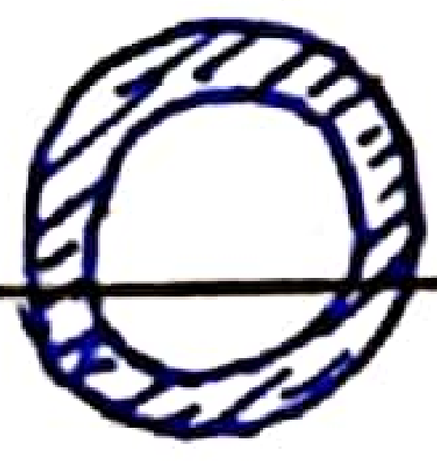
不涨落时 $\sqrt{\rho+\delta\rho}\,e^{i\theta} = \psi$ 如一 连续的 序参数.   ($\sqrt{\rho+\delta\rho}\,e^{i\delta\theta}$

对应作用量展开 $\mathcal{L}(\delta\rho,\delta\theta)$ 中 $\partial_t\delta\theta$ $\partial_t\delta\rho$ $\delta\rho\,\partial_t\delta\rho$ 可以 不存在

$\delta\rho\,\partial_t\delta\rho$ $(\partial_x\delta\rho)^2$. $(\partial_x\delta\theta)^2$ $\partial_x\delta\rho$ $(\delta\theta)^2$ 不能存在

$U(1)$ → ψ → $\psi e^{i\theta}$ 故 $(\delta\theta)^2$ 不能存在

旋涡.



$dy = Dy\,dl$ $\frac{dy}{dl} = D\cdot y$ $y \propto e^{Dl}$

D 阶项.   注意: 涨落引起的 $\underset{\Downarrow}{A}\,y\,dl$
修正项

$dy = (D+A)y\cdot dl$ $y \propto e^{(D+A)l}$

相变 由 D+A 决定   D+A 从正变负 时 relevant 到 irrelevant：BKT相变

No

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X. Y- Model.

$H = -J\sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j) = -J\sum \cos(\nabla\theta \cdot dl) = -J\sum(1 - \frac{1}{2}(\nabla\theta\cdot l)^2)$

$= \text{const} + \frac{J}{2}\int dx (\nabla\vec{\theta})^2$ 2维情况 $[\theta]$ [illegible]

$H = \frac{J}{2}\int dx (\nabla\theta)^2$ 故 $H = \frac{J}{2}\int dx\, p^2$ $L = \frac{J}{2}\int dx (\nabla\theta)^2$

$\frac{dL}{d\theta} = \frac{\partial}{\partial x_i}\frac{\partial L}{\partial(\partial_i\theta)} = J\nabla^2\theta = 0$ $\nabla^2\theta = 0.$

[illegible] $\theta = k\cdot x$ [illegible] $\theta \propto \frac{1}{2\pi}\ln|\vec{x}|.$

$\nabla^2 \ln|x| = \nabla^2\ln|\vec{r}| = \frac{\partial^2}{\partial x^2}\ln|\vec{r}| + \frac{\partial^2}{\partial y^2}\ln|\vec{r}| = \cancel{\frac{\partial}{\partial x}\frac{1}{2}} \frac{1}{2}\frac{\partial^2}{\partial x^2}\ln(x^2+y^2)$

$+ \frac{1}{2}\frac{\partial^2}{\partial y^2}\ln(x^2+y^2) = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) + \frac{1}{2}\frac{\partial}{\partial y}\frac{2y}{x^2+y^2} = \frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2}$

$+ \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$

$\psi = \sqrt{\rho}\, e^{i\delta\theta}$

$\vec{v} = -i\hbar\,\psi^*\nabla\psi/2m = -i\hbar\sqrt{\rho}\,e^{-i\delta\theta}\sqrt{\rho}\,e^{i\delta\theta}\, i\nabla\delta\theta/2m = \frac{\rho\hbar\nabla\delta\theta}{2m}$

$\theta = \frac{1}{2\pi}\ln|\vec{x}|$ $\vec{\nabla}\theta = \frac{1}{\sqrt{x^2+y^2}}\frac{2x\vec{x}+2y\cdot\vec{y}}{2\sqrt{x^2+y^2}} = \frac{x\hat{x}+y\cdot\hat{y}}{x^2+y^2} = \frac{\vec{r}}{r^2}$

$\oint \nabla\cdot\theta\cdot dl \propto \cancel{\oint\frac{1}{|r|}dr =} = ?\ \propto \oint \nabla\cdot\ln|\vec{x}|\, dl = \oint \frac{\vec{r}}{r^2}\cdot dl$

$H = \frac{J}{2}\int(\nabla\theta)^2 d^2\vec{r} = \int\frac{J}{2}n^2\frac{1}{r^2}\cdot d^2\vec{r} = \int\frac{J}{2}n^2\frac{1}{r^2}\cdot r\cdot dr\,d\theta = J\pi n^2\int_0^a \frac{1}{r}dr \overset{积出}{=} J\pi n^2\ln\frac{L}{a}$

Ref: ① X.G. Wen 第3.5 (中文3.5节)
② Shankar 第18.4 P352 RG of Sine-Gordon Model

$$S=\int\left[\frac{1}{2}(\nabla\phi)^2+\frac{y\Lambda^2}{2}\cos\beta\phi\right]d^2x$$

XY-spin + magnetic field

BKT 相变.

3. Ahamed Cooper Rooms The BKT Transition

XY Model.

$H=-J\sum_{ij}\cos(\theta_i-\theta_j) = -J\sum_{ij}\vec{S}_i\cdot\vec{S}_j$ (即一个矢量正则)

$=\text{const}+\frac{J}{2}\int d\vec{x}(\nabla\vec{\theta})^2$ 用 标度维度 $[\theta]=0$ (所以无量纲)

$dy=0\,dl+\underline{Ay\,dl}$ 一阶的主导项决定 irrelevant or relevant.

Marginal

解. $H=\frac{J}{2}\int dx(\nabla\theta)^2$

Thouless $\nabla^2\theta=0$. 平面波解 $\theta=\vec{k}\cdot\vec{x}$ 还有另一个解 $\theta\sim\frac{1}{2\pi}\ln|\vec{x}|$
[illegible]

~~$\nabla^2\theta=\delta^{(2)}(\vec{x})$~~ → $\underline{-\nabla^2G=\delta(\vec{x})}$ 2d Green Function

$\nabla^2\theta=0\longrightarrow-\nabla^2G=\delta(\vec{x})$ 的问题 [illegible]

$\theta=\frac{1}{2\pi}\ln|\vec{x}|$ 正好对应 2D - coulomb gas

(3d. $\oint\vec{E}\cdot d\vec{S}=Q$, $E\propto\frac{\vec{r}}{r^3}\sim\nabla(\frac{1}{r})$

2d $\underline{\frac{1}{2\pi i}\oint\frac{1}{z}dz}=\frac{1}{i}\oint d(\frac{\ln|z|}{2\pi})$ $\frac{1}{2\pi}\ln|z|$ 为一势)

$\nabla^2\theta=0$ 有两种解法: { 平面波 (类似 phonon/magnon ; vertex

~~[illegible]~~

平面波 的解 ([illegible])

$\nabla^2\theta=0\to-\nabla^2\theta=\delta(\vec{x})\to$ 点源 [illegible] ([illegible] $(\nabla\theta)^2$)

$\vec{v}=-i\hbar\psi^*\nabla\psi/2m=\frac{\hbar}{2m}(\nabla\theta)$, 其中 $\vec{v}$ 为 current 速度.

$H=\frac{J}{2}\int(\nabla\theta)^2$ $H=\int\frac{1}{2}Jv^2$ $\begin{cases}J\sim m\\ \nabla\theta\sim v\end{cases}$

绕数: $\oint\nabla\theta\,dl=2\pi n$ $\frac{1}{2\pi i}\oint\frac{1}{z}dz$. 令 $z=re^{i\theta}\Rightarrow\frac{1}{2\pi}\oint\theta'dl$

$\frac{1}{2\pi}\int\vec{v}\cdot d\vec{l}=n$. 则 $\vec{v}=\frac{(\text{const})}{r}$ $\nabla\theta=(\frac{n}{r})$

$H=\frac{J}{2}\int(\nabla\theta)^2=\frac{J}{2}n^2\int\frac{1}{r^2}d^2r$ 积分

$=\frac{J}{2}n^2 2\pi\int_a^L\frac{1}{r}dr=\frac{J}{2}n^2 2\pi\ln\frac{L}{a}=J\pi n^2\ln\frac{L}{a}$

$$\theta(x) = \frac{1}{\sqrt{V}} \sum_k (\theta_k e^{ikx} + \bar{\theta}_k e^{ikx})$$

$$\nabla \cdot \theta(x) = \frac{i}{\sqrt{V}} \sum_k k (\theta_k e^{ik\cdot x} - \bar{\theta}_k e^{-ikx})$$

$$\int dx (\nabla\theta(x))^2 = -\frac{1}{V} \sum_{kk'} \int k\cdot k' (\theta_k e^{ikx} + \bar{\theta}_k e^{-ikx})(\theta_{k'} e^{ik'\cdot x} + \bar{\theta}_{k'} e^{-ik'\cdot x}) dx$$

$$= -\frac{1}{V} \sum_{kk'} k k' \int (\theta_k \bar{\theta}_{k'} e^{i(k-k')x} dx + \bar{\theta}_k \theta_{k'} e^{i(k'-k)x} \cdot dx = \sum_k k^2 \theta_k \bar{\theta}_k + k^2 \bar{\theta}_k \theta_k$$

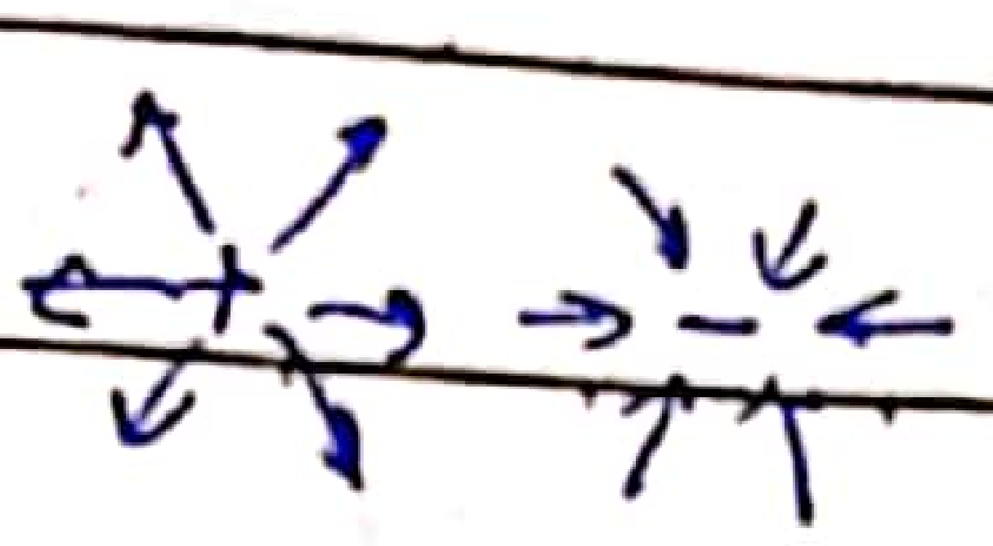
$\psi = \sqrt{\rho} e^{i\theta}$ ρ固定 $$\langle \psi^\dagger(x) \psi(0) \rangle = \frac{\int D\theta\, e^{-i\theta(x) + i\theta(0)} e^{\cancel{i\theta} + S}}{\int D\theta\, e^{\cancel{i\theta} + S}}$$

$$S = -\frac{\beta}{2} J \int dx (\nabla\theta)^2 \qquad \frac{\int D\theta\, e^{-i\theta(x) + i\theta(0) - \frac{\beta J}{2} \int dx (\nabla\theta)^2}}{\int D\theta\, e^{-\frac{\beta J}{2} \int dx (\nabla\theta)^2}}$$

c) Fourier Transformation ⇒ $$\frac{\int D\bar{\theta}_k D\theta_k\, e^{-i\frac{1}{\sqrt{V}} \sum_k (\theta_k e^{ikx} + \bar{\theta}_k e^{-ik\cdot x}) + i\frac{1}{\sqrt{V}} \sum_k (\theta_k + \bar{\theta}_k) + S}}{\int D\bar{\theta}_k D\theta_k\, e^{S}}$$

$$= \frac{\int D\bar{\theta}_k D\theta_k\, e^{-i\frac{1}{\sqrt{V}} \sum_k (\theta_k e^{ikx} + \bar{\theta}_k e^{-ikx}) + i\frac{1}{\sqrt{V}} \sum_k (\theta_k + \theta_{-k}) - \frac{\beta J}{2} \sum_k (k^2 \bar{\theta}_k \theta_k + \theta_k \bar{\theta}_k k^2)}}{\int D\bar{\theta}_k D\theta_k\, e^{-\frac{\beta J}{2} \sum_k (k^2 \bar{\theta}_k \theta_k + \theta_k \bar{\theta}_k)}}$$

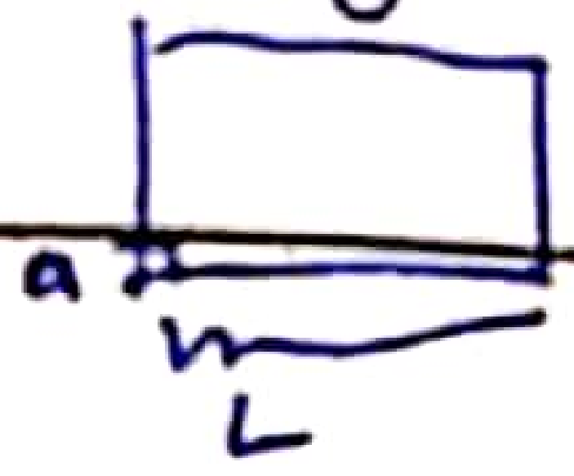
这就是vortex. 类似一个正负电荷



正负成为一个dipole

一个vortex $U=J\pi\ln\frac{L}{a}$

Entropy



$(\frac{L}{a})^2$ 种可能, 熵为 $2k_B\ln\frac{L}{a}$

$F=U-TS=J\pi\ln\frac{L}{a}-2k_BT\ln\frac{L}{a}$. T小一点 产生一个vortex F↑ T大一点 产生一个vortex F↓

$J\pi=2k_BT$

在BKT中正负自由. [illegible]正负自由

~~Ising Model $H=-\sum_i\sigma_i\sigma_{i+1}$~~

(Ising Model中 $H=-\sum_i\sigma_i\sigma_{i+1}$ 基态 $N=2L$ 激发: m=0 $\frac{L}{2}$个↑ 和 $\frac{L}{2}$↓ E(1+1对个, L−1↓) 和 S(L+1个 L−1↓)

$F=E-TS$. [illegible] T_c 相变(消失)

综上讨论: (why d=2 is special)

(有限温关联: 涨落发散 <=> 不稳定性 Wagner−Mermin (60年代))

1d $\int dk\frac{1}{e^{\beta vk^2}-1}$ × 2d $\int d^2k\frac{1}{e^{\beta vk^2}-1}$ × 不稳定
↓
(长程(不...)

periodic 形成 →

$H=\frac{J}{2}\int(\nabla\theta)^2 \Leftrightarrow \frac{k^2}{2}J$

3d $\int d^3k\frac{1}{e^{\beta vk^2}-1}$ => finite 有BEC

$\psi=\sqrt{\rho}e^{i\theta}$ => ρ const. $\langle\psi^\dagger(x)\psi(0)\rangle=\frac{\int D\theta\, e^{-i\theta(x)+i\theta(0)}e^{-S}}{\int D\theta\, e^{-S}}$

$S=-\frac{\beta}{2}J\int dx(\nabla\theta)^2$

$$\frac{\int D\theta\, e^{-i\theta(x)+i\theta(0)-\frac{\beta J}{2}\int d^2x(\nabla\theta)^2}}{\int D\theta\, e^{-\frac{\beta J}{2}\int d^2x(\nabla\theta)^2}} \overset{FT}{\Longleftrightarrow} \int D\bar\theta_k D\theta_k\, e^{-i\frac{1}{\sqrt V}\sum_k(\theta_k e^{ik\cdot x}+\bar\theta_k e^{-ik\cdot x})+i\frac{1}{\sqrt V}(\theta_k+\bar\theta_k)-\frac{\beta J}{2}\sum_k(\bar\theta_k\theta_k k^2+\theta_k\bar\theta_k k^2)}$$

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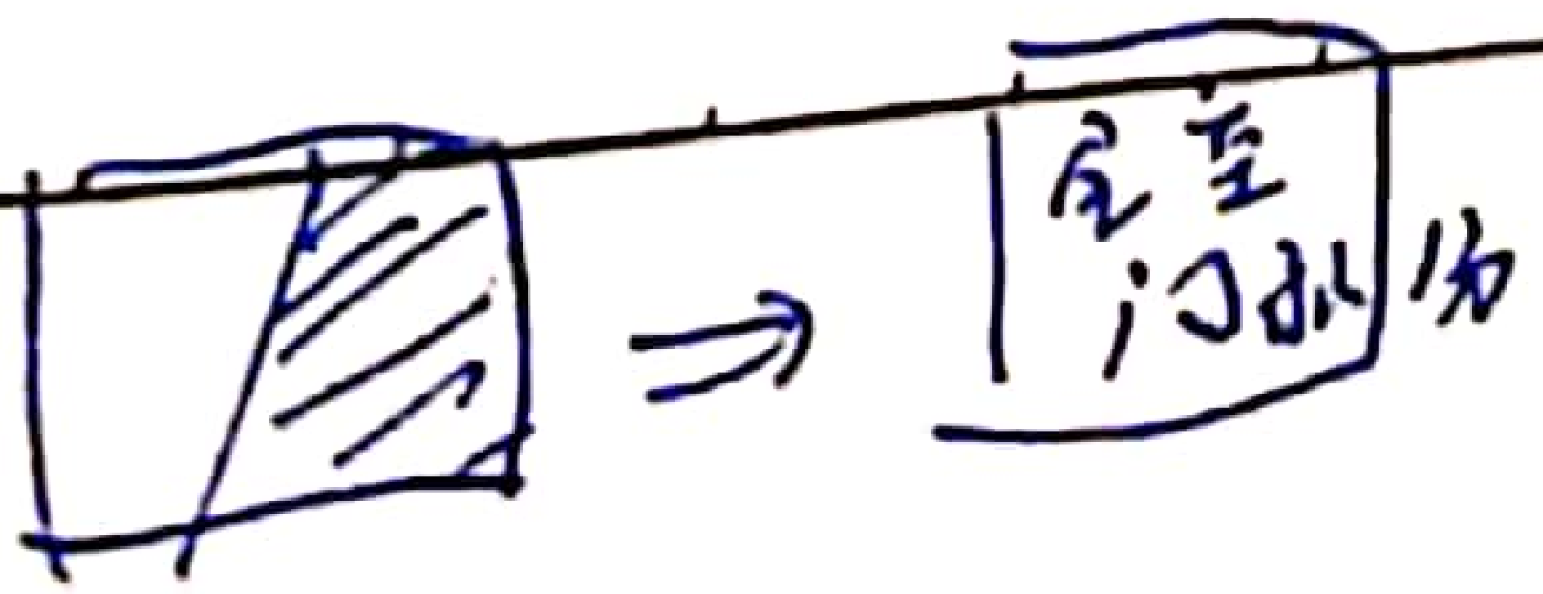

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计算关联函数 $e^{-\frac{\lambda^2}{V}\sum_k \frac{e^{ik\cdot x}}{\beta J k^2}}$

1. [illegible] $\langle e^{-i\lambda(\theta(x)-\theta(0))}\rangle$ $(\lambda=1)$

$\langle e^{u}\rangle = e^{\langle u\rangle + \frac{1}{2}\langle u^2\rangle_c}$

$\langle u\rangle = -i\lambda\langle \theta(x)-\theta(0)\rangle = 0$ （因为[illegible]）

$\frac{1}{2}\langle u^2\rangle_c = \frac{1}{2}(i\lambda)^2\langle(\theta(x)-\theta(0))^2\rangle = -\frac{\lambda^2}{2}(\underline{\langle\theta^2(x)\rangle} + \underline{\langle\theta^2(0)\rangle} - \langle\theta(x)\theta(0)\rangle$
$-\langle\theta(0)\theta(x)\rangle) = -\lambda^2(\langle\theta^2(0)\rangle - \langle\theta(0)\theta(x)\rangle)$ 这两项加起来为 $2\langle\theta^2(0)\rangle$

1; $\langle AB\rangle^2 \le \langle A^2\rangle\langle B^2\rangle$ $\langle\theta(0)\theta(x)\rangle^2 \le \langle\theta(0)\rangle^4$ $\langle\theta(0)\rangle^2 \ge \langle\theta(0)\theta(x)\rangle$

$\langle\theta(0)^2\rangle - \langle\theta(0)\theta(x)\rangle \ge 0$ $\frac{1}{2}\langle u^2\rangle_c \le 0$ 所以 二阶[illegible]一定 ≤ 0

1d. $\frac{1}{V}\sum_k \frac{e^{ik\cdot x}}{\beta J k^2} = \frac{1}{2\pi\beta J}\int dk \frac{e^{ik\cdot x}}{k^2}$ $kx = k'$ $k = \frac{k'}{x}$

$$\overset{\shortparallel}{\frac{1}{2\pi\beta J}}\int dk \frac{e^{ik}}{k^2}x = \frac{|x|}{2\pi\beta J}\int dk \frac{e^{ik}}{k} = \frac{A|x|}{2\pi\beta J}$$

$\mathrm{sgn}\,x = \frac{x}{\sqrt{x^2}}$

3d. $\int dk^3 \frac{e^{ik\cdot x}}{k^2} = \int d\theta d\varphi\, k^2 \sin\theta \frac{e^{ikx\cos\theta}}{k^2} = 2\pi\int d\cos\theta\, e^{ikx\cos\theta} dk$

$= -2\pi\int \frac{e^{ikx\cos\theta}}{ikx} dk \Big|_{\theta=0}^{\theta=\pi} = -\frac{2\pi}{ikx}\int(e^{-ikx} - e^{ikx})\,dk$

$= \int \frac{2\pi}{ikx} 2i\sin kx\, dk = \int 4\pi \frac{\sin kx}{kx} dk = \frac{\pi\,\mathrm{sing}(x)}{2x} = \frac{\pi}{2x}\mathrm{sgn}[x] = \frac{\pi}{2X}$

YCP

$$\frac{1}{V}\sum_k \frac{e^{ik\cdot(x-y)}}{\beta J k^2} \quad \nearrow 0$$

计算相关函数

$$= e^{-\frac{\lambda^2}{V}\sum_k \left(\frac{e^{i\vec{k}\cdot\vec{x}}}{\beta J\cdot k^2}\right)} \quad \nearrow 1$$

另一种计算方法. $\langle e^{-i\lambda(\theta(x)-\theta(0))}\rangle$

$$\langle e^{u}\rangle = e^{\langle u\rangle + \frac{1}{2}\langle u^2\rangle_c}$$

$$\frac{1}{2}(i\lambda)^2\langle(\theta(x)-\theta(0))^2\rangle = -\frac{\lambda^2}{2}\Big(\langle\theta(x)^2\rangle + \langle\theta(0)^2\rangle - \langle\theta(x)\theta(0)\rangle - \langle\theta(0)\theta(x)\rangle\Big)$$

$$= -\lambda^2\big(\underbrace{\langle\theta(0)^2\rangle - \langle\theta(0)\theta(x)\rangle}_{\geqslant 0}\big)$$

$\langle AB\rangle$ 与 [illegible] $\leq \langle A^2\rangle\langle B^2\rangle$

= 指数部分一定 < 0

1d. $$\frac{1}{V}\sum_k \frac{e^{ik\cdot x}}{\beta J k^2} = \frac{1}{2\pi\beta J}\int dk\,\frac{e^{ikx}}{k^2} = \underbrace{\frac{|x|}{2\pi\beta J}\int dk\,\frac{e^{il}}{k^2}}_{\text{定积分(发散)}}$$

$$= \frac{A|x|}{2\pi\beta J}$$

关联函数: $g(x) \propto e^{-(\ \ )|x|}$

关联与长程关联.　　(注 C.N. Yang off-diagonal long-range order)

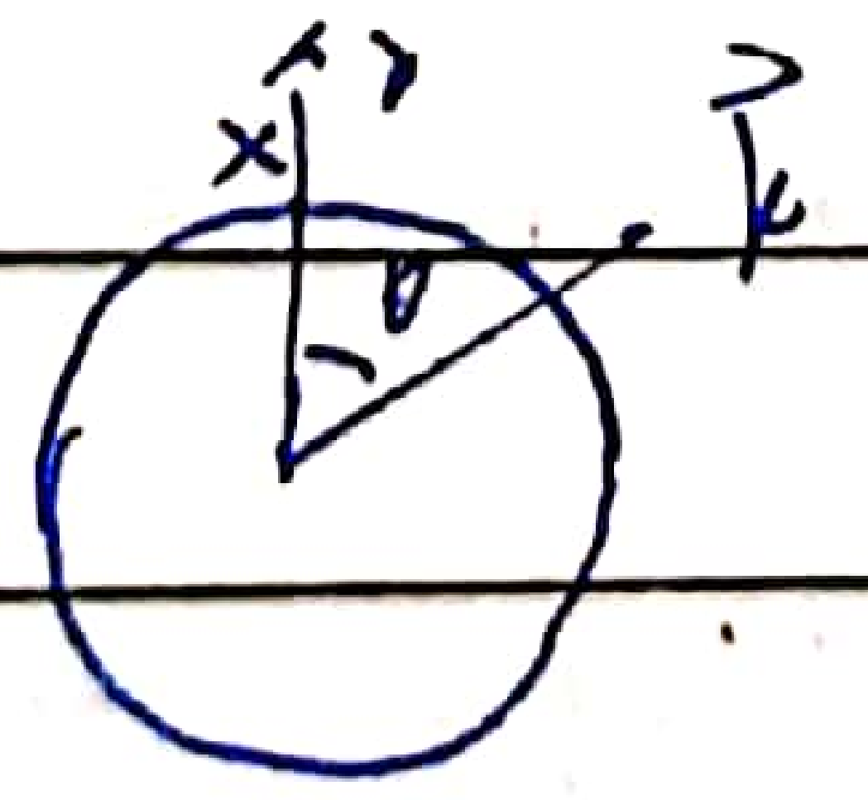
计算 $\langle\psi^{+}(x)\psi(0)\rangle$　　$\psi(x) = \bar{\psi}(x) + \delta\psi(x)$

$\bar{\psi}\psi + \langle\delta\psi(x)\delta\psi(0)\rangle$　　其中 $\langle\delta\phi(x)\rangle = 0$

~~$\bar{\psi}$~~ $\neq e^{-(\ \ )|x|}$ 故 $\bar{\psi} = 0$ ∴ 一维系统无长程关联

3d. $g(x) \to$ const.　$\bar{\psi} \neq 0$　$g(x) \sim e^{-cT}$

$$\int d^3k\,\frac{e^{ik\cdot x}}{k^2}$$



$$\int d\theta\, d\phi \sin\theta\, dk\, e^{ikx\cos\theta} = \int \frac{2\pi}{ikx}\, dk\, e^{ikx\cos\theta}\Big|_{\pi}^{0}$$

$$= \int \frac{2\pi}{ikx}\, dk\,(e^{ikx} - e^{-ikx}) = \int 4\pi\,\frac{\sin kx}{kx}\, dk \to \text{有限 (计算)}$$

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2d $$\int d^2k \frac{e^{ikx}}{k^2} = \int dk\, d\theta\, k \frac{e^{ikx\cos\theta}}{k^2} = \int dk\, d\theta \frac{e^{ikx\cos\theta}}{k}$$

$$= \int \frac{dk}{k} J_0(kx) = \int_0^\infty \frac{dk}{k} J_0(k)$$ 发散

引入红外和紫外截断 $k=a$ 和 $k=L$ 则积分变为 $\int_{ax}^{aL} \frac{dk}{k} J_0(k)$

$$= \frac{\beta}{2\pi J} \ln \frac{x}{L} \qquad g(x) = e^{-\frac{\beta}{2\pi J}\ln\frac{x}{L}} = \left(\frac{x}{L}\right)^{-\frac{\beta}{2\pi J}}$$

8. $$\int D\phi\, e^{-\frac{\beta J}{2}\int dx\left[(\nabla\phi)^2 + h\cos\phi\right]} \Rightarrow n\int D\theta\, e^{-\frac{\beta J}{2}\int dx\left[n^2(\nabla\theta)^2 + h\cos n\theta\right]}$$

令 $\phi = n\theta$ $$\frac{\beta J}{2} n^2 = \frac{k}{2} \qquad \frac{\beta J}{2} h = -g$$

$$\int D\theta\, e^{-\int dx\left[\frac{k}{2}(\nabla\theta)^2 - g\cos(n\theta)\right]}$$

$$\theta(x) = \theta_>(x) + \theta_<(x)$$

$$\int D\theta_< D\theta_>\, e^{-\int dx\left[\frac{k}{2}(\nabla\theta_<(x) + \nabla\theta_>(x))^2 - g\cos n(\theta_< + \theta_>)\right]}$$

$$= \int D\theta_<\, e^{-\int dx\left[\frac{k}{2}[\nabla\theta_<(x))^2 - g\cos n\theta_<\right]} \int D\theta_>\, e^{-\int dx\left[\frac{k}{2}(\nabla\theta_>(x))^2 - g\{\cos n(\theta_< + \theta_>) - \cos n\theta_<\}\right]}$$ #

$$\cos(\theta + x) - \cos\theta = \cos\theta\cos x - \sin\theta\sin x - \cos\theta = \cos\theta\left(1 - \frac{x^2}{2}\right) - x\sin\theta - \cos\theta$$

$$= -\frac{x^2}{2}\cos\theta - x\sin\theta$$

$$\# = \int D\theta_<\, e^{-\int dx\left[\frac{k}{2}(\nabla\theta_<(x))^2 - g\cos n\theta_<\right]} \int D\theta_>\, e^{-\int dx\frac{k}{2}(\nabla\theta_>(x))^2} \quad \langle e^{u}\rangle$$

$$u = g\int dx\left\{\cos n(\theta_< + \theta_>) - \cos n\theta_<\right\}$$

$$= g\int dx\left\{-\frac{n^2\theta_>^2}{2}\cos n\theta_< - n\theta_>\sin n\theta_<\right\}$$

n=1 时 $$g\int dx\left\{-\frac{\theta_>^2}{2}\cos\theta_< - \theta_>\sin\theta_<\right\}$$

YCP

讨论2d. $\int d^2k \frac{e^{ikx}}{k^2} = \int_0^{2\pi} d\theta \int_0^\infty k\,dk \frac{e^{ikx\cos\theta}}{k^2}$

$= \int_0^\infty \frac{dk}{k} \int_0^{2\pi} d\theta \frac{e^{ikx\cos\theta}}{\cancel{2\pi}}$ Bessel function

$\parallel$

$\int_0^\infty \frac{J_0(kx)}{kx} d(kx)$ 的积分发散

引入紫外截断 a 红外截断 L $\int_a^L \frac{J_0(kx)}{k} dk = \int_{ax}^{Lx} \frac{J_0(kx)}{kx} d(kx)$

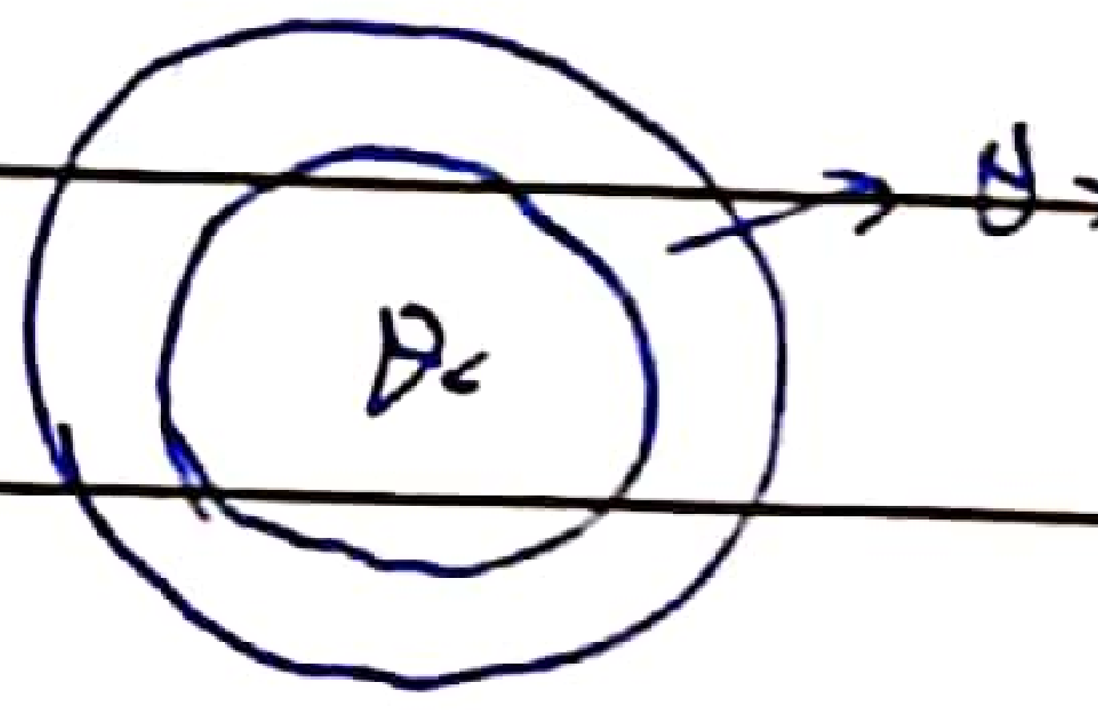
$= \int_{ax}^{Lx} \frac{J_0(y)}{y} dy$

$= \frac{\beta}{2\pi J} \ln(\frac{x}{L})$ $g(x) = (\frac{x}{L})^{-\frac{\beta}{2\pi J}}$

2d. 代数衰减

RG of BKT. phase Transition

$\theta(x) = \theta(<) + \theta(>) = \theta_< + \theta_>$



我们求 $dg = D_g dl + A_g dl$

$Z = \int D\phi\, e^{-\frac{\beta}{2}\int d^dx[(\nabla\phi)^2 + h\cos\phi]}$ $\downarrow \phi = n\theta$

$\overset{积掉}{\equiv} \int D\theta\, e^{-\int d^dx[\frac{K}{2}(\nabla\theta)^2 - g\cos(n\theta)]}$ $~~(\frac{1}{2}\theta_>^2\cos\theta_< + \theta_>\sin\theta_<)~~\, g$

$\parallel$

$= \int D\theta_< e^{-\int d^dx[(\frac{K}{2}(\nabla\theta_<)^2 - g\cos(n\theta_<)]} \int D\theta_> e^{-\int d^dx \frac{K}{2}(\nabla\theta_>)^2 - g\underline{[\cos(n\theta_< + n\theta_>) - \cos n\theta_<]}}$

$\cos(\theta + x) - \cos\theta$

$= \cos\theta(1 - \frac{x^2}{2}) - \sin\theta\sin x - \cos\theta$

$= -\frac{x^2}{2}\cos\theta - \cancel{\sin}\, x\sin\theta$

$= \int D\theta_< e^{-\int d^dx[\frac{K}{2}(\nabla\theta_<)^2 - g\cos n\theta_<]} \langle e^u \rangle$

$u = g\int d^dx[\frac{1}{2}\theta_>^2\cos\theta_< + \theta_>\sin\theta_<]$ $e^{\langle u\rangle + \frac{1}{2}\langle u^2\rangle_c}$

No

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$$\langle u\rangle = g\int dx\,\cos\theta_<\,\frac{1}{2}\underbrace{\langle\theta_>^2\rangle}_{\Downarrow \; \frac{1}{V}\sum_{k\in\text{shell}}\frac{1}{\frac{1}{2}\beta J k^2}} + g\int dx\,\sin\theta_<\langle\theta_>\rangle$$

$$= \frac{g}{2}\langle\theta_>^2\rangle\int dx\,\cos\theta_<$$

$$\langle u^2\rangle = g^2\int dx\,dx'\left\langle\left(\cos\theta_<\frac{1}{2}\theta_>^2 + \underline{\sin\theta_<\,\theta_>}\right)\left(\cos\theta_<\frac{1}{2}\theta_>^2 + \underline{\sin\theta_<\,\theta_>}\right)\right\rangle$$

$$= g^2\int dx\,dx'\,\sin\theta_<(x)\sin\theta_<(x')\langle\theta_>(x)\theta_>(x')\rangle$$

$$\simeq \frac{g^2}{2}\int dx\,dx'\,g(x-x')\sin\theta_<(x)\sin\theta_<(x')$$

$$= \frac{g^2}{2}\int dx\,dy\;g(y)\sin\theta_<(x)\sin\theta_<(x+y)$$

$$\sin\theta_<(x+y) = \sin(\theta_<(x) + \partial\theta_<(x)\,y)$$

$$= \sin\theta_<(x)\cos\partial\theta_<(x)y + \sin\partial\theta_<(x)y\cos\theta_<(x)$$

$$= \sin\theta_<(x)\left(1 - \underline{\frac{1}{2}(\partial\theta_<(x)y)^2}\right) + \sin\partial\theta_<(x)y\cos\theta_<(x)$$

$$-\frac{g^2}{4}\int dx\,dy\;g(y)\sin^2\theta_<(x)\,y^2(\nabla\theta_<(x))^2$$

$$= -\frac{g}{4}\int dy\,g(y)\,y^2\int dx\,\sin^2\theta_<(x)\,(\nabla\theta_<(x))^2$$

$$= -\frac{g}{4}\int dy\,g(y)\,y^2\int dx\,\frac{1-\cos2\theta_<(x)}{2}(\nabla\theta_<(x))^2$$

$$= -\frac{g}{8}\int dy\,g(y)\,y^2\int dx\,(\nabla\theta_<(x))^2 + \frac{g}{8}\int dy\,g(y)\,y^2\int dx\,\cos\theta_<(x)\,(\nabla\theta_<(x))^2$$

$\Rightarrow$ 低温时 [illegible] g 高次 　二阶 [illegible] g [illegible]

一阶贡献 　对 cos θ 做展开

$$\langle u\rangle = g\int dx\, \cos\theta_< \underbrace{\tfrac{1}{2}\langle \theta_>^2\rangle}_{\text{计算它}} + ~~g\int dx\, \sin\theta_< \langle\theta_>\rangle~~$$

$$\Downarrow$$

$$\frac{1}{V}\sum_{k\in \text{shell}} \frac{1}{\frac{1}{2}\beta J k^2}$$

$$= \frac{g}{2}\langle \theta_>^2(x)\rangle \int dx\, \cos\theta_<$$

二阶贡献:

[illegible]

$x' \backslash x$	$\theta_>^2$	$\theta_>$
$\theta_>^2$	$\theta_>^4$	$\theta_>^3$
$\theta_>$	$\theta_>^3$	$\boxed{\theta_>^2}$

$$\frac{g^2}{2}\int dx\,dx'\, \sin\theta_<(x)\sin\theta_<(x')\underbrace{\langle \theta_>(x)\theta_>(x')\rangle}_{\Downarrow}$$

$$g(x-x') \propto \frac{1}{V}\sum_{k\in\text{shell}} \frac{1}{\frac{1}{2}\beta J k^2} e^{ik\cdot(x-x')}$$

这项对什么有贡献?

$$\frac{g^2}{2}\int dx\,dx'\, \underbrace{g(x-x')}_{}\sin\theta_<(x)\sin\theta_<(x')$$

(g 指数衰减: 只有 x−x' 小时才有贡献)

$$= \frac{g^2}{2}\int dx\, g(y)\sin\theta_<(x)\underbrace{\sin\theta_<(x+y)}_{}$$

$$\|$$

~~$\sin\theta(x) + (\partial\theta_<)y$~~

$$\sin(\theta_<(x) + \partial\theta_< y)$$

$\sin(\theta+x) = \sin\theta\cos x + \cos\theta\sin x$

$$= -\frac{g^2}{4}\int dx\,dy\, g(y)(\sin^2\theta_<)y^2(\nabla\theta_<)^2$$

$$= -\frac{g^2}{4}\left(\int dy\, g(y)y^2\right)\int dx\, \underset{\frac{1-\cos 2\theta_<}{2}}{\underset{\downarrow}{\sin^2}}\theta_<\ (\nabla\theta_<)^2$$

$$= -\frac{g^2}{8}\underbrace{\left(\int dy\, g(y)y^2\right)\int dx\, (\nabla\theta_<)^2}_{\text{对 J 的修正}} - \underbrace{\int dx\,[\cos 2\theta_<(\nabla\theta_<)^2]}_{\text{是高阶项 可以忽略}}$$

证明: $\tilde{f}(k) = \int_n f(n) e^{2\pi i n k}$ $\quad \sum_k \tilde{f}(k) = \int_n f(n) \sum_k e^{2\pi i n k}$

$= \int_n f(n) \sum_m \delta(n-m) = \sum_m \int_n f(n) \delta(n-m) = \sum_m f(m)$

eg. $f(n) = e^{-an^2 - bn}$ $\quad \tilde{f}(k) = \int_n f(n) e^{2\pi i n k} = \int_n e^{-an^2 - bn + 2\pi i n k}$

$= \int dn\, e^{-an^2 - (b - 2\pi i k) n} = \int dn\, e^{-(\sqrt{a} n + \frac{b - 2\pi i k}{2\sqrt{a}})^2} e^{\frac{(b - 2\pi i k)^2}{4a}}$

$= \sqrt{\frac{\pi}{a}} e^{\frac{(b - 2\pi i k)^2}{4a}}$