The classical ground state is given by a uniform θ field: $\theta(\boldsymbol{x}, t) = \text{constant}$. The excitations are described by the fluctuations of θ which satisfy the equation of motion

$$(-\frac{1}{2V_0}\partial_t^2 + \frac{\rho_0}{2m}\partial_{\boldsymbol{x}}^2)\theta = 0$$

The above equation is a wave equation. It describes a wave with a linear dispersion

$$\omega = v|\boldsymbol{k}|, \qquad v^2 = \frac{\rho_0 V_0}{m} \tag{3.3.12}$$

where v is the wave velocity.

In many cases, it is not enough to just know the low-energy effective action. It is also important to know how the fields (or operators) in the original theory are represented by the fields (or operators) in the low-energy effective theory. In the following, we will use the density operator as an example to illustrate the representation of the physical operators (or fields) in the original theory by the fields in the effective theory. Firstly, we add a source term $-A_0\rho = -A_0(\rho_0 + \delta\rho)$ that couples to the density in the original Lagrangian. Then, we carry through the same calculation to obtain the effective theory. We find that the effective theory contains an additional term (to linear order in A_0) $-A_0(\rho_0 - \frac{\partial_t\theta}{V_0})$. Thus, the density operator is represented by

$$\rho = \rho_0 - \frac{\partial_t\theta}{V_0} \tag{3.3.13}$$

in the effective theory. Similarly, we find that the boson current density $\boldsymbol{j} = \text{Re}\varphi^\dagger\frac{\partial_{\boldsymbol{x}}}{im}\varphi$ becomes

$$\boldsymbol{j} = \frac{\rho_0}{m}\partial_{\boldsymbol{x}}\theta \tag{3.3.14}$$

in the effective theory. Equations (3.3.13), (3.3.14), and (3.3.10) allow us to use path integrals to calculate the density and current correlations within the simple low-energy effective theory. Those correlations can then be compared with experimental results, such as compressibility and conductivity. The relationship between ρ and θ also tells us that the low-energy fluctuations described by θ are simply the density–current fluctuations.

Problem 3.3.3.

(a) There is another way to obtain the low-energy effective theory for θ. We express $\delta\rho$ in terms of $\partial_t\theta$, $\partial_{\boldsymbol{x}}\theta$, etc. by solving the equation of motion for the $\delta\rho$ field. Then, we substitute $\delta\rho$ back into the action to obtain an effective action that contains only θ. Show that this method produces the same XY-model action (3.3.10).

(b) If we substitute $\delta\rho$ into the density and current operators, then we will reproduce eqn (3.3.13) and eqn (3.3.14), respectively. Show that ρ and $\boldsymbol{j}$ satisfy the conservation law $\partial_t\rho + \partial_{\boldsymbol{x}}\cdot\boldsymbol{j} = 0$.

(c) Use the expression for the density operator given in eqn (3.3.13) to calculate the superfluid density–density correlation function in momentum–frequency space.