diverges, which implies that $\xi = \infty$. We see that the Z_n-symmetry-breaking transition really happens when $\kappa_l = \kappa_c$, where

$$\kappa_c = \frac{n^2}{8\pi} - \sqrt{\frac{3n^4}{32\pi^4}} g_l$$

If κ_l is slightly above κ_c, then we find that

$$g_{min}^2 = 2\left(\frac{32\pi^4}{3n^4}\right)^{1/2} g_l(\kappa_l - \kappa_c)$$

As $\bar{g}_{min}$ is much less than $|\delta\bar{\kappa}_l|$ and $\delta\bar{\kappa}_\xi$, eqn (3.5.14) becomes

$$\sqrt{\frac{3}{32\pi^2}} \frac{1}{g_{min}} = \ln(\frac{\xi}{l})$$

We find that

$$\xi = l e^{\left(C/(\kappa_l - \kappa_c)\right)^{1/2}}, \qquad C = \frac{n^2}{2\pi\bar{g}_l}\left(\frac{3}{32\pi^2}\right)^{3/2}$$

3.5.7 Fixed points and phase transitions

- Fixed points and universal properties.
- A fixed point with no relevant perturbations corresponds to a stable phase. A fixed point with one relevant perturbation corresponds to the transition point between two stable phases.

Running coupling constants and fixed points (or universality classes) are probably the two most important concepts in RG theory. In this section, we are going to discuss them in a general setting. Let us consider a theory with two coupling constants g_1 and g_2. When combined with the cut-off scale l, we can define the dimensionless coupling constants $\tilde{g}_a = g_a l^{\lambda_a}$, $a = 1, 2$. As we integrate out short-distance fluctuations, the dimensionless coupling constants may flow. One of the possible flow diagrams is given in Fig. 3.11(a).

What can we learn from such a flow diagram? First, we note that the flow has two attractive fixed points A and B. If $(\tilde{g}_1, \tilde{g}_2)$ is anywhere below the DCD$'$ line, then, after a long flow, the system will be described by $(\tilde{g}_1(A), \tilde{g}_2(A))$. So the system is described by the fixed point A at long distances. This picture demonstrates the principle of universality. The long-distance behavior of a system does not depend on the short-distance details of the system. All of the systems below the DCD$'$ line share a common long-distance behavior described by the fixed-point theory at A. One of the common long-distance properties is the algebraic decay exponent in the correlation function. All of the systems below the DCD$'$ line have

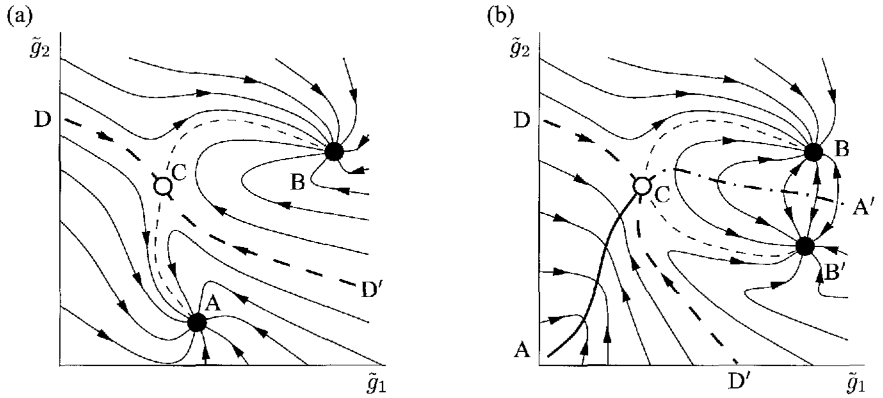

FIG. 3.11. (a) A and B are two stable fixed points representing two phases. C is an unstable fixed point with one relevant operator/direction. The transition between phase A and phase B is continuous. The critical point is described by the unstable fixed point C. (b) The fixed point/line structure of the model (3.5.3). CA is a stable fixed line. B and B′ are two stable fixed points. CA, B, and B' represent three phases. C is the critical point representing the transition between the A phase (with algebraic long-range correlations) and the B/B′ phase (with no long-range correlations). The transition is the KT transition. CA′ is an unstable fixed line, describing the transition between the B phase and the B′ phase.

the same decay exponent in the corresponding correlations. Those common properties are called universal properties. All of the systems that flow to the same fixed point form a universality class.

The systems above the DCD′ line flow to a different fixed point and form a different universality class. Those systems have different universal properties (at long distances). In particular, the decay exponents are different.

The universality classes and phases are closely related. We see that, as $(\tilde{g}_1, \tilde{g}_2)$ moves across the DCD′ line, the long-distance behavior and the long-wavelength fluctuations of the system change suddenly. As a result, the free energy of the system has a singularity at the DCD′ line. Thus, the DCD′ line is a phase transition line that separates two phases. Under this picture, we can say that the systems below the DCD′ line form one phase and the systems above the DCD′ line form the other phase. Phase and universality class mean the same thing here.

Let us start with a system exactly on the fixed point A. We add some perturbations to move the coupling constant $(\tilde{g}_1, \tilde{g}_2)$ away from $(\tilde{g}_1(A), \tilde{g}_2(A))$. As $(\tilde{g}_1, \tilde{g}_2)$ flows back to $(\tilde{g}_1(A), \tilde{g}_2(A))$, the perturbations flow to zero at long distances. Thus, the perturbations are irrelevant perturbations. As all perturbations around a stable fixed point flow to zero, *the effective theory at a stable fixed point contains no relevant or marginal perturbations.*

Now let us consider the long-distance properties of the transition point (or the critical point). If we start anywhere on the DCD′ line, then we can see that the system flows to the fixed point C. Thus, the long-distance behavior of the critical point is described by the unstable fixed point C. Here again, we see universality. No matter where we cross the transition line, the long-distance behavior of the transition point is always the same.

The fixed point C has one (and only one) unstable direction. A perturbation in that direction will flow away from the fixed point. Therefore, the fixed-point theory for C has one, and only one, relevant perturbation. In general, *a critical point describing a transition between two phases has one, and only one, relevant perturbation.* If an unstable fixed point has two relevant perturbations, then the fixed point will describe a tri-critical point.

The model (3.5.3) contains a marginal perturbation. Its flow diagram is more complicated. (See Fig. 3.11(b), where $(\tilde{g}_1, \tilde{g}_2)$ corresponds to $(\tilde{\eta}, \tilde{g})$.) The system has three phases. The phase below the DCD′ line is controlled by the stable fixed line AC. This phase has algebraic long-range correlations. The exponent of the algebraic long-range correlations depends on the position on the AC line. The phase above the DCA′ line is controlled by the stable fixed point B. It has no long-range correlation and is characterized by, say, $\langle\cos(\theta)\rangle < 0$. The phase to the right of the D′CA′ line is controlled by the stable fixed point B′. It has no long-range correlation either, and is characterized by $\langle\cos(\theta)\rangle > 0$. The transition between phase AC and phase B (or phase B′) is controlled by the unstable fixed point C, and the transition between phase B and phase B′ is controlled by the unstable fixed line CA′. The critical exponents depend on the position on the CA′ line.

From the above two simple examples, we see that we can learn a lot about the phases and phase transitions from the RG flow diagram of a system. In Section 3.3.2, we discussed phases and phase transitions from the point of view of symmetry breaking. In this section, we see that phases and phase transitions can also be understood based on an RG picture. Here, I would like to point out that the RG picture (although less concrete) is more fundamental than the symmetry-breaking picture. The symmetry-breaking picture assumes that the two stable fixed points in Fig. 3.11(a) have different symmetries and the phase transition line DCD′ is a symmetry-breaking transition line. This symmetry-breaking picture is not always true. We can construct explicit examples where the fixed points A and B have the same symmetry and the phase transition line DCD′ does not change any symmetry (Coleman and Weinberg, 1973; Halperin *et al.*, 1974; Fradkin and Shenker, 1979; Wen and Wu, 1993; Senthil *et al.*, 1999; Read and Green, 2000; Wen, 2000).