(of a new type, because the definition of on order parameter is difficult) occurs. This phase transition is called the Kosterlitz–Thouless transition, being an extremely important phase transition, where the crucial point are topological defects. Because this idea strongly underlies gauge theory on a lattice, we will discuss it in detail in the next sections.

3.3 Kosterlitz–Thouless Transition

We now understand the properties of symmetry breaking at low temperature and of the order parameter characterizing the phase transition discussed in Sect. 3.1. In the present and the following sections, we examine a phase transition where it is difficult to define such an order parameter.

We consider a classical XY model on a two-dimensional orthogonal lattice, described by the following Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \quad , \tag{3.3.1}$$

where $\langle ij \rangle$ indicates nearest-neighbour sites. We already demonstrated in Sects. 3.1 and 3.2 that the correlation length of the correlation function is characterized by a power behaviour at low temperature

$$\left\langle \mathrm{e}^{\mathrm{i}(\theta_i - \theta_j)} \right\rangle \sim |\boldsymbol{R}_i - \boldsymbol{R}_j|^{-T/2\pi J} \quad , \tag{3.3.2}$$

and exponential decay at high temperature

$$\left\langle \mathrm{e}^{\mathrm{i}(\theta_i - \theta_j)} \right\rangle \sim \mathrm{e}^{-|\boldsymbol{R}_i - \boldsymbol{R}_j|/\xi} \quad . \tag{3.3.3}$$

For the derivation of (3.3.2), the assumption has been made that in

$$\begin{aligned} \cos(\theta_i - \theta_j) &\approx 1 - \frac{1}{2}(\theta_i - \theta_j)^2 \\ &\approx 1 - \frac{1}{2}\left[(\boldsymbol{R}_i - \boldsymbol{R}_j) \cdot \boldsymbol{\nabla}\theta \left(\frac{\boldsymbol{R}_i + \boldsymbol{R}_j}{2}\right)\right]^2 \end{aligned} \tag{3.3.4}$$

the difference in θ between neighbouring sites is small compared with π (the spin wave approximation).

In this approximation, the Hamiltonian becomes quadratic and we always obtain (3.3.2), and no phase transition would occur. Therefore, we conclude that the high-temperature phase transition described by (3.3.3) occurs, because configurations start to be excited where spins at neighbouring sites do make a difference in θ of magnitude π, and can no longer be described by a continuous function $\theta(\boldsymbol{R})$. Indeed, for the derivation of (3.3.3) in Sect. 3.1 we used the high-temperature expansion where (i) the angle θ is defined on the lattice and (ii) the range of θ is limited to the interval from 0 to 2π (reflecting the 2π periodicity in θ).

So, what kind of distribution will it be that cannot be described by a continuous function $\theta(\boldsymbol{R})$? We have to handle three points to answer this question. First, when θ_i is expressed by $\theta(\boldsymbol{R})$ in the continuum limit, owing to (i), $\theta(\boldsymbol{R})$ may also have singularities $\boldsymbol{R} = \boldsymbol{R}_0$. That is, when the singularity is placed inside the plaquette, where originally no spin has been defined, we do not run into difficulties. Second, owing to (ii), $\theta(\boldsymbol{R})$ may also be multi-valued. Of course, this multi-valueness is limited at one point $\boldsymbol{R}$ to be an integer multiplied by 2π.

The last point is that in the continuum limit, besides the singularity at $\boldsymbol{R} = \boldsymbol{R}_0$, the function $\theta(\boldsymbol{R})$ is determined by the Hamiltonian

$$\mathcal{H} = \frac{J}{2} \int (\nabla\theta(\boldsymbol{R}))^2 d^2\boldsymbol{R} \tag{3.3.5}$$

as a solution of the variational equation $\delta\mathcal{H} = 0$. The trivial solution which for all sides i, $\theta_i = \theta_0$, obeys $\delta\mathcal{H} = 0$. However, we are looking for different, non-homogeneous solutions. It is evident that $\delta\mathcal{H}$ is given explicitly by the Laplace equation

$$\nabla^2\theta(\boldsymbol{R}) = 0 \ . \tag{3.3.6}$$

In two dimensions, the Laplace equation can be derived by the Cauchy–Riemann equation. Recall that both the real part and the imaginary part of a regular function of $z = x + \mathrm{i}y$ ($\boldsymbol{R} = (x, y)$) obey (3.3.6).

We conclude that the solution can be given by

$$\theta(\boldsymbol{R}) = \theta(z) = \pm \operatorname{Im} \ln(z - z_0) \ . \tag{3.3.7}$$

Here, $z_0 = x_0 + \mathrm{i}y_0$ is the position of the singularity, and by moving once anticlockwise around the singularity, the phase of $z - z_0$ gains 2π, and $\theta(\boldsymbol{R})$ just changes by $\pm 2\pi$. In such a way, a whirl emerges around the point $z = z_0$, being the so-called vortex.

Next, we derive the excitation energy corresponding to the solution (3.3.7). It is sufficient to insert (3.3.7) into (3.3.5), with the result

$$\begin{aligned} E_{\text{vortex}} &= \frac{J}{2} \int_a^{R_\mathrm{c}} 2\pi R \, \mathrm{d}R \frac{1}{R^2} \\ &= \pi J \ln \frac{R_\mathrm{c}}{a} \ . \end{aligned} \tag{3.3.8}$$

Here, a is the smallest size where the continuum limit (3.3.5) is valid, that is, the lattice spacing. R_c is the size of the sample. Then, the vortex energy is logarithmically diverging with respect to the sample size!

Therefore, can we forget about the configuration (3.3.5) in the limit $R_\mathrm{c} \to \infty$? In this context, recall the discussion that followed equation (3.1.5). That is, not the energy E, but the free energy $F = E - TS$ has to be considered. Therefore, we have to calculate the entropy S_{vortex} of one vortex. With W being the number of all possible microstates, the entropy is given by $S =$

$\ln W$. From the number $W \sim R_c^2/a^2$ of possibilities of placing the centre of the vortex, we again obtain a logarithmic dependence:

$$S_{\text{vortex}} = \ln\left(\frac{R_c}{a}\right)^2 = 2\ln\frac{R_c}{a} \quad . \tag{3.3.9}$$

Using (3.3.8) and (3.3.9) we obtain for the free energy of the vortex

$$\begin{aligned} F_{\text{vortex}} &= E_{\text{vortex}} - TS_{\text{vortex}} \\ &= (\pi J - 2T)\ln\frac{R_c}{a} \quad . \end{aligned} \tag{3.3.10}$$

From the sign of the coefficient of $\ln(R_c/a)$, we conclude that at $T_c = \pi J/2$, a phase transition occurs between the phases where a vortex does or does not occur due to thermal excitation. This transition is the so-called Kosterlitz–Thouless transition (KT transition), the unique phase transition where a vortex, i.e. a topological defect, plays the main role.

However, the above discussion is incomplete because only one vortex has been considered, and also the discussion whether thermal excitation arises or not is incomplete. In order to clarify this point, we have to consider a system with many vortices, and so for a more detailed mathematical investigation, we will introduce the so-called duality mapping.

We return to (3.3.1) and discuss again the partition function

$$Z = \int \mathrm{d}\theta_1 \cdots \int \mathrm{d}\theta_N \, \exp\left[\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right] \quad . \tag{3.3.11}$$

Consider one pair $\langle ij \rangle$ corresponding to one link. $\exp[\beta J \cos(\theta_i - \theta_j)]$ is 2π-periodic in $\theta_i - \theta_j$, and for every $\theta_i - \theta_j = 2\pi m + \varepsilon$ ($\varepsilon \ll \pi$), the integral becomes Gaussian, $\mathrm{e}^{\beta J}\mathrm{e}^{-(\beta J/2)\varepsilon^2}$.

We replace $\exp[\beta J \cos(\theta_i - \theta_j)]$ by a function that fulfils these two properties and that can be handled more easily:

$$\mathrm{e}^{\beta J \cos(\theta_i - \theta_j)} \to \sum_{m=-\infty}^{\infty} \mathrm{e}^{\beta J} \exp\left[-\left(\frac{\beta J}{2}\right)(\theta_i - \theta_j - 2\pi m)^2\right] \quad . \tag{3.3.12}$$

Near every minimum $2\pi m$, the right-hand side of (3.3.12) equals the above approximation. Furthermore, using Poisson's equation

$$\sum_{m=-\infty}^{\infty} h(m) = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\phi \, h(\phi)\, \mathrm{e}^{2\pi i l \phi} \quad , \tag{3.3.13}$$

(3.3.12) becomes

$$
\begin{aligned}
&\mathrm{e}^{\beta J\cos(\theta_i-\theta_j)} \\
&\quad\to \sum_{l_{ij}=-\infty}^{\infty} \int \mathrm{d}\phi\, \mathrm{e}^{\beta J} \exp\left[-\left(\frac{\beta J}{2}\right)(\theta_i-\theta_j-2\pi\phi)^2+2\pi \mathrm{i} l_{ij}\phi\right] \\
&\quad= \frac{1}{\sqrt{2\pi\beta J}} \sum_{l_{ij}=-\infty}^{\infty} \mathrm{e}^{\beta J} \exp[\mathrm{i} l_{ij}(\theta_i-\theta_j)] \exp[-l_{ij}^2/2\beta J] \quad . \qquad (3.3.12')
\end{aligned}
$$

Inserting (3.3.12′) into (3.3.11), up to some constant factor, we obtain

$$
Z=\int \mathrm{d}\theta_1 \cdots \int \mathrm{d}\theta_N \sum_{\{l_{ij}\}} \exp\left(-\sum_{\langle ij\rangle}\left[\frac{l_{ij}^2}{2\beta J}-\mathrm{i} l_{ij}\cdot(\theta_i-\theta_j)\right]\right) \quad . \qquad (3.3.14)
$$

Here, l_{ij} is defined on every link, and we interpret it as a vector field $l_\mu(r)$ ($\mu = x, y$) that is directed from the starting point r, the left-hand side or the lower side of the link between i and j, to the other side of the link. Then, the argument of the exponent in (3.3.14) becomes

$$
-\sum_{r,\mu}\left[\frac{l_\mu(r)^2}{2\beta J}-\mathrm{i} l_\mu(r)\cdot(\theta(r)-\theta(r+\mu))\right] \quad . \qquad (3.3.15)
$$

Here, r runs over all lattice points, and μ is the sum in x and y. We can rewrite the second term in (3.3.15) as

$$
-\mathrm{i}\sum_{r,\mu} l_\mu(r)\cdot(\theta(r)-\theta(r+\mu)) = -\mathrm{i}\sum_{r,\mu}(l_\mu(r)-l_\mu(r-\mu))\theta(r) \qquad (3.3.16)
$$

and can therefore perform the $\theta(r)$-integration from 0 to 2π. In addition to numerical factors, we obtain from (3.3.14)

$$
Z=\sum_{\{l_\mu(r)\}} \exp\left(-\sum_{r,\mu}\frac{l_\mu(r)^2}{2\beta J}\right)\prod_r \delta_{\sum_\mu(l_\mu(r)-l_\mu(r-\mu)),0} \quad . \qquad (3.3.17)
$$

In (3.3.17), the constraint given by the delta function

$$
\sum_\mu (l_\mu(r)-l_\mu(r-\mu))=0 \qquad (3.3.18)
$$

is the discrete version of $\operatorname{div} \boldsymbol{l}=0$ of the vector $\boldsymbol{l}(r)=(l_x(r), l_y(r))$.

Then, by analogy, a vector field $\boldsymbol{n}(r)$ satisfying $\boldsymbol{l}(r)=\operatorname{rot}\boldsymbol{n}(r)$ should exist, and because $\boldsymbol{l}(r)$ is two-dimensional in this case, using only the z-component of $\boldsymbol{n}$, we write $l_x(r)=\partial_y n(r)$, $l_y(r)=-\partial_x n(r)$. Indeed, the discrete version of these equations

$$
\begin{aligned}
l_x(r) &= n(r)-n(r-y) \quad , \\
l_y(r) &= -n(r)+n(r-x) \quad ,
\end{aligned} \qquad (3.3.19)
$$

inserted into the left-hand side of (3.3.18) leads to

$$\begin{aligned} & l_x(r) - l_x(r-x) + l_y(r) - l_y(r-y) \\ & \quad = [n(r) - n(r-y)] - [n(r-x) - n(r-x-y)] \\ & \qquad + [-n(r) + n(r-x)] - [-n(r-y) + n(r-x-y)] \\ & \quad = 0 \ . \end{aligned} \tag{3.3.20}$$

Next, we consider the number of degrees of freedom. With N being the number of lattice points, the number of different $l_\mu(r)$ is $2N$. The number of conditions (3.3.18) equals the number of points r, that is, N, and therefore only $2N - N = N$ vectors are independent. Indeed, because the number of $n(r)$ is also given by N, the result is consistent. Inserting (3.3.19) into (3.3.17), we obtain

$$Z = \sum_{\{n(r)\}} \exp\left(- \sum_{r,\mu} \frac{1}{2\beta J} (n(r) - n(r-\mu))^2 \right) \ . \tag{3.3.21}$$

Interpreting $n(r)$ as the height of the atom layer at position r, then (3.3.21) describes a model where with increasing height difference at neighbouring positions, the energy becomes larger. The model describes the roughening transition of the surface. Notice that in (3.3.21), β appears in the denominator, and therefore the high-temperature (low-temperature) phase of the original XY model corresponds to the low-temperature (high-temperature) phase of this model.

Let us again rewrite the sum (3.3.21) running over integers $n(r)$ using the Poisson equation (3.3.13):

$$Z = \int_{-\infty}^{\infty} \prod_r \mathrm{d}\phi(r) \sum_{m(r)=-\infty}^{\infty} \exp\left[-\frac{1}{2\beta J} \sum_{r,\mu} (\Delta_\mu \phi(r))^2 + 2\pi \mathrm{i} \sum_r m(r)\phi(r) \right] \ . \tag{3.3.22}$$

Here, we defined $\Delta_\mu \phi(r) = \phi(r) - \phi(r-\mu)$. The integral (3.3.22) can be performed when $\phi(r)$ is Fourier transformed, with the result

$$Z = Z_{\mathrm{SW}} \sum_{m(r)=-\infty}^{\infty} \exp\left\{ -2\pi^2 \beta J \sum_{r,r'} m(r) G(r-r') m(r') \right\} \ . \tag{3.3.23}$$

Here, Z_{SW} is the sum of states of the spin waves, and $m(r)$ indicates the presence of $m(r)$ vortices $(= 0, \pm 1, \pm 2, \ldots)$ at position r. Furthermore, $G(r-r')$ is given by

$$G(r-r') = \int_{-\pi}^{\pi} \frac{\mathrm{d}k_x}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}k_y}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}(r-r')}}{(4 - 2\cos k_x - 2\cos k_y)} \ . \tag{3.3.24}$$

For large $|r-r'|$, it behaves like

$$G(r-r') \approx -\frac{1}{2\pi} \ln\left(\frac{|r-r'|}{a}\right) - \frac{1}{4} + G(0) \tag{3.3.25}$$

and $G(0)$ is estimated by the logarithmic dependence $\ln R_c$, where the lower range of $k = \sqrt{k_x + k_y}$ is given by π/R_c.

Splitting $G(r-r')$ into two parts,

$$G(r-r') = G(0) + G'(r-r') \ , \tag{3.3.26}$$

where by definition only the first term contains divergent terms, we can write (3.3.23) as

$$\begin{aligned} Z = Z_{\mathrm{SW}} \sum_{m(r)=-\infty}^{\infty} & \exp\left\{-2\pi^2 \beta J G(0)\left[\sum_r m(r)\right]^2\right\} \\ & \times \exp\left\{-2\pi^2 \beta J \sum_{r,r'} m(r) G'(r-r') m(r')\right\} \end{aligned} \tag{3.3.27}$$

and conclude that there is only a contribution to the sum of states in the case when the term $[\sum_r m(r)]^2$ that is multiplied by $G(0)$ vanishes. As mentioned, the absolute value of $m(r)$ indicates the vortex number, and the sign its direction. Interpreting $m(r)$ as an electric charge at position r, and identifying the logarithmic potential (3.3.25) with the Coulomb potential in two dimensions, then $\sum_r m(r) = 0$ can be interpreted as the neutrality condition of the whole system. That is, (3.3.27) signifies that the XY model can be split into a degree of freedom of spin waves and a degree of freedom of vortices; the latter is equivalent to a two-dimensional Coulomb gas.

From this fact, we can deduce the following physical picture. At low temperature, even when vortices are excited, they must emerge as a $+/-$ pair, forming a dipole, but no free charge, and therefore the system is in the insulator phase (dielectric substance). The lowering of the Coulomb force due to the dielectric constant ε_0 does not affect it being a long-distance force, and therefore the fact that the charge must be bound as a plus or minus charge pair is a self-consistent description. However, when the temperature becomes higher, the number of charges becomes larger and larger, and therefore the screening effect gains importance. Therefore, conversely, because of the existence of free charges, the Coulomb force becomes a short-range force due to screening, and therefore free charges can exist, and the metallic state emerges self-consistently.

The KT transition is the phase transition between the metallic state and the insulator state. We need to obtain a more quantitative picture:

$$\begin{aligned} & \sum_{r,r'} m(r) G'(r-r') m(r') \\ & \quad = \sum_{r \neq r'} m(r) G'(r-r') m(r') \end{aligned}$$

$$\approx -\frac{1}{4}\sum_{r\neq r'} m(r)m(r') - \frac{1}{2\pi}\sum_{r\neq r'} m(r)\ln\left(\frac{|r-r'|}{a}\right)m(r')$$
$$= \frac{1}{4}\sum_{r} m(r)^2 - \frac{1}{2\pi}\sum_{r\neq r'} m(r)\ln\left(\frac{|r-r'|}{a}\right)m(r') \quad . \tag{3.3.28}$$

Here, we have used $\sum_r m(r) = 0$, $G'(0) = 0$ and the approximation (3.3.25) of $G'(r-r')$. Therefore, from (3.3.27) we obtain

$$Z = Z_{\mathrm{SW}} \sum_{m(r)=-\infty}^{\infty} \exp\left[\ln y \cdot \sum_r m(r)^2 - \pi\beta J \sum_{r\neq r'} m(r)\ln\left(\frac{|r-r'|}{a}\right)m(r')\right] . \tag{3.3.29}$$

Here, y is the so-called fugacity, given by $\mathrm{e}^{\beta\mu}$, with μ being the chemical potential. In the present case, μ is given by $\mu = -\pi^2 J/2$.

When y is small enough, the absolute value of $m(r)$ cannot become very large. We now discuss the properties of the system in this dilute state limit. In order to do so, we return to the step (3.3.22) and add by hand the term $\ln(y)\sum_r m(r)^2$:

$$Z = \int \prod_r \mathrm{d}\phi(r) \sum_{m(r)=-\infty}^{\infty} \exp\left[-\frac{1}{2\beta J}\sum_{r,\mu}(\Delta_\mu\phi(r))^2 + \ln y \cdot \sum_r m(r)^2 + 2\pi\mathrm{i}\sum_r m(r)\phi(r)\right] . \tag{3.3.30}$$

Here, because y is small, we only sum over $0, \pm 1$ in the sum of $m(r)$. We obtain approximately

$$\sum_{m(r)=0,\pm 1} \exp\left[\ln y \cdot m(r)^2 + 2\pi\mathrm{i}m(r)\phi(r)\right] = 1 + 2y\cos(2\pi\phi(r))$$
$$\cong \exp\left[2y\cos(2\pi\phi(r))\right] \quad . \tag{3.3.31}$$

Then, (3.3.30) becomes

$$Z \cong \int \prod_r \mathrm{d}\phi(r) \exp\left[-\frac{1}{2\beta J}\sum_{r,\mu}(\Delta_\mu\phi(r))^2 + 2y\sum_r \cos(2\pi\phi(r))\right] . \tag{3.3.32}$$

Performing the continuum limit at this stage, we obtain the so-called Sine–Gordon model, given by

$$Z \cong \int \mathcal{D}\phi(\boldsymbol{r}) \exp\left[-\int \mathrm{d}^2\boldsymbol{r}\left(\frac{1}{2\beta J}|\nabla\phi(\boldsymbol{r})|^2 - 2y\cos(2\pi\phi(\boldsymbol{r}))\right)\right]$$
$$= \int \mathcal{D}\phi(\boldsymbol{r}) \exp\left[-\int \mathrm{d}^2\boldsymbol{r}\left(\frac{1}{2}|\nabla\phi(\boldsymbol{r})|^2 - 2y\cos(2\pi\sqrt{\beta J}\,\phi(\boldsymbol{r}))\right)\right] . \tag{3.3.33}$$

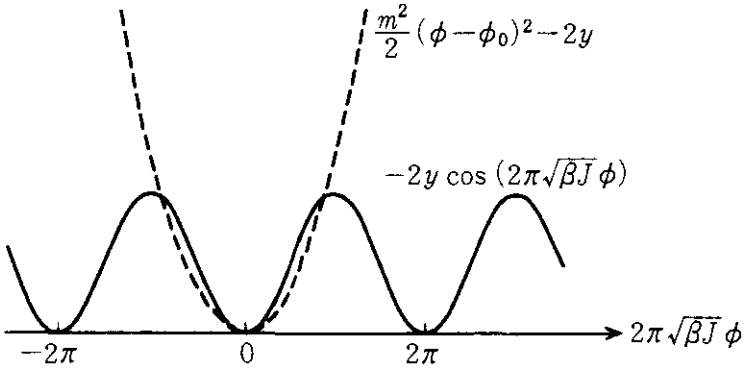

Fig. 3.8. The potential of the phase ϕ

Here, we redefined the integration variable $\phi(\boldsymbol{r})$ in a suitable manner. With $\phi(\boldsymbol{r}) = \phi_0$ (constant) being the minimum of the action, we obtain an infinite number of possibilities $2\pi\sqrt{\beta J}\phi_0 = 2\pi m$ (m constant), as indicated in Fig. 3.8. The question is: Will $\phi(\boldsymbol{r})$ rest in the vicinity of one valley or not?

We apply the variation method discussed in (2.2.20) and (2.2.21) to discuss this question. We choose as the trial action the action S_0 with the dotted line in Fig. 3.8 as the potential, obeying the quadratic equation

$$S_0 = \int \mathrm{d}^2\boldsymbol{r} \left\{ \frac{1}{2}|\nabla\phi(\boldsymbol{r})|^2 + \frac{m^2}{2}(\phi(\boldsymbol{r}) - \phi_0)^2 \right\} \quad . \tag{3.3.34}$$

Using the curvature m^2 of the potential as the variation parameter, it can be determined whether $\phi(\boldsymbol{r})$ is bounded in the vicinity of ϕ_0 or not, depending on whether m is zero or finite. Let us compute $f(m) = -T\ln Z_0 + T\langle S - S_0\rangle$, the right-hand side of (2.2.21). In order to do so, we perform a Fourier transformation of S_0 in (3.3.34). With $\phi_{\boldsymbol{k}}$ being the Fourier component of $\phi(\boldsymbol{r}) - \phi_0$, we can write

$$\begin{aligned} S_0 &= \sum_{\boldsymbol{k}} \frac{1}{2}(\boldsymbol{k}^2 + m^2)\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}} \\ &= \sum_{\boldsymbol{k}} \frac{1}{2}(\boldsymbol{k}^2 + m^2)\left\{(\mathrm{Re}\,\phi_{\boldsymbol{k}})^2 + (\mathrm{Im}\,\phi_{\boldsymbol{k}})^2\right\} \\ &= \sum_{\boldsymbol{k}:\mathrm{half}} (\boldsymbol{k}^2 + m^2)\left\{(\mathrm{Re}\,\phi_{\boldsymbol{k}})^2 + (\mathrm{Im}\,\phi_{\boldsymbol{k}})^2\right\} \quad . \end{aligned} \tag{3.3.35}$$

Notice that because $\phi(\boldsymbol{r})$ is real, the relation $\phi_{\boldsymbol{k}}^* = \phi_{-\boldsymbol{k}}$ holds, and that when the degrees of freedom are assigned both to the real and imaginary parts for every $\boldsymbol{k}$, then it is sufficient to sum only over one half of the $\boldsymbol{k}$'s.

With this action S_0 in the exponential, performing the Gauss integral with the integration measure

$$\prod_{\boldsymbol{k}:\mathrm{half}} \int \mathrm{d}(\mathrm{Re}\,\phi_{\boldsymbol{k}})\,\mathrm{d}(\mathrm{Im}\,\phi_{\boldsymbol{k}})$$

gives the right expression for $\langle\ \rangle_0$, leading to

$$\begin{aligned}\langle\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}\rangle_0 &= \left\langle(\mathrm{Re}\,\phi_{\boldsymbol{k}})^2\right\rangle_0 + \left\langle(\mathrm{Im}\,\phi_{\boldsymbol{k}})^2\right\rangle_0 \\ &= \frac{1}{2(\boldsymbol{k}^2+m^2)} + \frac{1}{2(\boldsymbol{k}^2+m^2)} \\ &= \frac{1}{\boldsymbol{k}^2+m^2}\ . \end{aligned} \tag{3.3.36}$$

Using the following equation derived from the Gauss integral

$$\begin{aligned}\langle\exp[ia(\phi(\boldsymbol{r})-\phi_0)]\rangle_0 &= \exp\left[-\frac{1}{2}a^2\left\langle(\phi(\boldsymbol{r})-\phi_0)^2\right\rangle_0\right] \\ &= \exp\left[-\frac{1}{2}a^2\sum_{\boldsymbol{k}}\langle\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}\rangle_0\right] \\ &= \exp\left[-\frac{1}{2}a^2\sum_{\boldsymbol{k}}\frac{1}{\boldsymbol{k}^2+m^2}\right] \end{aligned} \tag{3.3.37}$$

we can compute every term in $f(m)$, and the result is (when performing the k integration, a cut-off k_c of magnitude of the inverse lattice constant is introduced, and terms with higher powers in m^2/k_c^2 as well as terms independent of m^2 are ignored)

$$f(m) \cong T\left[m^2 - 2y\left(\frac{m^2}{k_c^2}\right)^{(\pi/2)\beta J}\right]\ . \tag{3.3.38}$$

Notice that the term $\sum_{\boldsymbol{k}} 1/(\boldsymbol{k}^2+m^2)$ in (3.3.37) causes the $\ln(k_c^2/m^2)$ dependence, leading finally to the particular exponent in the second term in (3.3.38).

The behaviour of $f(m)$ is determined by the relation between the powers of the first and second terms. That is, as shown in Fig. 3.9, for the case $(\pi/2)\beta J > 1$, the minimum is at $m^2 = 0$, whereas for $(\pi/2)\beta J < 1$, a minimum at a finite value of m^2 emerges. Therefore, the temperature obeying the equation $(\pi/2)\beta J = 1$ corresponds to the phase transition point $T = T_c = \pi J/2$. At temperatures lower than T_c, the cos term in (3.3.33) can effectively be ignored and the system can be described by the spin waves only (in the

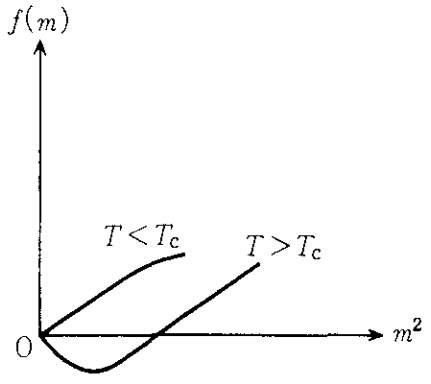

Fig. 3.9. The free energy $f(m)$ as obtained by the variation method. T_c is given by $T_c = \pi J/2$

picture of the Coulomb gas this corresponds to the insulator phase). On the other hand, at higher temperatures the potential arising due to the cos term leads to confining of phase ϕ, and as a result, the Coulomb force becomes a short-range force with the range $1/m$. This corresponds to the metallic phase, where the Coulomb force is screened.

In this manner, using a simple variational method, we have discussed whether the cos term effectively plays a role at low energy. This approach is quite similar to the discussion of relevant/irrelevant terms in the theory of renormalization. Indeed, an excellent paper [G.10] has been written about the application of Wilson's style renormalization group to the model (3.3.33), which is recommended to the reader.

We end this section by discussing briefly the three-dimensional XY model. In three dimensions, it is also possible to perform the duality transformation starting from the reformulation (3.3.12). However, $l_\mu(r)$ introduced below (3.3.14) is now a three-component vector $\boldsymbol{l}(r)$, and therefore the vector field $\boldsymbol{n}$ that describes $\boldsymbol{l}$ through $\boldsymbol{l} = \operatorname{rot} \boldsymbol{n}$ also has three components. Therefore, the integers vector $\boldsymbol{m} = m_\mu(r)$ appearing when rewriting the sum for every component in an integral using the Poisson equation also has three components. For details, the literature [19,20] is recommanded. As can be found there, the sum of states finally reads

$$Z = \prod_{r,\mu} \sum_{m_\mu(r)=-\infty}^{\infty} \prod_{r} \delta_{\Delta_\mu m_\mu(r),0} \exp\left[-4\pi^2 \beta J \sum_{r,r',\mu} m_\mu(r) v(r-r') m_\mu(r') \right]. \tag{3.3.39}$$

Here, $v(r-r')$ is the three-dimensional Coulomb potential. From the property of the delta function, it follows that at every site the number of ingoing and outgoing vector fields $\boldsymbol{m}(r)$ are equal (that is, there is no source present).

Therefore, when $m_\mu(r)$ at one link is given by 1, on both neighbouring sites it must be joined with links with $m_{\mu'} = \pm 1$. In this manner, a string of joined links grows, and this string cannot have a starting point; therefore the only possibility is the construction of a closed loop. Comparing the above consideration with the two-dimensional case, where topological defects (vortices) have been points, in the three-dimensional space, topological defect lines (vortex lines) arise that are creating loops themselves. Equation (3.3.39) describes the statistical mechanics of loops created by segments interacting via the Coulomb force. It is important to notice how the form of the topological defects changes depending on the dimension. In the next section, when the gauge field is discussed, we will meet a similar case.

However, it is a mistake to think that a far-reaching force acts between the loops just because the Coulomb interaction acts on the vortex segments. When m_μ in the direction from site i to site $i+\mu$ is given by $+1$, then in the direction from $i+\mu$ to i we have $m_\mu(i) = -1$. Therefore, from a wider point of view, the $+$ and $-$ contributions of m_μ almost cancel. Therefore, we expect an effective short-distance potential for $v(r-r')$.

Having this in mind, we choose one vortex line and try to sketch roughly the phase transition in the three-dimensional XY model. Consider the free energy F of a vortex line of length L. With (3.3.39), the energy is given by

$$E \cong 4\pi^2 Jv(0)L \ . \tag{3.3.40}$$

Here, $v(0)$ is a representative finite short-range value of $v(r-r')$ (≈ 0.253). On the other hand, because the entropy S is given by the logarithm of the number of possibilities of constructing a string of length L, it can roughly be estimated as follows. We ignore the condition that the string must be closed, then because at every site the string has the possibility of proceeding further in five directions, which are all directions different from the one it came from, we conclude $W \propto 5^L$. Therefore we obtain

$$F \cong (4\pi^2(0.253)J - T\ln 5)L \tag{3.3.41}$$

and has critical temperature $T_c = 4\pi^2(0.253)J/\ln 5$. Above this temperature, vortex rings with infinite radius exist, leading to a short-range spin correlation function; on the other hand, at low temperature, all vortex rings have a finite radius, having no influence at a scale above this radius. Above, we described the phase transition of the three-dimensional XY model from the point of view of condensation of topological defects.

3.4 Lattice Gauge Theory and the Confinement Problem

In solid state physics, the existence of a crystal lattice is fundamental. In the tight-binding approximation, at every lattice point i, an atomic wave function φ_i is defined, and the wave function of the electrons in the crystal is determined by linear superposition of the atomic wave functions. This approximation is useful because the originally infinitely many degrees of freedom in the continuous space around the lattice point i are represented only by φ_i, and when the number of lattice points is N, in this manner the problem is reduced to a system with N degrees of freedom.

For simplicity, we consider a two-dimensional square lattice as shown in Fig. 3.10. On the lattice points are orbits, and between neighbouring orbits there is the hopping integral t. The transition process of an electron from i to j is described by the Hamiltonian $-t_{ji}a_j^\dagger a_i$. $a_i^\dagger$ and a_i are the creation and annihilation operators of the orbit φ_i. We require that the theory is invariant when an arbitrary phase $\mathrm{e}^{\mathrm{i}\theta_i}$ is multiplied by φ_i at every point i. This corresponds to the transformation $a_i \to a_i \mathrm{e}^{-\mathrm{i}\theta_i}$, $a_i^\dagger \to a_i^\dagger \mathrm{e}^{\mathrm{i}\theta_i}$. The Hamiltonian is invariant if t_{ij} transforms as $t_{ij} \to t_{ij}\mathrm{e}^{\mathrm{i}(\theta_i-\theta_j)}$. For the case that t_{ij} is a simple number, no such transformation can emerge. It is necessary to think of t_{ij} as a field causing such a phase transformation. This field is nothing but the electromagnetic field.