15.1 The Geometry of Gauge Invariance

In Section 4.1 we wrote down the Lagrangian of Quantum Electrodynamics and noted the curious fact that it is invariant under a very large group of transformations (4.6), allowing an independent symmetry transformation at every point in spacetime. This invariance is the famous *gauge symmetry* of QED. From the modern viewpoint, however, gauge symmetry is not an incidental curiosity, but rather the fundamental principle that determines the form of the Lagrangian. Let us now review the elements of the theory, taking the modern viewpoint.

We begin with the complex-valued Dirac field $\psi(x)$, and stipulate that our theory should be invariant under the transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x). \tag{15.1}$$

This is a phase rotation through an angle $\alpha(x)$ that varies arbitrarily from point to point. How can we write a Lagrangian that is invariant under this transformation? As long as we consider terms in the Lagrangian that have no derivatives, this is easy: We simply write the same terms that are invariant to global phase rotations. For example, the fermion mass term

$$m\bar{\psi}\psi(x)$$

is permitted by global phase invariance, and the local invariance gives no further restriction.

The difficulty arises when we try to write terms including derivatives. The derivative of $\psi(x)$ in the direction of the vector n^μ is defined by the limiting procedure

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x)\right]. \tag{15.2}$$

However, in a theory with local phase invariance, this definition is not very sensible, since the two fields that are subtracted, $\psi(x+\epsilon n)$ and $\psi(x)$, have completely different transformations under the symmetry (15.1). The quantity $\partial_\mu \psi$, in other words, has no simple tranformation law and no useful geometric interpretation.

In order to subtract the values of $\psi(x)$ at neighboring points in a meaningful way, we must introduce a factor that compensates for the difference in phase transformations from one point to the next. The simplest way to do this is to define a scalar quantity $U(y,x)$ that depends on the two points and has the transformation law

$$U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \tag{15.3}$$

simultaneously with (15.1). At zero separation, we set $U(y,y)=1$; in general, we can require $U(y,x)$ to be a pure phase: $U(y,x) = \exp[i\phi(y,x)]$. With this definition, the objects $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation law, and we can subtract them in a manner that is meaningful despite the local symmetry. Thus we can define a sensible derivative, called the *covariant derivative*, as follows:

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \big[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\big]. \tag{15.4}$$

To make this definition explicit, we need an expression for the comparator $U(y,x)$ at infinitesimally separated points. If the phase of $U(y,x)$ is a continuous function of the positions y and x, then $U(y,x)$ can be expanded in the separation of the two points:

$$U(x+\epsilon n, x) = 1 - ie\,\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2). \tag{15.5}$$

Here we have arbitrarily extracted a constant e. The coefficient of the displacement ϵn^μ is a new vector field $A_\mu(x)$. Such a field, which appears as the infinitesimal limit of a comparator of local symmetry transformations, is called a *connection.* The covariant derivative then takes the form

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x). \tag{15.6}$$

By inserting (15.5) into (15.3), one finds that A_μ transforms under this local gauge transformation as

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x). \tag{15.7}$$

To check that all of these expressions are consistent, we can transform $D_\mu\psi(x)$ according to Eqs. (15.1) and (15.7):

$$\begin{aligned} D_\mu \psi(x) \to &\Big[\partial_\mu + ie\big(A_\mu - \frac{1}{e}\partial_\mu \alpha\big)\Big]\, e^{i\alpha(x)}\psi(x) \\ &= e^{i\alpha(x)}\big(\partial_\mu + ieA_\mu\big)\psi(x) = e^{i\alpha(x)} D_\mu \psi(x). \end{aligned} \tag{15.8}$$

Thus the covariant derivative transforms in the same way as the field ψ, exactly as we constructed it to in the original definition (15.4).

We have now recovered most of the familiar ingredients of the QED Lagrangian. From our current viewpoint, however, the definition of the covariant derivative and the transformation law for the connection A_μ follow from the postulate of local phase rotation symmetry. Even the very existence of the vector field A_μ is a consequence of local symmetry: Without it we could not write an invariant Lagrangian involving derivatives of ψ.

More generally, our present analysis gives us a way to construct all possible Lagrangians that are invariant under the local symmetry. In any term with derivatives of ψ, replace these with covariant derivatives. According to Eq. (15.8), these transform in exactly the same manner as ψ itself. Therefore any combination of ψ and its covariant derivatives that is invariant under a global phase rotation (and only these combinations) will also be locally invariant.

To complete the construction of a locally invariant Lagrangian, we must find a kinetic energy term for the field A_μ: a locally invariant term that depends on A_μ and its derivatives, but not on ψ. This term can be constructed

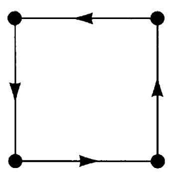

Figure 15.1. Construction of the field strength by comparisons around a small square in the $(1, 2)$ plane.

either integrally, from the comparator $U(y, x)$, or infinitesimally, from the covariant derivative.

Working from $U(y, x)$, we will need to extend our explicit formula (15.5) to the next term in the expansion in ϵ. Using the assumption that $U(y, x)$ is a pure phase and the restriction $(U(x, y))^\dagger = U(y, x)$, it follows that

$$U(x + \epsilon n, x) = \exp\left[-ie\epsilon n^\mu A_\mu(x + \tfrac{\epsilon}{2}n) + \mathcal{O}(\epsilon^3)\right]. \tag{15.9}$$

(Relaxing these restrictions introduces additional vector fields into the theory; this is an unnecessary complication.) Using this expansion for $U(y, x)$, we link together comparisons of the phase direction around a small square in spacetime. For definiteness, we take this square to lie in the $(1, 2)$-plane, as defined by the unit vectors $\hat{1}$, $\hat{2}$ (see Fig. 15.1). Define $\mathbf{U}(x)$ to be the product of the four comparisons around the corners of the loop:

$$\begin{aligned}\mathbf{U}(x) \equiv U(x, x + \epsilon\hat{2})&U(x + \epsilon\hat{2}, x + \epsilon\hat{1} + \epsilon\hat{2}) \\ &\times U(x + \epsilon\hat{1} + \epsilon\hat{2}, x + \epsilon\hat{1})U(x + \epsilon\hat{1}, x).\end{aligned} \tag{15.10}$$

The transformation law (15.3) for U implies that $\mathbf{U}(x)$ is locally invariant. In the limit $\epsilon \to 0$, it will therefore give us a locally invariant function of A_μ. To find the form of this function, insert the expansion (15.9) to obtain

$$\begin{aligned}\mathbf{U}(x) = \exp\Big\{ - i\epsilon e\big[&-A_2(x + \tfrac{\epsilon}{2}\hat{2}) - A_1(x + \tfrac{\epsilon}{2}\hat{1} + \epsilon\hat{2}) \\ &+ A_2(x + \epsilon\hat{1} + \tfrac{\epsilon}{2}\hat{2}) + A_1(x + \tfrac{\epsilon}{2}\hat{1})\big] + \mathcal{O}(\epsilon^3)\Big\}.\end{aligned} \tag{15.11}$$

When we expand the exponent in powers of ϵ, this reduces to

$$\mathbf{U}(x) = 1 - i\epsilon^2 e\big[\partial_1 A_2(x) - \partial_2 A_1(x)\big] + \mathcal{O}(\epsilon^3). \tag{15.12}$$

Therefore the structure

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15.13}$$

is locally invariant. Of course, $F_{\mu\nu}$ is the familiar electromagnetic field tensor, and its invariance under (15.7) can be checked directly. The preceding construction, however, shows us the geometrical origin of the structure of $F_{\mu\nu}$. Any function that depends on A_μ only through $F_{\mu\nu}$ and its derivatives is locally invariant. More general functions, such as the vector field mass term

$A_\mu A^\mu$, transform under (15.7) in ways that cannot be compensated and thus cannot appear in an invariant Lagrangian.

A related argument for the invariance of $F_{\mu\nu}$ can be made using the covariant derivative. We have seen above that, if a field has the local transfomation law (15.1), then its covariant derivative has the same transformation law. Thus the second covariant derivative of ψ also transforms according to (15.1). The same conclusion holds for the commutator of covariant derivatives:

$$[D_\mu, D_\nu]\psi(x) \to e^{i\alpha(x)}[D_\mu, D_\nu]\psi(x). \tag{15.14}$$

However, the commutator is not itself a derivative at all:

$$\begin{aligned} [D_\mu, D_\nu]\psi &= [\partial_\mu, \partial_\nu]\psi + ie\big([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\big)\psi - e^2[A_\mu, A_\nu]\psi \\ &= ie\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)\cdot\psi. \end{aligned} \tag{15.15}$$

That is,

$$[D_\mu, D_\nu] = ieF_{\mu\nu}. \tag{15.16}$$

On the right-hand side of (15.14), the factor $\psi(x)$ accounts for the entire transformation law, so the multiplicative factor $F_{\mu\nu}$ must be invariant. One can visualize the commutator of covariant derivatives as the comparison of comparisons across a small square; fundamentally, therefore, this argument is equivalent to that of the previous paragraph.

Whatever the method of proving the invariance of $F_{\mu\nu}$, we have now assembled all of the ingredients we need to write the most general locally invariant Lagrangian for the electron field ψ and its associated connection A_μ. This Lagrangian must be a function of ψ and its covariant derivatives, and of $F_{\mu\nu}$ and its derivatives, and must be invariant to global phase transformations. Up to operators of dimension 4, there are only four possible terms:

$$\mathcal{L}_4 = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu})^2 - c\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} - m\bar{\psi}\psi. \tag{15.17}$$

By adjusting the normalization of the fields ψ and A_μ, we have set the coefficients of the first two terms to their standard values. This normalization of A_μ requires the arbitrary scale factor e in our original definition (15.5) of A_μ. The third term violates the discrete symmetries P and T, so we may exclude it if we postulate these symmetries.† Then $\mathcal{L}_4$ contains only two free parameters, the scale factor e and the coefficient m.

By using operators of dimension 5 and 6, we can form many additional gauge-invariant combinations:

$$\mathcal{L}_6 = ic_1\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi + c_2(\bar{\psi}\psi)^2 + c_3(\bar{\psi}\gamma^5\psi)^2 + \cdots. \tag{15.18}$$

More allowed terms appear at each higher order in mass dimension. But all of these terms are nonrenormalizable interactions. In the language of Section 12.1, they are irrelevant to physics in four dimensions in the limit where the cutoff is taken to infinity.

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†The general systematics of P, C, and T violation are discussed in Section 20.3.

We have now reached a remarkable conclusion. We began by postulating that the electron field obeys the local symmetry (15.1). From this postulate, we showed that there must be an electromagnetic vector potential. Further, the symmetry principle implies that the most general Lagrangian in four dimensions that is renormalizable (or relevant, in Wilson's sense) is the general form $\mathcal{L}_4$. If we insist that this Lagrangian also be invariant under time reversal or parity, we are led uniquely to the Maxwell-Dirac Lagrangian that is the basis of quantum electrodynamics.

15.2 The Yang-Mills Lagrangian

If the simple geometrical constructions of the previous section yield Maxwell's theory of electrodynamics, then surely it must be possible to construct other interesting theories by starting with more general geometrical principles. Yang and Mills proposed that the argument of the previous section could be generalized from local phase rotation invariance to invariance under any continuous symmetry group. In this section, we will introduce this generalization of local symmetry. For most of the discussion, we will consider our local symmetry to be the three-dimensional rotation group, $O(3)$ or $SU(2)$, since in this case the necessary group theory should be familiar. At the end of the section, we will generalize further to the case of an arbitrary local symmetry.

Consider, then, the following generalization of the phase rotation (15.1): Instead of a single fermion field, we start with a doublet of Dirac fields,

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \tag{15.19}$$

which transform into one another under abstract three-dimensional rotations as a two-component spinor:

$$\psi \to \exp\Big(i\alpha^i \frac{\sigma^i}{2}\Big)\psi. \tag{15.20}$$

Here σ^i are the Pauli sigma matrices, and, as usual, a sum over repeated indices is implied. It is important to distinguish this abstract transformation from a rotation in physical three-dimensional space; in their original paper, Yang and Mills considered (ψ_1, ψ_2) to be the proton-neutron doublet as it is transformed under isotopic spin. As in the case of a phase rotation, it is not hard to construct Lagrangians for ψ that are invariant to (15.20) as a global symmetry.

We now promote (15.20) to a local symmetry, by insisting that the Lagrangian be invariant to this transformation for α^i an arbitrary function of x. Write this transformation as

$$\psi(x) \to V(x)\psi(x), \qquad \text{where } V(x) = \exp\Big(i\alpha^i(x)\frac{\sigma^i}{2}\Big). \tag{15.21}$$