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2020.11.11 $\begin{cases} \text{Gauss积分} \\ \text{只有二阶累积量 } \langle x^4\rangle_0^c = 0 \end{cases}$

作业1. ~~Gauss~~ $\dfrac{\int dx\, x^2 e^{-\frac{a}{2}x^2 - bx^4}}{\int dx\, e^{-\frac{a}{2}x^2 - bx^4}} = ?$ (Mathematica) $\dfrac{1}{a} - \dfrac{12b}{a^3} + \dfrac{384b^2}{a^5} + \cdots$

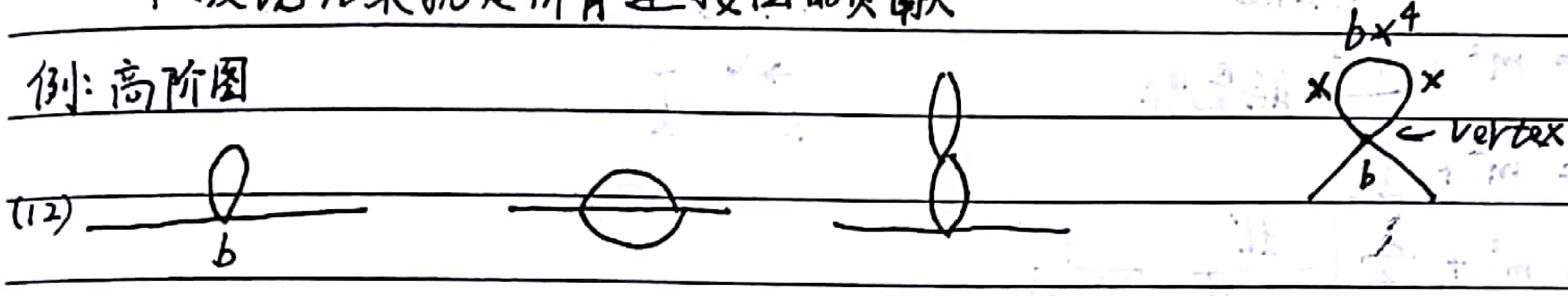
① 求解表达式，

② 用图表示每一个解（1、2、3阶，后面不用了）

③ 设 $a=1$，展开到 10阶，比较严格结果和数值结果

可以发现结果就是所有连接图的贡献

例：高阶图



*微扰能奏效的一般都是只在二阶以内：$\dfrac{1}{a} - \dfrac{12b}{a^3}$

因为与 b 关系还不大

$$\frac{\int D\phi\, e^{-H_0-u}\hat{O}}{\int D\phi\, e^{-H_0-u}} = \sum_{\ell=0}^{+\infty} \frac{(-1)^\ell}{\ell!}\langle \hat{O} u^\ell\rangle_0^c$$

如：$\dfrac{\int D\phi\, \hat{O}\, e^{i\int dx\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right]}}{\int D\phi\, e^{i\int dx\left[\underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2}_{H_0} - \underbrace{\frac{\lambda}{4!}\phi^4}_{u}\right]}}$

上次作业：

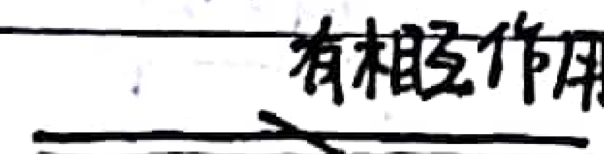
$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x^4$$
$$= \left(\frac{p^2}{2m_R} + \frac{1}{2}m_R\omega_R^2x^2\right) + \delta H$$

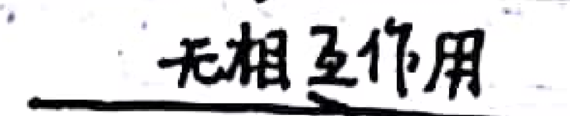
有相互作用后，散射会重整化质量、频率等参数

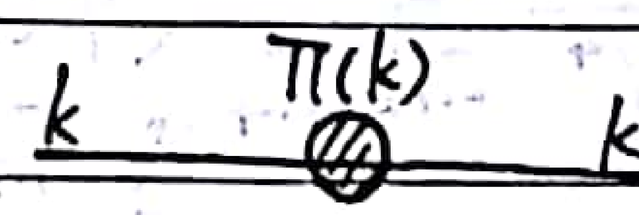
Ref: Shankar书 Ch14

Peskin Ch10

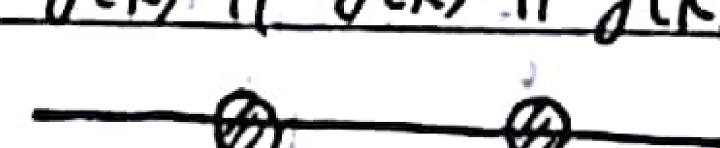
上一节课讲到：

$G(k) = \langle \phi^*(k)\, \phi(k)\rangle =$ 

$=$ 

$+$ 

$G(k) = g(k) + g(k)\,\Pi(k)\,g(k) + g(k)\,\Pi\, g(k)\,\Pi\, g(k)$



~~IF Π是小量~~

~~$= g(k)\cdot(1+\Pi(k)g(k))$~~

$= g(k)\dfrac{1}{1-\Pi(k)g(k)}$

Dyson eq $= \dfrac{1}{g^{-1}(k) - \Pi(k)}$ 在Peskin 4.3节

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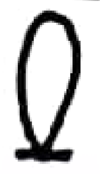
$\Pi(k) =$ 

$$G(k) = \frac{1}{g^{-1}(k) - \Pi(k)}$$

$$g(k) = \frac{1}{k^2 - m^2}$$

$$G(k) = \frac{1}{k^2 - m_R^2}$$

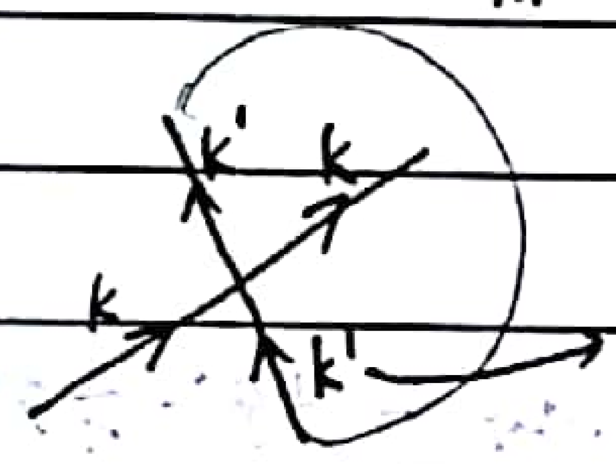
$$\Leftrightarrow \quad \cancel{k^2} - m_R^2 = \cancel{k^2} - m^2 - \Pi(k)$$

$\Leftrightarrow \quad m_R^2 = m^2 + \Pi$ 自能

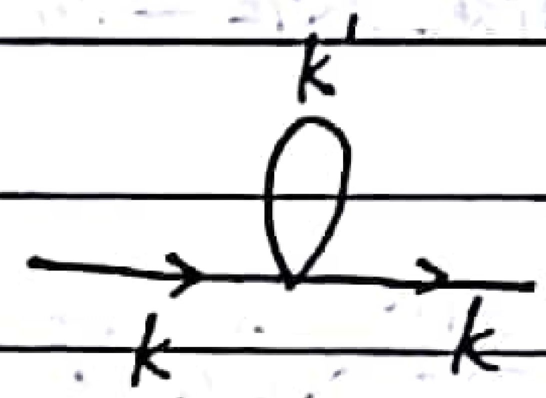
$= m^2 + \underline{\delta m^2}$ 能量修正

$= m^2 +$ 

$$= m^2 + \frac{\lambda}{2} \int \frac{dk'}{(2\pi)^d} \frac{1}{k^2 + m^2}$$



$\forall$ 一个动量的粒子散射总的贡献 $\Rightarrow$



* 只有维度为 1 时不发散，其它都是发散的

$$\int \frac{dk}{k^2 + m^2} \sim \int \frac{k^{d-1} dk}{k^2 + m^2} \sim \int k^{d-3} dk \quad \begin{cases} d=1, \text{不发散} \\ d=2, \int^\Lambda \frac{1}{k} dk \sim \ln\Lambda \\ d=3, \int^\Lambda dk \sim \Lambda \\ d=4, \int^\Lambda k\,dk \sim \Lambda^2 \end{cases}$$

解决问题的唯一方法就是认为 $m = m(\Lambda)$, $\lambda = \lambda(\Lambda)$.

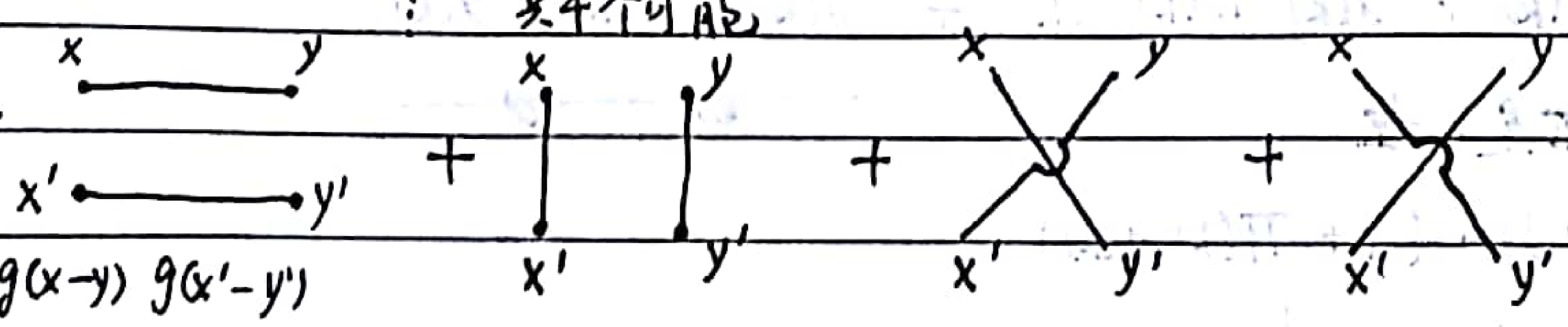
四体之间的散射

$$\langle \phi(x)\, \phi(y)\, \phi(y')\, \phi(x') \rangle$$

$$= \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} \langle \hat{O} u^l \rangle_0$$

$l=0$ 时，$\langle \hat{O} \rangle_0^c = \frac{1}{V^4} \sum_{k_1 k_2 k_3 k_4} \langle \phi(k_1)\, \phi(k_2)\, \phi(k_3)\, \phi(k_4) \rangle_0^c \, e^{i(k_1 x + k_2 y + k_3 y' + k_4 x')}$

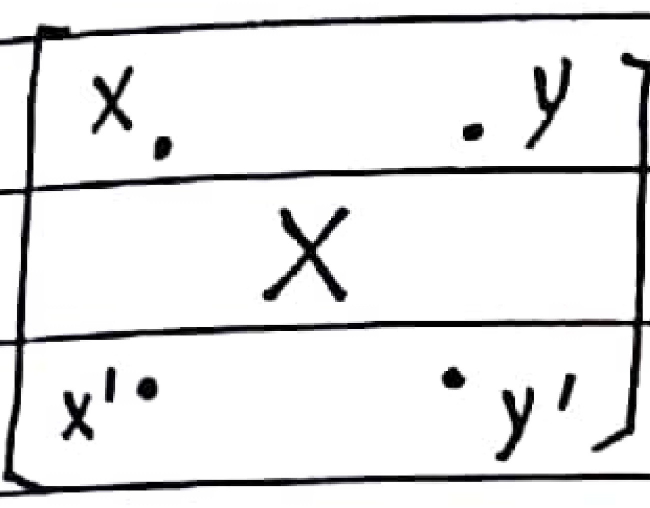
$k_2 = -k_1$, $k_4 = -k_3$ 或 $k_3 = -k_1$, $k_4 = -k_2$ 或 ... 共4个可能

= 

$l=1$时，$\langle \hat{O}\frac{(-1)^l}{l}u\rangle_0^c = -\langle\phi(x)\phi(y)\,\phi(y')\phi(x')\,\frac{i\lambda}{4!}\int dz\,\phi^4(z)\rangle_0^c$

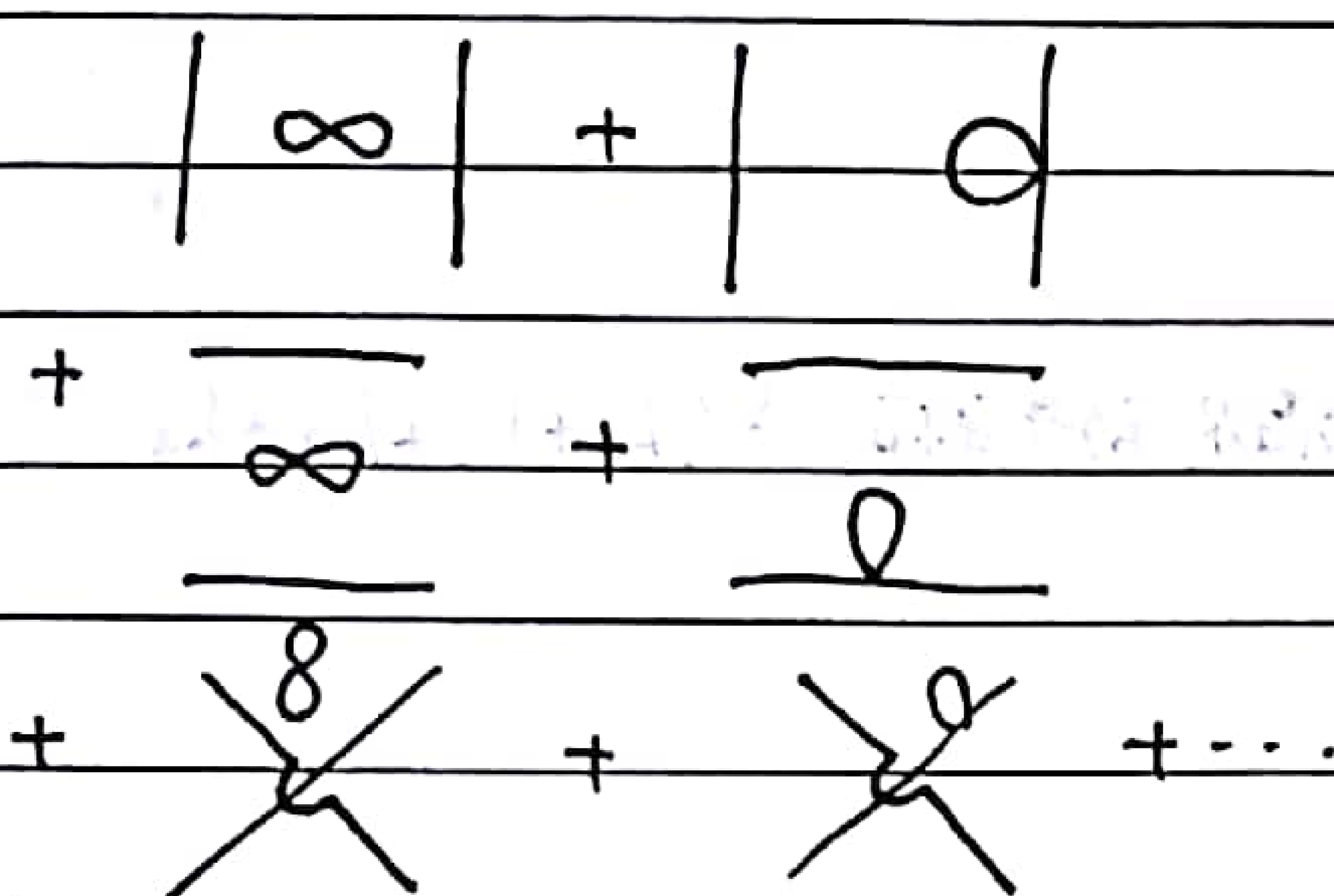
$\left(S=i\int dx\left[\frac{1}{2}(\partial\phi)^2-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4\right] \Rightarrow -H_0-u,\ u=i\frac{\lambda}{4!}\int\phi^4dz\right)$

$=-\frac{i\lambda}{4!}\int\langle\phi(x)\phi(y)\phi(y')\phi(x')\phi^4(z)\rangle_0^c\,dz$



连起来.

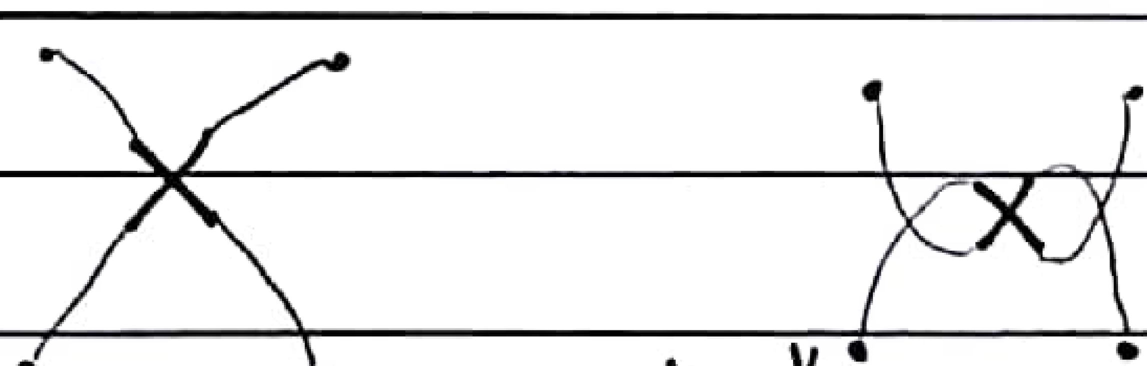
如：



这些非连通图没有贡献

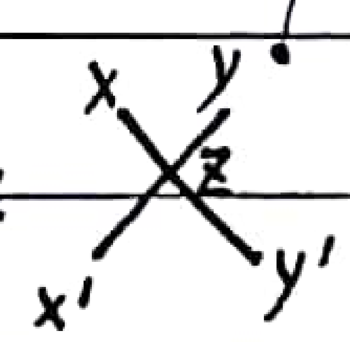
真正有贡献的必须 x，x'，y，y' 四点都参与与 X 的散射过程.

如



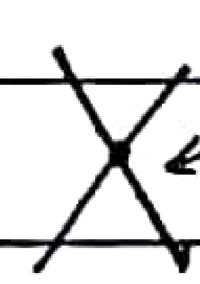
，系数权重为 4!

连接图 $=-\frac{i\lambda}{4!}\,4!\int dz$



Peskin书上 94-95

记 =

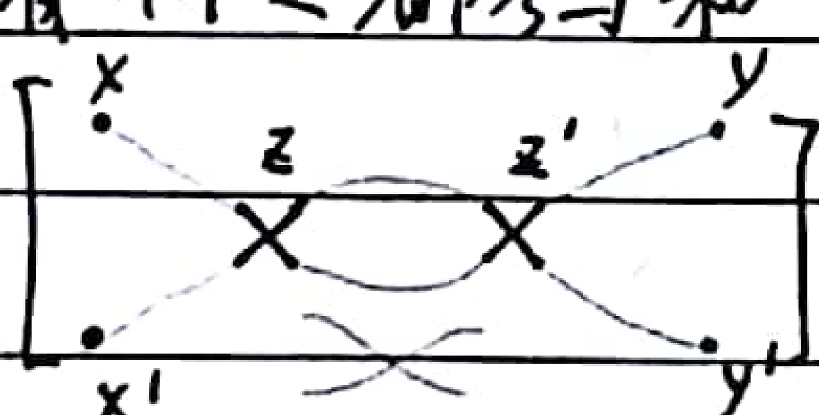


← 权重 $(-i\lambda)$

$l=2$时，$\left(\frac{i\lambda}{4!}\right)^2\frac{1}{2!}\int\langle\phi(x)\,\phi(y)\,\phi(y')\,\phi(x')\,\phi^4(z)\,\phi^4(z')\rangle_0^c\,dz\,dz$

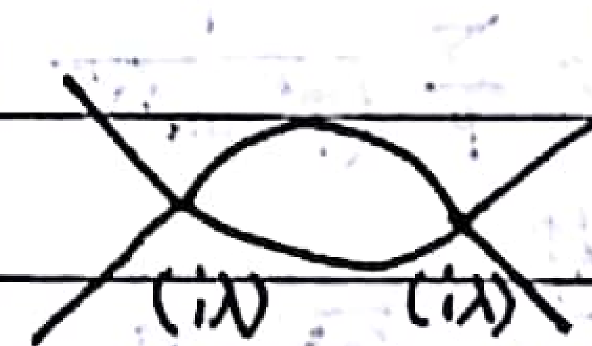
同样，只有4个点都参与和 X 和 X 的散射过程才有贡献

①



12种 2种 12种

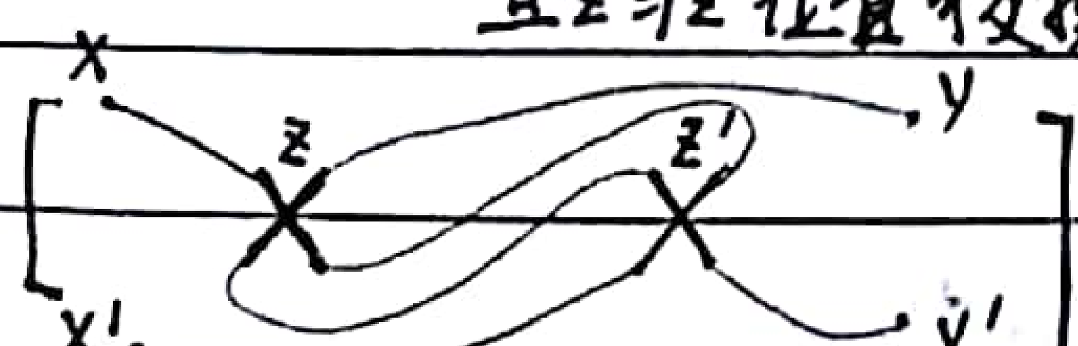
Feynman图表示 $(\frac{1}{2})$



$=\frac{1}{2}(i\lambda)^2\int\cdots$

且z与z'位置可交换，×2.

②



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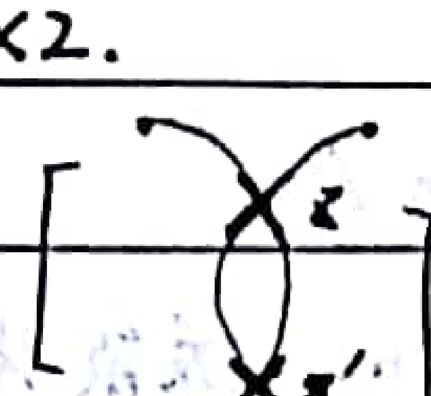
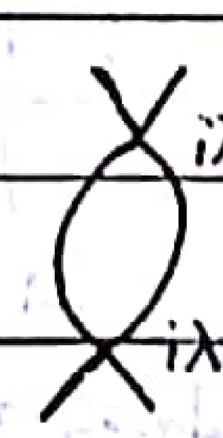
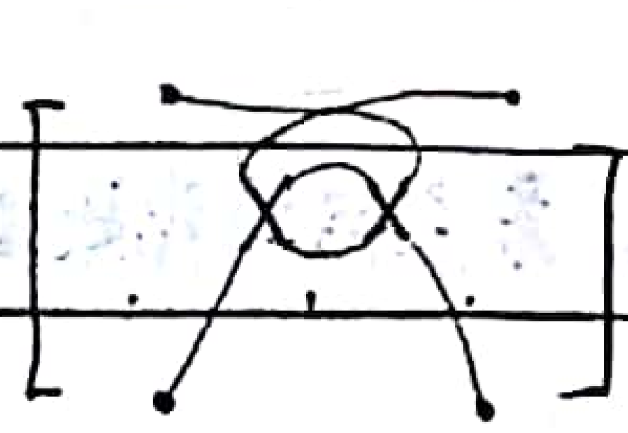


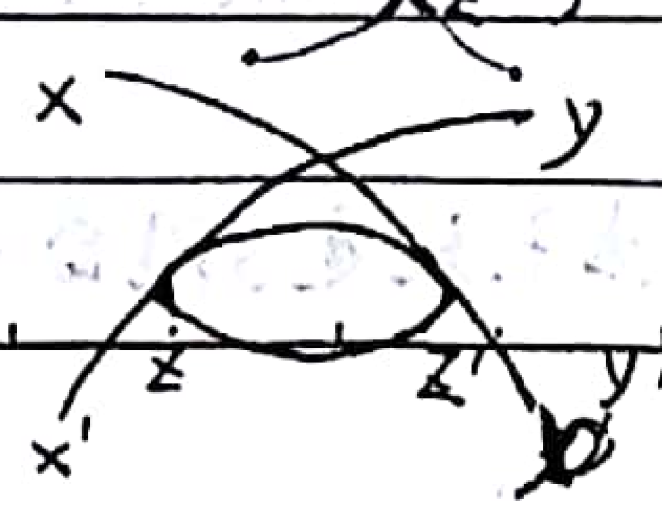
$=\frac{1}{2}$



③



$=\frac{1}{2}$



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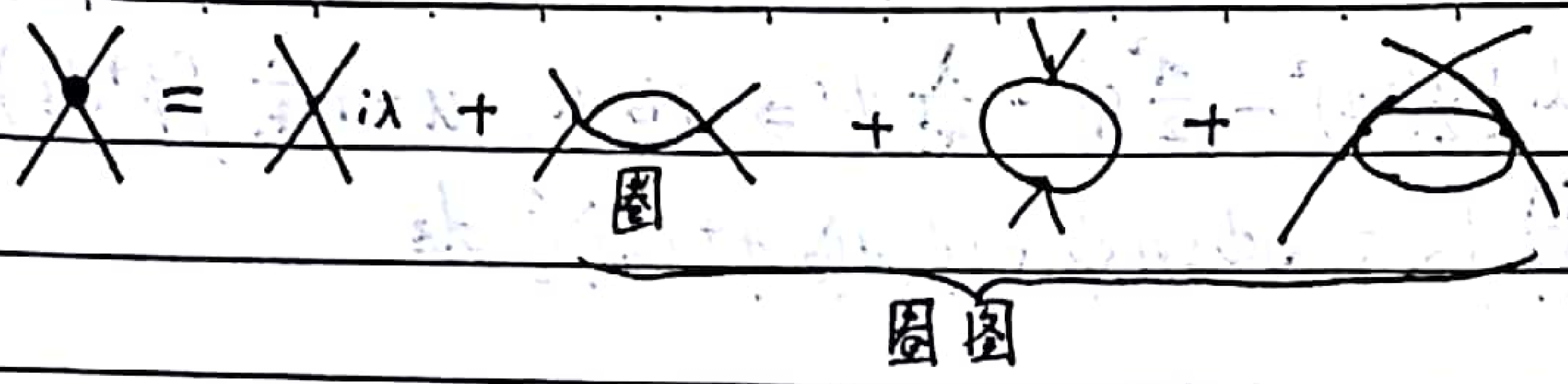
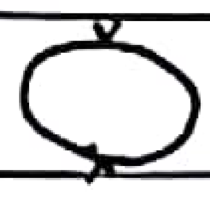
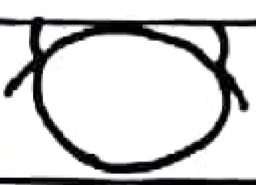
$l=2$（续）

iM 

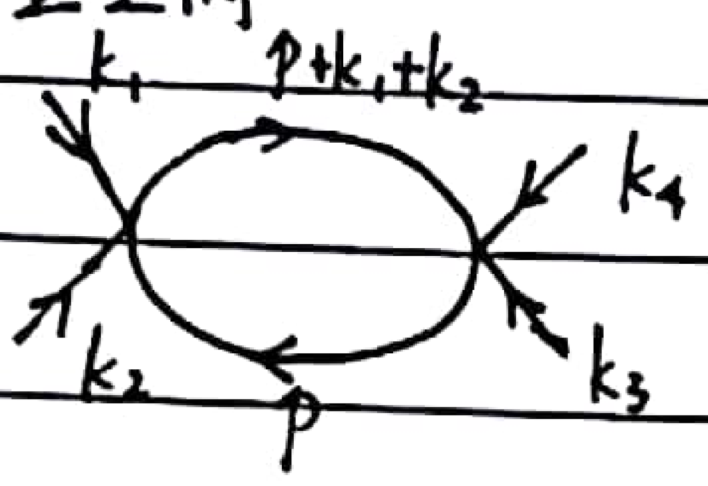
$iM = -i\lambda + \frac{1}{2}$  $+ \frac{1}{2}$  $+ \frac{1}{2}$ 

☆计算

转到动量空间

$\boxed{\text{第一个圈图}}$

$\frac{1}{V}\delta(k_1+k_2+k_3+k_4)\sum_p$

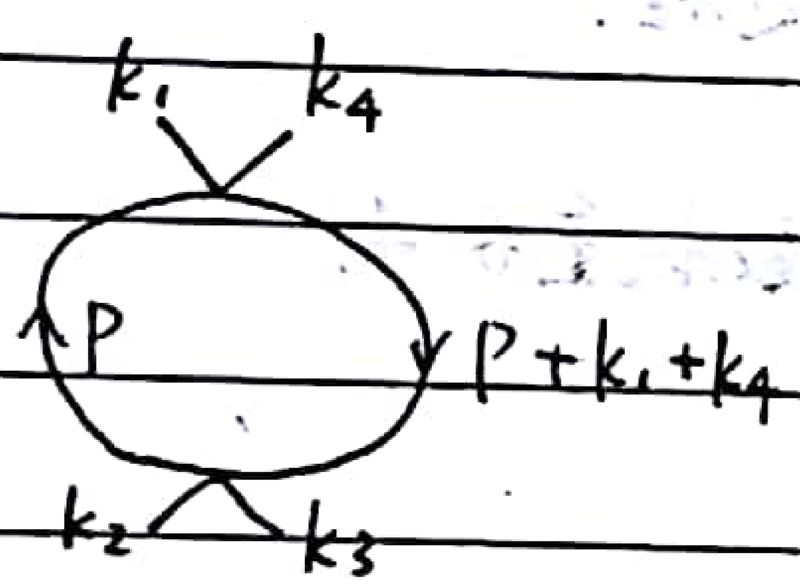


— 满足动量守恒 $\delta(k_1+k_2+k_3+k_4)$

实验上观测到的—这是对所有 p 求和

$$= \frac{1}{V}\sum_p \frac{1}{(p^2+m^2)}\,\frac{1}{(p+k_1+k_2)^2+m^2}\,(-i\lambda)^2$$

$\boxed{\text{第二个圈图}}$



这种积分不好做，有很多 trick

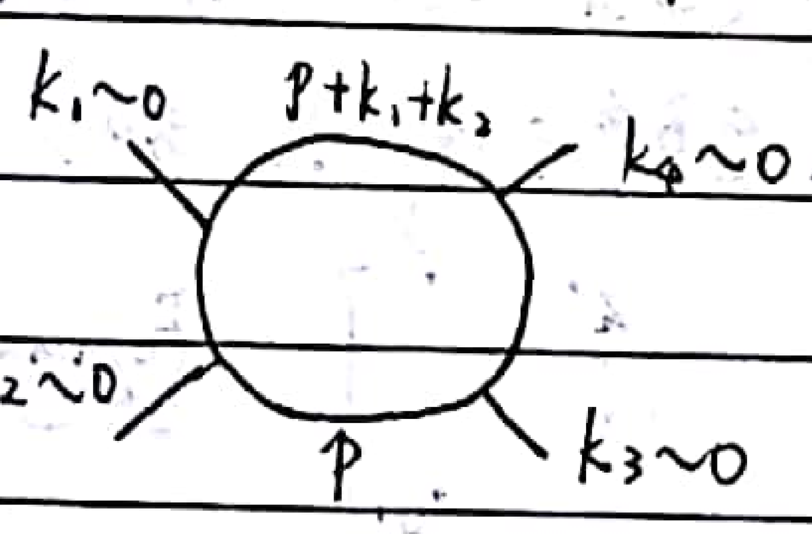
如：Feynman trick：$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$

但在凝聚态物理学家这里讨论一个特殊情况：$k_1 \sim k_2 \sim k_3 \sim k_4 \sim 0$（Shankar 书上讨论方法）

此时三个圈图等同

积分式 $\overset{k_i \to 0}{=\!=\!=} \frac{1}{V}\sum_p \frac{1}{(p^2+m^2)^2}(-i\lambda)^2$

$$= \int \frac{dp}{(2\pi)^d}\frac{1}{(p^2+m^2)}(-i\lambda)^2$$



此时变为 ln 发散

$\int \frac{p^{d-1}}{p^4}dp$　① $d=4$，$\int \frac{p^3}{p^4}dp \sim \ln\Lambda$

$\ln(2^{10}) = 10\ln 2$（发散得非常慢）

② $d=3$，收敛了

凝聚态中很多体系就是3维的（低速，不必考虑相对论效应）

如磁性等…

粒子物理中 GeV 量级，$\Lambda\sim10^6$GeV

但在凝聚态物理远远达不到，所以圈图只是在原先结果上的小修正.

☆计算 ① Feynman trick ×

② $k_1,k_2,k_3,k_4\sim0$

$$\begin{cases} M=-\lambda+\delta\lambda \\ \delta\lambda=-3\lambda^2\frac{1}{V}\sum_p\frac{1}{(p^2+m^2)^2} \end{cases}$$

$$=\begin{cases} d=4,\ -3\lambda^2\int\frac{dp}{(2\pi)^d}\frac{1}{(p^2+m^2)^2}\sim\ln\Lambda \\ d<4,\ \text{finite} \end{cases}$$

③ 维度正规化

$d=4$ 发散，就在 $d=4-\varepsilon$ 上做

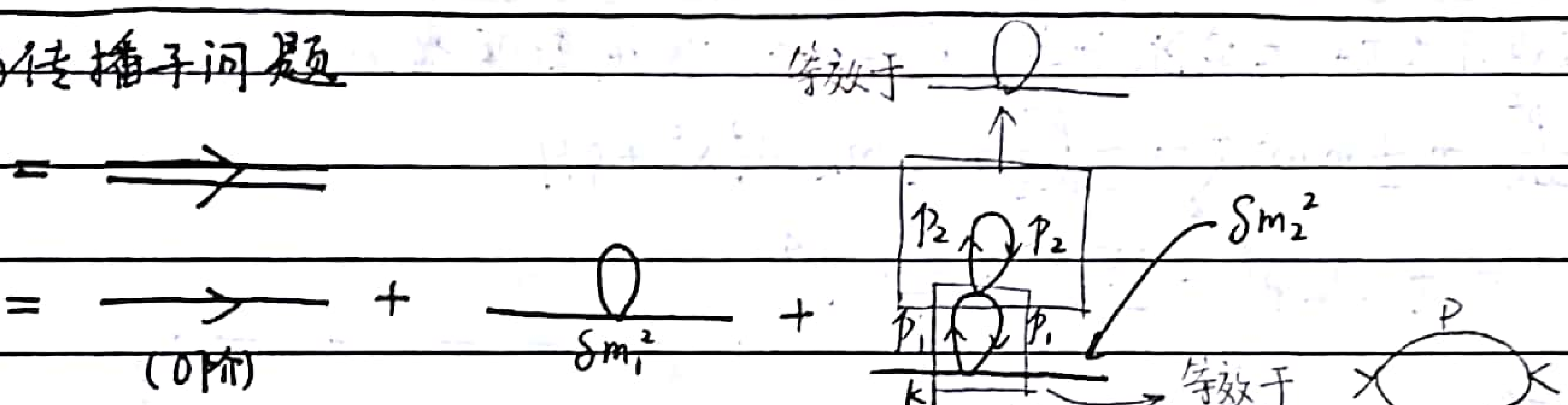
可以保证 Gauge Invariance.

Read Peskin P248 ~~在我们讨论的 ϕ^4-theory 中不存在规范问题~~

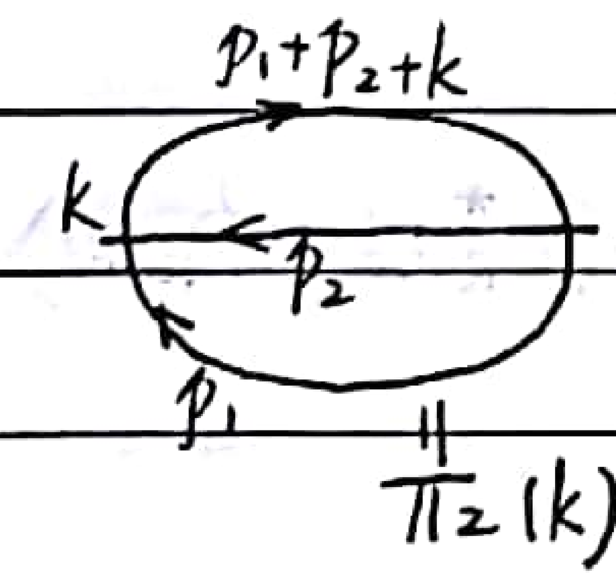
(eq 7.79) 但在 QED中，不同的处理方法有可能破坏规范变换不变性.

~~再回到单粒子的传播子问题~~

$\langle\phi^*(k)\,\phi(k)\rangle=$



$\propto(-i\lambda)^2\frac{1}{V^2}\sum_{p_1,p_2}\frac{1}{(p_1^2+m^2)^2}\frac{1}{(p_2^2+m^2)}=\text{const}$



$\boxed{m_R^2=m^2+\delta m_1^2+\delta m_2^2+\Pi_2(k)}$ 会出现问题

$$G(k)=\frac{1}{k^2-m_R^2}$$

$$=\frac{1}{k^2-m^2-(\underbrace{\delta m_1^2+\delta m_2^2}_{\delta m^2}+\Pi_2(k))}$$ 作为方便，假设 $\Pi_2(k)=A+Bk^2$

$$=\frac{1}{(1-B)k^2-(m^2+\delta m^2+A)}$$

$$=\frac{Z}{k^2-M^2}$$ $Z\to$ 分子不再为1，重整化；$k^2-M^2\to$ pole

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☆ $\Pi_2(k) \Rightarrow$ pole $\quad Z = \frac{1}{1-B} \neq 1$

☆ 高阶效应：$\underbrace{k^3, k^4}_{(\partial\phi)^4}, k^5 \cdots$ 由于守称有时可以消掉.

总结:

① $\begin{cases} \underbrace{M}_{\text{实验可观测结果}} = -\lambda + \delta\lambda \\ \delta\lambda = -3\lambda^2 \frac{1}{V} \sum_p \frac{1}{(p^2+m^2)^2} \propto \lambda^2 \int \frac{dp}{(2\pi)^d} \frac{1}{(p^2+m^2)^2} \end{cases}$

② $\begin{cases} m_R^2 = m^2 + \delta m^2 \\ \delta m^2 \propto \lambda \int \frac{dp}{(2\pi)^d} \frac{1}{p^2+m^2} \end{cases}$

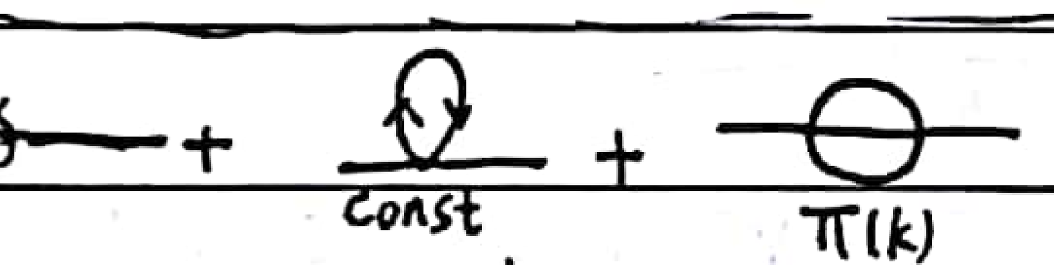
注意：(1) $\frac{\partial}{\partial m^2}(\delta m^2) \propto \delta\lambda$　　类似

(2) $g\,\delta^{(2)}(0)$: $\quad g \sum_k \frac{1}{E+k^2} = 1$

以上理论不太好，在实际计算中更好的是 波函数重整化.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \underset{+\lambda x^4}{x^2} = \frac{p^2}{2m_R} + \frac{1}{2} m_R \omega_R^2 x^2 + \delta H$$

修改 $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$

考虑删掉 

$Z = \frac{1}{1-B} \neq 1$

$\begin{cases} \langle \phi^*(k)\ \phi(k) \rangle = \left(\frac{Z}{k^2-m_R^2}\right) \quad \Leftrightarrow \quad \langle \phi_R^*(k)\ \phi_R(k) \rangle = \left(\frac{1}{k^2-m_R^2}\right) \\ \text{令}\ \phi(k) = \sqrt{Z}\ \phi_R(k) \end{cases}$

令 $\phi = \sqrt{Z}\ \phi_R$

名词：bare coupling $\quad (m, \lambda)$, (m_0, λ_0)

$\begin{cases} \text{裸质量} \\ \text{裸耦合常数} \end{cases}$ $\quad \Lambda \to \infty,\ m \to \infty,\ \lambda \to \infty$

$\begin{cases} \text{Renormalized coupling } (m_R, \lambda_R)\text{. 有限} \\ \text{实验可测} \end{cases}$

$$\mathcal{L}_R = \frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{m_R^2}{2}\phi_R^2 - \frac{\lambda_R}{4!}\phi_R^4$$

$$\mathcal{L} = \mathcal{L}_R + \underbrace{\delta\mathcal{L}}_{\text{无穷大的微扰}}$$

$\delta\mathcal{L} = \frac{1}{2}\delta Z(\partial_\mu \phi_R)^2 - \frac{1}{2}\delta m\, \phi_R^2 - \frac{\delta\lambda}{4!}\phi_R^4$

$\delta Z = Z - 1 \Leftrightarrow 1 + \delta Z = Z$

$\delta m = Z m^2 - m_R^2$

$\delta\lambda = Z^2\lambda - \lambda_R$

Peskin 书 §10.2 P322 – P328

实验上观测到的是 λ_R, m_R, 且与能标基本无关.

未知的是：m = ?，z = ?，λ = ?，以及和Λ的关系.

实验中可以观测到一个质量，但由于有相互作用，测到的未必是真实质量

发散项互相抵消，最后给出有效值(实验可观测)

$$H_0 \propto \int dx\, \frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{m_R^2}{2}\phi_R^2$$

$$u = -\underbrace{\int dx\, \delta\mathcal{L}}_{\text{重整化余部分}} + \underbrace{\int dx\, \frac{\lambda_R}{4!}\phi_R^4}_{\text{相互作用部分}}$$

$$\delta H = \lambda x^4 + \delta_p p^2 + \delta m\, x^2.$$

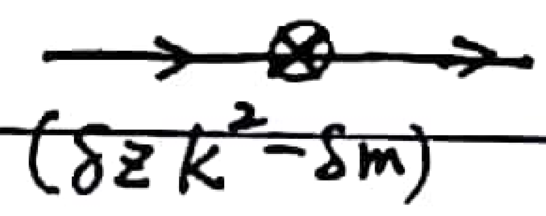
重整化后形式：

$$\langle \phi_R^*(k)\phi_R(k)\rangle = \xrightarrow{k} = \sum_\ell \frac{(-1)^\ell}{\ell!}\langle \hat{U}^\ell\rangle_0^c$$

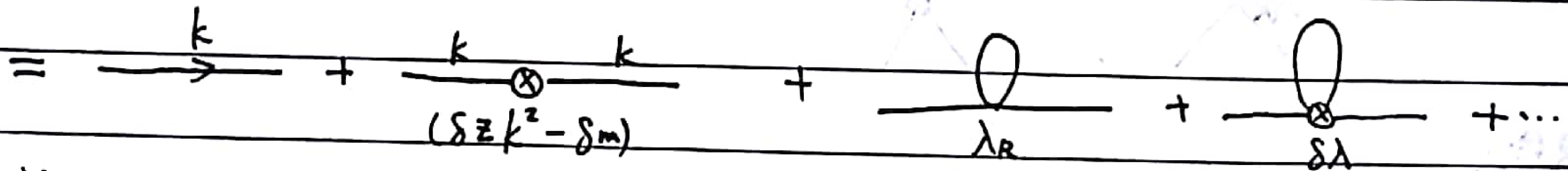
$$\delta\mathcal{L} = \frac{1}{2}\delta z\,(\partial_\mu\phi_R)^2 - \frac{1}{2}\delta m\,\phi_R^2 - \frac{\delta\lambda}{4!}\phi_R^4$$

$$\overset{\text{动量空间}}{=} \underbrace{\sum_k (\delta z k^2 - \delta m)\phi_R^*(k)\phi_R(k)} - \frac{\delta\lambda}{4!}\int dx\, \phi_R^4$$

$$\langle \phi^*(k)\phi(k)\phi^*(q)\phi(q)(\delta z q^2 - \delta m)\rangle$$

$=$ 

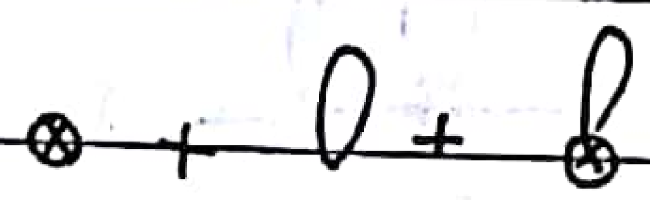
类似也有λ部分

$=$ 

在线性以下区别不大

真正有区别是在二阶以上，但比较复杂，此处不讨论

$$\frac{1}{k^2-m_R^2} = \frac{1}{k^2-m_R^2} + \frac{1}{k^2-m_R^2}(\delta z k^2 - \delta m)\frac{1}{k^2-m_R^2} + (\lambda_R + \delta\lambda)\int \frac{dp}{(2\pi)^d}\frac{1}{p^2+m_R^2}$$

$\Pi(k) =$ 

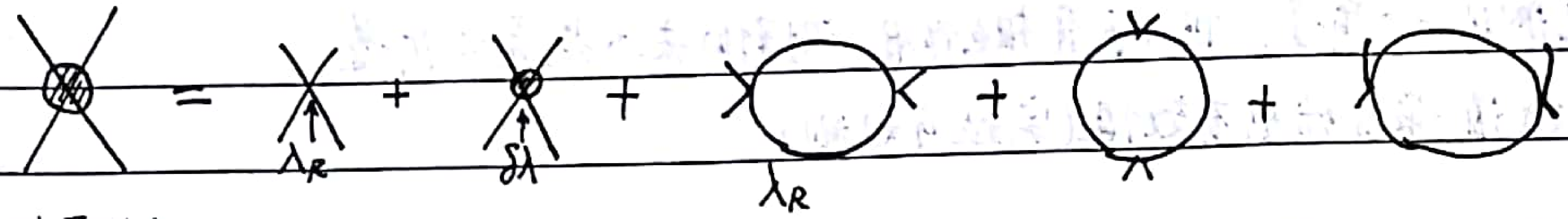
$$= \delta z k^2 - \delta m + (\lambda_R + \delta\lambda)\int \frac{dp}{(2\pi)}\frac{1}{p^2+m_R^2}$$

逻辑上，H_0包含的已经是观测到的结果了，所以无穷大项之间应该严格抵消.

Date.　　No.

再看

四体之间散射一阶微扰

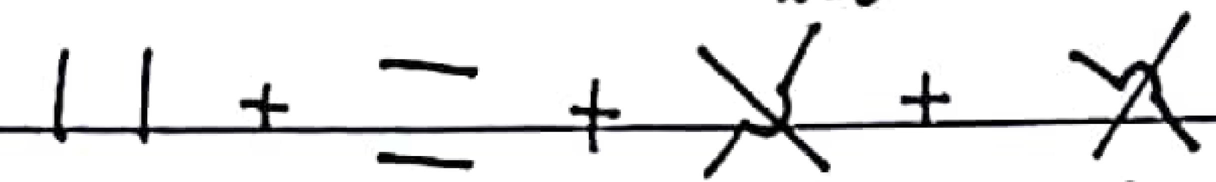


回到原始定义上

~~$\langle \phi^*(k)\phi(k)\rangle = \dots$~~

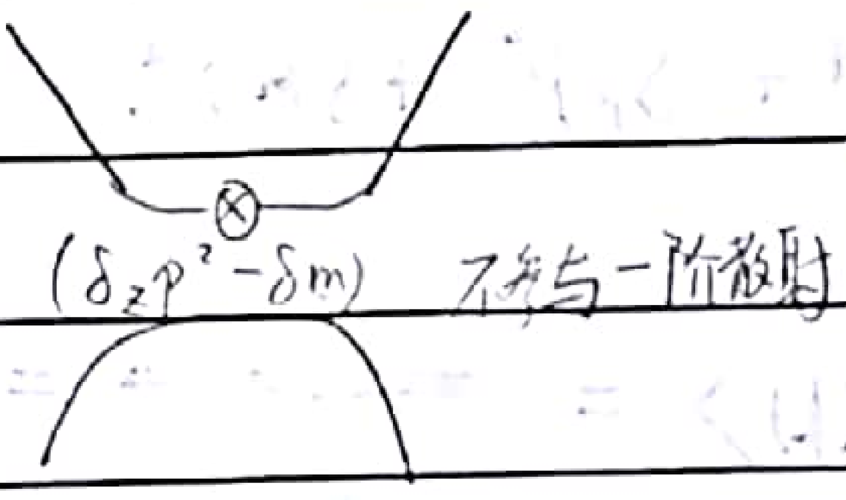
$$\langle \phi(k_1)\,\phi(k_2)\,\phi(k_3)\,\phi(k_4)\rangle = \sum_{l=0}^{+\infty} \langle \hat{O}\,U^l\rangle_0^c \,\frac{(-1)^l}{l!}$$

$l=0$,



$l=1$, $\langle \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\cdot U\rangle_0^c$

U代表3项
$$\begin{cases} (\delta z\, p^2-\delta m)\,(p^2-m^2)\,\phi^*(p)\phi(p) \\ \lambda_R \int \phi^4(z)\,dz \\ \delta\lambda \int \phi^4(z)\,dz \end{cases}$$

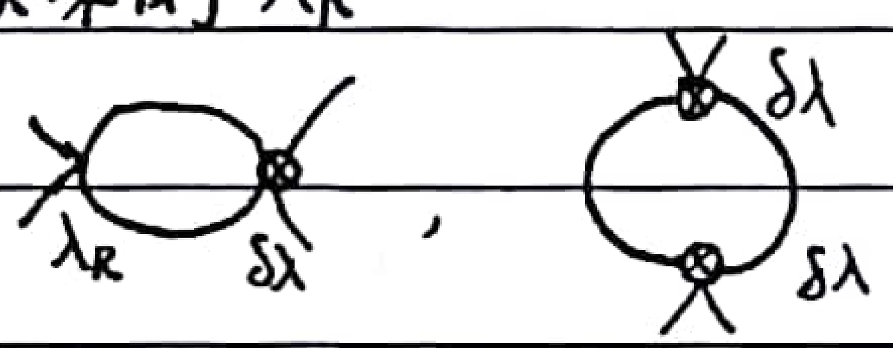


$l=2$

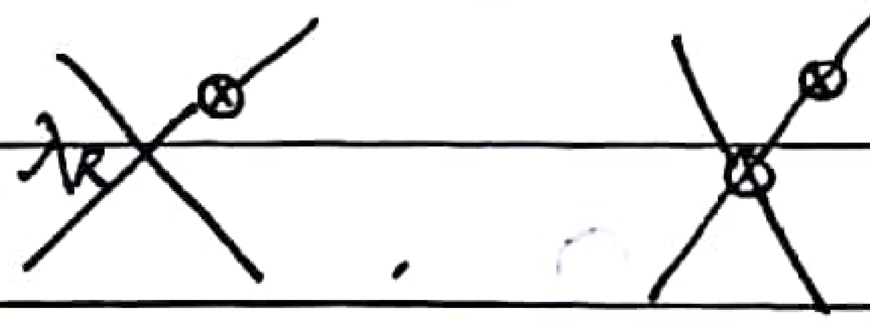
$$\begin{pmatrix} ①(\delta z\, p^2-\delta m)\,(p^2-m^2)\,\phi^*(p)\phi(p) \\ ②\lambda_R \int \phi^4(z)\,dz \\ ③\delta\lambda \int \phi^4(z)\,dz \end{pmatrix}^2$$

主要贡献来自于 λ_R

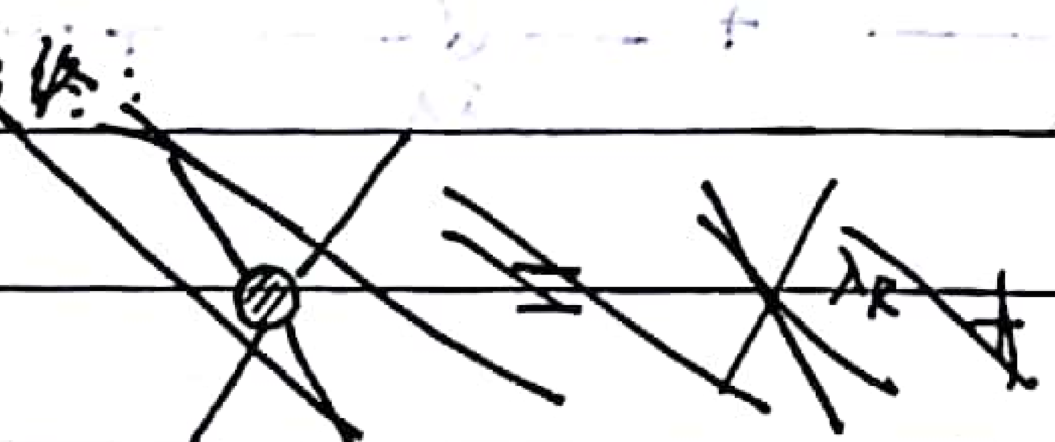
还会出现



①单独不会有贡献，但①②在一起会有贡献

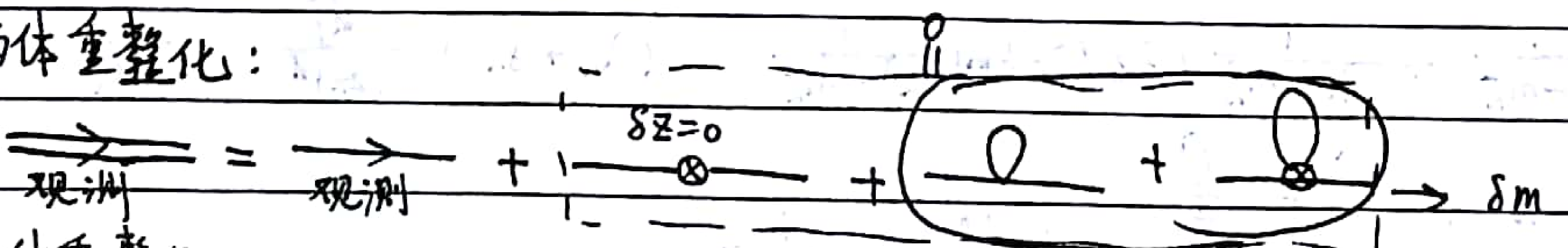


~~最终：~~

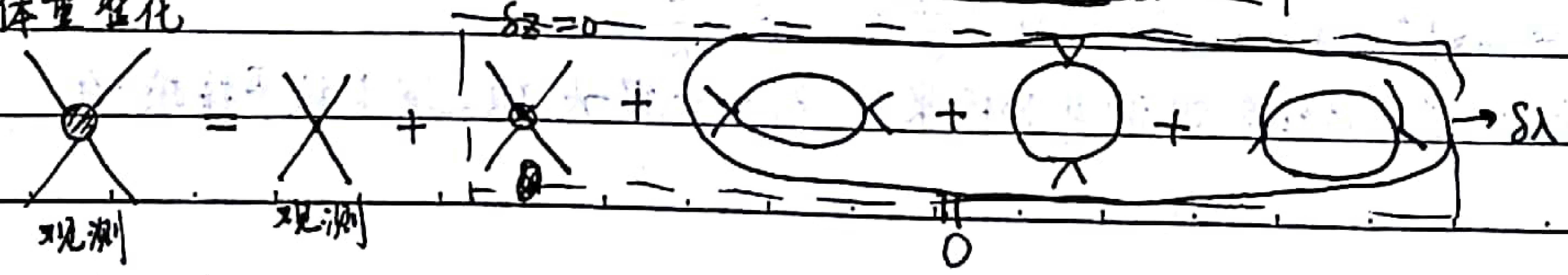


总结：

两体重整化：



四体重整化



下一节RG.