13
Renormalization Group for the ϕ^4 Model

We will now see the full power of the RG as applied to critical phenomena. The treatment, here and elsewhere, will emphasize the key ideas and eschew long and detailed calculations. In this and the next chapter I will focus on issues I found confusing rather than complicated. For example, a five-loop Feynman diagram is complicated but not confusing; I know what is going on. On the other hand, the relationship between renormalization of continuum field theories with one or two couplings and Wilson's program with an infinite number of couplings used to confuse me.

Because of universality, we can choose any member to study the whole class. For the Ising model the action has to have Z_2 symmetry, or invariance under sign reversal of the order parameter:

$$S(\phi) = S(-\phi). \tag{13.1}$$

(An infinitesimal symmetry-breaking term of the form $h\phi$ will be introduced to find exponents related to magnetization. Having done that, we will set $h = 0$ in the rest of the analysis.)

I will, however, discuss an action that enjoys a larger $U(1)$ symmetry:

$$S(\phi, \phi^*) = S(\phi e^{i\theta}, \phi^* e^{-i\theta}), \tag{13.2}$$

where θ is arbitrary. (We can also see this as $O(2)$ symmetry of S under rotations of the real and imaginary parts of ϕ.) I discuss $U(1)$ because the computations are very similar to the upcoming discussion of non-relativistic fermions, which also have $U(1)$ symmetry. I will show you how a minor modification of the $U(1)$ results yields the exponents of the Z_2 (Ising) models.

13.1 Gaussian Fixed Point

The Gaussian fixed point is completely solvable: we can find all the eigenvectors and eigenvalues of the linearized flow matrix T. It also sets the stage for a non-trivial model of magnetic transitions amenable to approximate calculations.

The partition function for the Gaussian model is

$$Z = \int \left[\mathcal{D}\phi^*(k)\right][\mathcal{D}\phi(k)]\, e^{-S^*(\phi,\phi^*)}, \tag{13.3}$$

where we have an asterisk on S^* because it will prove to be a fixed point. (The asterisk on ϕ^*, of course, denotes the conjugate.) The action is

$$S^*(\phi,\phi^*) = \int_0^{\Lambda} \phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d}, \tag{13.4}$$

where the limits on the integral refer to the magnitude of the momentum k. This action typically describes some problem on a lattice. There are no ultraviolet singularities in this problem with a natural cut-off in momentum $k \simeq \frac{\pi}{a}$, where a is the lattice constant. We want to invoke the RG to handle possible infrared singularities at and near criticality. In that case we may focus on modes near the origin. We will begin with a ball of radius $\Lambda \ll \frac{1}{a}$ centered at the origin and ignore the shape of the Brillouin zone for $k \simeq \frac{1}{a}$.

The number of dimensions d is usually an integer, but we must be prepared to work with continuous d. In general, we will need to make sense of various quantities like vectors, dot products, and integrals in non-integer dimensions. For our discussions, we just need the integration measure for rotationally invariant integrands:

$$d^dk = k^{d-1}dkS_d, \tag{13.5}$$

where

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{13.6}$$

is the "area" of the unit sphere in d dimensions. We will rarely need this precise expression.

As a first step we divide the existing modes into slow and fast ones based on k:

$$\phi_{\mathrm{s}} = \phi(\boldsymbol{k}) \quad \text{for } 0 \le k \le \Lambda/s \quad \text{(slow modes)}, \tag{13.7}$$

$$\phi_{\mathrm{f}} = \phi(\boldsymbol{k}) \quad \text{for } \Lambda/s \le k \le \Lambda \quad \text{(fast modes)}, \tag{13.8}$$

where $s > 1$ is the scale parameter that decides how much we want to eliminate. We are going to eliminate the modes between the new cut-off Λ/s and the old one Λ.

The action itself separates into slow and fast pieces in momentum space:

$$S^*(\phi,\phi^*) = \int_0^{\Lambda/s} \phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + \int_{\Lambda/s}^{\Lambda} \phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} \tag{13.9}$$

$$= S^*(\text{slow}) + S^*(\text{fast}), \tag{13.10}$$

and the Boltzmann weight factorizes over slow and fast modes. Thus, integrating over the fast modes just gives an overall constant $Z(\text{fast})$ multiplying the Z for the slow modes:

$$Z = \int \left[\mathcal{D}\phi_{\mathrm{s}}^*(k)\right][\mathcal{D}\phi_{\mathrm{s}}(k)]e^{-S^*(\phi_{\mathrm{s}})} \int \left[\mathcal{D}\phi_{\mathrm{f}}^*(k)\right][\mathcal{D}\phi_{\mathrm{f}}(k)]e^{-S_0^*(\phi_{\mathrm{f}})} \tag{13.11}$$

$$\equiv \int \left[\mathcal{D}\phi_{\mathrm{s}}^*(k)\right][\mathcal{D}\phi_{\mathrm{s}}(k)]e^{-S^*(\phi_{\mathrm{s}})}Z(\text{fast}) \tag{13.12}$$

$$= \int \left[\mathcal{D}\phi_{\mathrm{s}}^*(k)\right][\mathcal{D}\phi_{\mathrm{s}}(k)]e^{-S^*(\phi_{\mathrm{s}})+\ln Z(\text{fast})}. \tag{13.13}$$

We can ignore $\ln Z(\text{fast})$ going forward, because it is independent of ϕ_{s} and will drop out of all ϕ_{s} correlation functions.

The action after mode elimination,

$$S'^*(\phi,\phi^*) = \int_0^{\Lambda/s} \phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d}, \tag{13.14}$$

is Gaussian, but not quite the same as the action we started with,

$$S^*(\phi,\phi^*) = \int_0^{\Lambda} \phi^*(\mathbf{k})k^2\phi,(\mathbf{k})\frac{d^dk}{(2\pi)^d} \tag{13.15}$$

because the allowed region for k is different.

We remedy this by defining a new momentum,

$$k' = sk, \tag{13.16}$$

which runs over the same range as k did before elimination. The action now becomes

$$S'^*(\phi,\phi^*) = \int_0^{\Lambda} \phi^*\left(\frac{\mathbf{k}'}{\mathbf{s}}\right)\left[\frac{k'^2}{s^2}\right]\phi\left(\frac{\mathbf{k}'}{\mathbf{s}}\right)\frac{d^dk'}{s^d(2\pi)^d} \tag{13.17}$$

$$= s^{-(d+2)}\int_0^{\Lambda} \phi^*\left(\frac{\mathbf{k}'}{\mathbf{s}}\right)k'^2\phi\left(\frac{\mathbf{k}'}{\mathbf{s}}\right)\frac{d^dk'}{(2\pi)^d}. \tag{13.18}$$

Due to this constant rescaling of units, the cut-off remains fixed and we may set $\Lambda = 1$ at every stage. (Of course, the cut-off decreases in fixed laboratory units. What we are doing is analogous to using the lattice size of the block spins as the unit of length as we eliminate degrees of freedom.)

To take care of the factor $s^{-(d+2)}$ we introduce a rescaled field:

$$\phi'(k') = s^{-(\frac{d}{2}+1)}\phi\left(\frac{k'}{s}\right) \equiv \zeta^{-1}(s)\phi\left(\frac{k'}{s}\right). \tag{13.19}$$

Notice that for every $\phi'(k')$ for $0 \le k' \le \Lambda$ there is a corresponding original field that survives elimination and is defined on a smaller sphere $(0 \le k \le \Lambda/s)$.

In terms of ϕ', the new action coincides with the original one in every respect:

$$S'^{*}(\phi', \phi'^{*}) = \int_0^{\Lambda} \phi'^{*}(\mathbf{k}') k'^2 \phi'(\mathbf{k}') \frac{d^d k'}{(2\pi)^d}. \tag{13.20}$$

Thus, S^* is a *fixed point* of the RG with the following three steps:

- Eliminate fast modes, i.e., reduce the cut-off from Λ to Λ/s.
- Introduce rescaled momenta, $k' = sk$, which now go all the way to Λ.
- Introduce rescaled fields $\phi'(k') = \zeta^{-1}\phi(k'/s)$ and express the effective action in terms of them. This action should have the same coefficient for the quadratic term. (In general, the field rescaling factor ζ^{-1} could be different from the $s^{-(1+d/2)}$ that was employed above.)

With this definition of the RG transformation, we have a mapping from actions defined in a certain k-space (a ball of radius Λ) to actions in the *same* space. Thus, if we represent the initial action as a point in a coupling constant space, this point will flow under the RG transformation to another point in the same space.

As the Gaussian action is a fixed point of this RG transformation, it must correspond to $\xi = \infty$, and indeed it does:

$$G(k) = \frac{1}{k^2} \leftrightarrow G(r) = \frac{1}{r^{d-2}}. \tag{13.21}$$

Here is our first critical exponent associated with the Gaussian fixed point:

$$\eta = 0. \tag{13.22}$$

This result is independent of d.

We now want to determine the flow of couplings near this fixed point. We will do this by adding various perturbations (also referred to as operators) and see how they respond to the three-step RG mentioned above. The perturbations will be linear, quadratic, and quartic in ϕ. (Operators with more powers of ϕ or more derivatives will prove highly irrelevant near $d = 4$, the region of interest to us.) We will then find the eigenvectors and eigenvalues of the linearized flow matrix T and classify them as relevant, irrelevant, or marginal. In general, the operators we add will mix under this flow and we must form linear combinations that go into multiples of themselves, i.e. the eigenvectors of T. The critical exponents and asymptotic behavior of correlation functions will follow from these.

13.1.1 Linear and Quadratic Perturbations

The perturbations can be an even or odd power of ϕ or ϕ^* (which I may collectively refer to as ϕ).

The uniform magnetic field couples linearly to ϕ and corresponds to an odd term $h\phi(0)$, where the argument (0) refers to the momentum. (Again, we must use the perturbation

$h\phi + h^*\phi^*$. We do not bother because the scaling is the same for both, and denote by h the coupling linear in the field.)

We have, from Eq. (13.19), which defines the rescaled field,

$$h\phi(0) = h\zeta\phi'(0\cdot s) = hs^{1+\frac{1}{2}d}\phi'(0) \stackrel{\text{def}}{=} h_s\phi'(0), \tag{13.23}$$

which means the renormalized h is

$$h_s = hs^{1+\frac{1}{2}d}. \tag{13.24}$$

Having found how an infinitesimal h gets amplified by RG, we will set $h = 0$ as we continue our analysis. We will not consider ϕ^3 because it is a *redundant operator* [1]. What this means is that if we began with

$$S = \int \left[h\phi + r_0\phi^2 + v\phi^3 + u\phi^4\right] d^d x, \tag{13.25}$$

the ϕ^3 term can be eliminated by a shift $\phi \to \phi - \frac{v}{4u}$.

Higher odd powers of ϕ or terms with an even number of extra derivatives will prove extremely irrelevant near $d = 4$, which will be our focus.

Now we turn to quadratic and quartic perturbations. First, consider the addition of the term

$$S_r = \int_0^\Lambda \phi^*(\mathbf{k}) r_0 \phi(\mathbf{k}) \frac{d^d k}{(2\pi)^d}, \tag{13.26}$$

which separates nicely into slow and fast pieces. Mode elimination just gets rid of the fast part, leaving behind the *tree-level term*

$$S_r^{\text{tree}} = \int_0^{\Lambda/s} \phi^*(\mathbf{k}) r_0 \phi(\mathbf{k}) \frac{d^d k}{(2\pi)^d}. \tag{13.27}$$

The adjective *tree-level* in general refers to terms that remain of an interaction upon setting all fast fields to 0. Keeping only this term amounts to eliminating fast modes by simply setting them to zero. Of course, the tree-level term will be viewed by us as just the first step.

If we express S_r^{tree} in terms of the new fields and new momenta, we find

$$S'_r = s^2 \int_0^\Lambda \phi'^*(\mathbf{k}') r_0 \phi'(\mathbf{k}') \frac{d^d k'}{(2\pi)^d} \tag{13.28}$$

$$\stackrel{\text{def}}{=} \int_0^\Lambda \phi'^*(\mathbf{k}') r_{0s} \phi'(\mathbf{k}') \frac{d^d k'}{(2\pi)^d}, \quad \text{which means} \tag{13.29}$$

$$r_{0s} = r_0 s^2. \tag{13.30}$$

In other words, after an RG action by a factor s, the coupling r_0 evolves into $r_{0s} = r_0 s^2$. Since S_r lacks the two powers of k that S_0 has, it is to be expected that r_0 will get amplified by s^2.

We have identified another relevant eigenvector in $r_0\phi^2$.

We may identify r_0 with t, the dimensionless temperature that takes us off criticality. Since the correlation length drops by $1/s$ under this rescaling of momentum by s,

$$\xi(r_0) = s^1\xi(r_{0s}) \tag{13.31}$$

$$= s^1\xi(r_0 s^2) \tag{13.32}$$

$$= (r_0)^{-\frac{1}{2}}\xi(1), \tag{13.33}$$

which means that

$$\nu = \frac{1}{2} \tag{13.34}$$

for all values of d. We do not need the RG to tell us this because we can find the propagator with a non-zero r_0 exactly:

$$G(k) \simeq \frac{1}{k^2+r_0} \leftrightarrow G(r) \simeq \frac{e^{-\sqrt{r_0}r}}{r^{d-2}} \equiv \frac{e^{-r/\xi}}{r^{d-2}}, \quad \text{i.e.,} \tag{13.35}$$

$$\xi = r_0^{-\frac{1}{2}}. \tag{13.36}$$

In general, the quadratic perturbation could be with coupling $r(k)$ that varies with k. Given the analyticity of all the couplings in the RG actions (no singularities in and no singularities out), we may expand

$$r(k) = r_0 + r_2k^2 + r_4k^4 + \cdots \tag{13.37}$$

and show that

$$r_{2s} = r_2, \tag{13.38}$$

$$r_{4s} = s^{-2}r_4, \tag{13.39}$$

and so on. Thus, r_2 is marginal, and adding it simply modifies the k^2 term already present in S_0. If you want, you could say that varying the coefficient of k^2 in S_0 gives us a line of fixed points, but this line has the same exponents everywhere because the coefficient of k^2 may be scaled back to unity by field rescaling. (The Jacobian in the functional integral will be some constant.) The other coefficients like r_4 and beyond are irrelevant.

Exercise 13.1.1 *Derive Eqs. (13.38) and (13.39).*

13.1.2 Quartic Perturbations

When we consider quartic perturbations of the fixed point, we run into a new complication: the term couples slow and fast modes and we have to do more than just rewrite the old perturbation in terms of new fields and new momenta. In addition, we will find that mode elimination generates corrections to the flow of r_0, the quadratic term.

Consider the action

$$
\begin{aligned}
S &= S^* + S_r + S_u &(13.40)\\
&= \int_0^\Lambda \phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + r_0\int_0^\Lambda \phi^*(\mathbf{k})\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d}\\
&\quad + \frac{u_0}{2!2!}\int_{|k|<\Lambda}\phi^*(\boldsymbol{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\prod_{i=1}^{3}\frac{d^dk_i}{(2\pi)^d} &(13.41)\\
&\equiv S_0 + S_{\mathrm{I}}, \quad \text{where} &(13.42)\\
S_0 &= S^* + S_r = \int_0^\Lambda \phi^*(\mathbf{k})(k^2+r_0)\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d}, &(13.43)\\
S_{\mathrm{I}} &= S_u \equiv \int_\Lambda \phi^*(4)\phi^*(3)\phi(2)\phi(1)u_0, &(13.44)\\
\boldsymbol{k}_4 &= \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3. &(13.45)
\end{aligned}
$$

Notice the compact notation used for the quartic interaction in Eq. (13.44). The subscript in S_{I} stands for *interaction*, which is what S_u is here.

Remember that S_0 *is not the fixed point action*, it is the *quadratic* part of the action and includes the r_0 term. With respect to the Gaussian fixed point S^*, it is true that S_r is a perturbation, but in field theories, S_r is part of the non-interacting action S_0 and only $S_u \equiv S_{\mathrm{I}}$ is viewed as a perturbation. In other words, S_r perturbs the Gaussian fixed point action, while S_{I} perturbs the non-interacting action. For what follows it is more expedient to use the decomposition $S = S_0 + S_{\mathrm{I}}$, where $S_{\mathrm{I}} = S_u$ is quartic.

13.1.3 Mode Elimination Strategy

I will now describe the strategy for mode elimination for the case

$$
S(\phi_{\mathrm{s}},\phi_{\mathrm{f}}) = S_0(\phi_{\mathrm{s}}) + S_0(\phi_{\mathrm{f}}) + S_{\mathrm{I}}(\phi_{\mathrm{s}},\phi_{\mathrm{f}}), \tag{13.46}
$$

where S_0 has been separated into slow and fast pieces whereas S_{I} cannot be separated in that manner.

Let us do the integration over fast modes:

$$
Z = \int [\mathcal{D}\phi_{\mathrm{s}}^*][\mathcal{D}\phi_{\mathrm{s}}]e^{-S_0(\phi_{\mathrm{s}})}\int [\mathcal{D}\phi_{\mathrm{f}}^*][\mathcal{D}\phi_{\mathrm{f}}]e^{-S_0(\phi_{\mathrm{f}})}e^{-S_{\mathrm{I}}(\phi_{\mathrm{s}},\phi_{\mathrm{f}})} \tag{13.47}
$$

$$
\stackrel{\text{def}}{=} \int [\mathcal{D}\phi_{\mathrm{s}}^*][\mathcal{D}\phi_{\mathrm{s}}]e^{-S_{\mathrm{eff}}(\phi_{\mathrm{s}})}, \tag{13.48}
$$

which defines the *effective action* $S_{\rm eff}(\phi_{\rm s})$. Let us manipulate its definition a little:

$$\begin{aligned} e^{-S_{\rm eff}(\phi_{\rm s})} &= e^{-S_0(\phi_{\rm s})}\int [\mathcal{D}\phi_{\rm f}^*][\mathcal{D}\phi_{\rm f}]e^{-S_0(\phi_{\rm f})}e^{-S_{\rm I}(\phi_{\rm s},\phi_{\rm f})} \\ &= e^{-S_0(\phi_{\rm s})}\frac{\int [\mathcal{D}\phi_{\rm f}^*][\mathcal{D}\phi_{\rm f}]e^{-S_0(\phi_{\rm f})}e^{-S_{\rm I}(\phi_{\rm s},\phi_{\rm f})}}{\int [\mathcal{D}\phi_{\rm f}^*][\mathcal{D}\phi_{\rm f}]e^{-S_0(\phi_{\rm f})}}\underbrace{\int [\mathcal{D}\phi_{\rm f}][\mathcal{D}\phi_{\rm f}^*]e^{-S_0(\phi_{\rm f})}}_{Z_{0\rm f}}. \end{aligned} \tag{13.49}$$

Dropping $Z_{0\rm f}$, which does not affect averages of the slow modes,

$$e^{-S_{\rm eff}(\phi_{\rm s})} = e^{-S_0(\phi_{\rm s})}\left\langle e^{-S_{\rm I}(\phi_{\rm s},\phi_{\rm f})}\right\rangle_{0\rangle} \stackrel{\rm def}{=} e^{-S_0-\delta S'}, \tag{13.50}$$

where $\langle\ \rangle_{0\rangle}$ denotes averages with respect to the fast modes with action $S_0(\phi_{\rm f})$.

Combining Eq. (13.50) with the *cumulant expansion*, which relates the *mean of the exponential to the exponential of the means*,

$$\left\langle e^{\Omega}\right\rangle = e^{\left[\langle\Omega\rangle+\frac{1}{2}(\langle\Omega^2\rangle-\langle\Omega\rangle^2)+\cdots\right]}, \tag{13.51}$$

we find

$$S_{\rm eff} = S_0 + \langle S_{\rm I}\rangle - \frac{1}{2}(\langle S_{\rm I}^2\rangle - \langle\ S_{\rm I}\rangle^2) + \cdots \tag{13.52}$$

It is understood that this expression has to be re-expressed in terms of the rescaled fields and momenta to get the final contribution to the action. We will do this eventually.

Exercise 13.1.2 *Verify the correctness of the cumulant expansion Eq. (13.51) to the order shown. (Expand e^{Ω} in a series, average, and re-exponentiate.)*

Since $S_{\rm I}$ is linear in u, Eq. (13.52) is a weak coupling expansion. It is now clear what has to be done. Each term in the series contains some monomials in fast and slow modes. The former have to be averaged with respect to the quadratic action $S_0(\phi_{\rm f})$ by the use of Wick's theorem. The result of each integration will yield monomials in the slow modes. When re-expressed in terms of the rescaled fields and momenta, each will renormalize the corresponding coupling. In principle, you have been given all the information to carry out this process. There is, however, no need to reinvent the wheel. There is a procedure involving Feynman diagrams that automates this process. These rules will not be discussed here since they may be found, for example, in Sections 3–5 of [2], or in many field theory books. Instead, we will go over just the first term in the series in some detail and comment on some aspects of the second term. Readers familiar with Feynman diagrams should note that while these diagrams have the same multiplicity and topology as the field theory diagrams, the momenta being integrated out are limited to the shell being eliminated, i.e., $\Lambda/s \le k \le \Lambda$.

The leading term in the cumulant expansion in Eq. (13.52) has the form

$$\langle S_{\mathrm{I}}\rangle = \frac{1}{2!2!}\left\langle \int_{|k|<\Lambda} (\phi_{\mathrm{f}}+\phi_{\mathrm{s}})_4^*(\phi_{\mathrm{f}}+\phi_{\mathrm{s}})_3^*(\phi_{\mathrm{f}}+\phi_{\mathrm{s}})_2(\phi_{\mathrm{f}}+\phi_{\mathrm{s}})_1 u_0)\right\rangle_{0\rangle}, \tag{13.53}$$

where the subscript $0\rangle$ stands for the average with respect to the quadratic action of the fast modes. The 16 possible monomials fall into four groups:

- Eight terms with an odd number of fast modes.
- One term with all fast modes.
- One term with all slow modes.
- Six terms with two slow and two fast modes.

We have no interest in the first two items: the first because they vanish by symmetry and the second because it makes a constant contribution, independent of ϕ_{s}, to the effective action.

The one term with all slow fields makes a tree-level contribution

$$S_u^{\mathrm{tree}} = \frac{1}{2!2!}u_0 \int_{k<\Lambda/s} \phi^*(\mathbf{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\prod_{i=1}^{3}\frac{d^dk_i}{(2\pi)^d} \tag{13.54}$$

to the action. The momentum and field have not been rescaled yet, as is evident from the cut-off.

That leaves us with six terms which have two fast and two slow fields. Of these, two are no good because both the fast fields are ϕ_{f}'s or ϕ_{f}^*'s, and these have zero average by $U(1)$ symmetry. This leaves us with four terms which schematically look like

$$\phi_{\mathrm{f}}^*\phi_{\mathrm{s}}^*\phi_{\mathrm{f}}\phi_{\mathrm{s}},\ \phi_{\mathrm{s}}^*\phi_{\mathrm{f}}^*\phi_{\mathrm{f}}\phi_{\mathrm{s}},\ \phi_{\mathrm{s}}^*\phi_{\mathrm{f}}^*\phi_{\mathrm{s}}\phi_{\mathrm{f}},\ \phi_{\mathrm{f}}^*\phi_{\mathrm{s}}^*\phi_{\mathrm{s}}\phi_{\mathrm{f}}. \tag{13.55}$$

All four terms make the same contribution to S_r (modulo dummy labels, which takes care of the $\frac{1}{2!2!}$ up front):

$$\delta S_r = u_0 \int \phi_{\mathrm{s}}^*(4)\phi_{\mathrm{s}}(2)dk_4dk_2\langle\phi_{\mathrm{f}}^*(3)\phi_{\mathrm{f}}(1)\rangle dk_3dk_1\delta(4+3-2-1), \tag{13.56}$$

where I am using a compact notation:

$$\delta(4+3-2-1) \equiv (2\pi)^d\delta(k_4+k_3-k_2-k_1), \tag{13.57}$$

$$dk \equiv \frac{d^dk}{(2\pi)^d}. \tag{13.58}$$

Using

$$\langle\phi_{\mathrm{f}}^*(3)\phi_{\mathrm{f}}(1)\rangle = \frac{\delta(3-1)}{k_3^2+r_0}, \tag{13.59}$$

we find (on changing some dummy variables and returning to the more explicit notation):

$$\delta S_r = \int_0^{\Lambda/s} \phi^*(k)\phi(k)\frac{d^dk}{(2\pi)^d}\left(u_0\int_{\Lambda/s}^{\Lambda}\frac{d^dk_3}{(2\pi)^d}\frac{1}{k_3^2+r_0}\right). \tag{13.60}$$

Let us see briefly how the above results follow in the diagrammatic approach. First, we associate with each of the 16 monomials contained in Eq. (13.53) a four-pronged X, as in Figure 13.1(a). The incoming arrows correspond to ϕ and the outgoing ones to ϕ^*. Each prong can stand for a ϕ_s or a ϕ_f, or its conjugate. Figure 13.1(a) shows the case where all four are slow. It contributes to u_0 at tree level. It merely has to be re-expressed in terms of new fields and momenta. Figure 13.1(b) is an example of the eight terms with an odd number of fast lines. These average to zero. Figure 13.1(c) describes the case with two fast modes (labels 1 and 3, with average $G(3)$) and two slow lines (labels 2 and 4), with both sets coming in complex conjugate pairs. The two fast lines are joined by the averaging, and the average is represented by the line joining them, the propagator $G(k_3)$. This is called the *tadpole diagram*. We are left with two slow fields, 2 and 4. This renormalizes the quadratic term S_r as per Eq. (13.60). Finally, Figure 13.1(d) describes the case where all lines are fast, come in complex-conjugate pairs, and average to two propagators which are integrated over. We do not consider this term since it contributes a constant independent of ϕ_s.

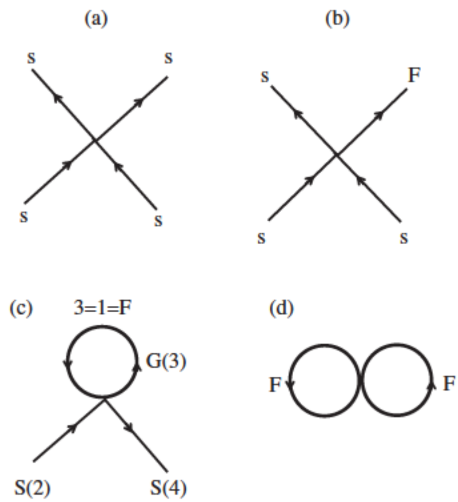

Figure 13.1 Diagrammatic description of the 16 monomials contained in Eq. (13.53). (a) The term with four slow fields, in complex-conjugate pairs. Contributes to u_0 at tree level. (b) One of the eight terms that have no average due to having odd powers of the fast field. (c) The tadpole graph that comes from averaging two fast fields (labeled 1 and 3, and with average $G(3)$) leaving behind a quadratic piece for the slow modes (labeled 2 and 4). It renormalizes r_0. (d) An ignorable ϕ_s-independent contribution with two fast pairs averaged.

Let us take stock. We began with

$$S = S^* + S_r + S_u \tag{13.61}$$

$$= \int_0^\Lambda \phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + r_0\int_0^\Lambda \phi^*(\mathbf{k})\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + \frac{u_0}{2!2!}\int_{|k|<\Lambda}\phi^*(\mathbf{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\prod_{i=1}^{3}\frac{d^dk_i}{(2\pi)^d} \tag{13.62}$$

$$\equiv S_0 + S_{\mathrm{I}}. \tag{13.63}$$

We ended up with

$$S_{\text{eff}} = \int_0^{\Lambda/s}\phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + r_0\int_0^{\Lambda/s}\phi^*(\mathbf{k})\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + \int_0^{\Lambda/s}\phi^*(\boldsymbol{k})\phi(\boldsymbol{k})\frac{d^dk}{(2\pi)^d}\left(u_0\int_{\Lambda/s}^{\Lambda}\frac{d^dk_3}{(2\pi)^d}\frac{1}{k_3^2+r_0}\right) + \frac{u_0}{2!2!}\int_{k<\Lambda/s}\phi^*(\mathbf{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\prod_{i=1}^{3}\frac{d^dk_i}{(2\pi)^d}. \tag{13.64}$$

Now for the long awaited rescaling to switch to new momenta k and new fields ϕ':

$$k' = sk, \tag{13.65}$$

$$\phi'(\boldsymbol{k}') = s^{-(\frac{d}{2}+1)}\phi\left(\frac{\boldsymbol{k}'}{s}\right) \equiv \zeta^{-1}(s)\phi\left(\frac{\boldsymbol{k}'}{s}\right). \tag{13.66}$$

I invite you to show that

$$S_{\text{eff}} = \int_0^{\Lambda}\phi^*(\mathbf{k})k^2\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + s^2r_0\int_0^{\Lambda}\phi^*(\mathbf{k})\phi(\mathbf{k})\frac{d^dk}{(2\pi)^d} + s^2\int_0^{\Lambda}\phi^*(\boldsymbol{k})\phi(\boldsymbol{k})\frac{d^dk}{(2\pi)^d}\left(u_0\int_{\Lambda/s}^{\Lambda}\frac{d^dk_3}{(2\pi)^d}\frac{1}{k_3^2+r_0}\right) + s^{4-d}\frac{u_0}{2!2!}\int_{k<\Lambda}\phi^*(\mathbf{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\prod_{i=1}^{3}\frac{d^dk_i}{(2\pi)^d}. \tag{13.67}$$

Exercise 13.1.3 *Carry out the rescaling of moment and fields and arrive at the preceding equation. (Only the momenta that are arguments of ϕ need rescaling, not k_3.)*

Exercise 13.1.4 *Suppose we begin with a quartic coupling $u(\boldsymbol{k}_1,\dots,\boldsymbol{k}_4)$ instead of a constant u_0. Expand it in powers of k_i^2 and show that the coefficients of the higher powers are highly irrelevant near $d = 4$. This is analogous to what happened when we considered $r(k)$ instead of r_0 in Exercise 13.1.1.*

It is common to parametrize s as

$$s = e^l. \tag{13.68}$$

(Sometimes one sets $s = e^t$, and we will too, in a later chapter not connected to critical phenomena. But here it would be inviting trouble since t is associated with deviation from criticality.) In particular, for infinitesimal scale change we write

$$s = e^{dl} \simeq 1 + dl. \tag{13.69}$$

Look at the k_3 integral in Eq. (13.67):

$$\begin{aligned} u_0 \int_{\Lambda/s}^{\Lambda} \frac{d^d k_3}{(2\pi)^d} \frac{1}{k_3^2 + r_0} &= u_0 \int_{\Lambda(1-dl)}^{\Lambda} \frac{k_3^{d-3} dk_3 S_d}{(2\pi)^d} \frac{1}{k_3^2 + r_0} \\ &= u_0 \frac{\Lambda^{d-3} S_d \Lambda dl}{(2\pi)^d} \frac{1}{\Lambda^2 + r_0} &(13.70) \\ &\stackrel{\text{def}}{=} u_0 \frac{A}{1 + r_0} dl, &(13.71) \end{aligned}$$

which defines the constant A whose precise expression will not matter. As explained in the discussion following Eq. (13.18), we may set $\Lambda = 1$. (It is being used as the unit as we renormalize, just the way the new lattice spacing was used as the unit of distance after a block spin operation or decimation.)

Since we want to go to first order in r_0 and u_0 we may neglect the r_0 in Eq. (13.71) and approximate:

$$u_0 \frac{A}{1 + r_0} \simeq u_0 A. \tag{13.72}$$

Adding this induced quadratic term to the one from tree level, rescaling the momenta and fields, we find the following quadratic term:

$$(1 + 2dl) \int_0^{\Lambda} \phi^*(\boldsymbol{k}) \phi(\boldsymbol{k})(r_0 + A u_0 dl) \frac{d^d k}{(2\pi)^d} \stackrel{\text{def}}{=} \int_0^{\Lambda} r_0(dl) \phi^*(\boldsymbol{k}) \phi(\boldsymbol{k}) \frac{d^d k}{(2\pi)^d}, \tag{13.73}$$

from which we deduce that

$$\begin{aligned} r_0(dl) &= (1 + 2dl)(r_0 + A u_0 dl) &(13.74) \\ \frac{dr_0}{dl} &= 2r_0 + A u_0. &(13.75) \end{aligned}$$

Now for the u_0 term in Eq. (13.66). We are working to first order in r_0 and u_0. Since the tree-level term for u_0 is already first order, we stop with

$$u_0(dl) = u_0 s^{(4-d)} = u_0(1 + (4 - d)dl), \tag{13.76}$$

and conclude that

$$\frac{du_0}{dl} = (4-d)u_0. \tag{13.77}$$

Here are our final flow equations and β-functions:

$$\beta_r = \frac{dr_0}{dl} = 2r_0 + Au_0, \tag{13.78}$$

$$\beta_u = \frac{du_0}{dl} = (4-d)u_0. \tag{13.79}$$

These flow equations have only one fixed point, the Gaussian fixed point at the origin:

$$K^* = (r_0^* = 0, u_0^* = 0). \tag{13.80}$$

Near the fixed point a coupling $K_\alpha = K_\alpha^* + \delta K_\alpha$ flows as follows:

$$\frac{dK_\alpha}{dl} = \beta_\alpha(K^* + \delta K) = 0 + \left.\frac{\partial \beta_a}{\partial K_\beta}\right|^* (K_\beta - K_\beta^*), \tag{13.81}$$

$$\frac{d\delta K_\alpha}{dl} = \left.\frac{\partial \beta_a}{\partial K_\beta}\right|^* \delta K_\beta, \quad \text{where} \tag{13.82}$$

$$\delta K_a = K_\alpha - K_\alpha^*. \tag{13.83}$$

In our problem where $K^* = (0,0)$, $\delta K_a = K_\alpha$. That is, $\delta r_0 = r_0$ and $\delta u_0 = u_0$. Starting with Eqs. (13.78) and (13.79), and taking partial derivatives with respect to r_0 and u_0, we arrive at

$$\begin{pmatrix} \frac{dr_0}{dl} \\ \frac{du_0}{dl} \end{pmatrix} = \begin{pmatrix} 2 & A \\ 0 & (4-d) \end{pmatrix} \begin{pmatrix} r_0 \\ u_0 \end{pmatrix}. \tag{13.84}$$

The 2×2 matrix is not Hermitian and for good reason. The 0 in the lower left reflects the fact that r_0 does not generate any u_0, while the non-zero element A in the upper right says that u_0 generates some r_0.

Because the lower-left element vanishes, the relevant and irrelevant eigenvalues are the diagonal entries themselves:

$$ad = 2 \text{ (or } \nu = \tfrac{1}{2}), \qquad \omega d = 4 - d, \tag{13.85}$$

in the notation of Eqs. (12.143)–(12.145).

Even though the flow got more complicated by the introduction of the irrelevant term, the relevant exponent did not get modified. The asymmetric matrix has distinct left and right eigenvectors. The right eigenvectors, which we have been using all along, are given, in the notation of Section 12.4, by

$$|a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\omega\rangle = \begin{pmatrix} -\frac{A}{d-2} \\ 1 \end{pmatrix}. \tag{13.86}$$

In terms of canonical operators, the eigenvectors correspond to

$$|a\rangle = 1 \cdot \phi^2 + 0 \cdot \phi^4, \tag{13.87}$$

$$|\omega\rangle = -\frac{A}{d-2}\phi^2 + 1 \cdot \phi^4. \tag{13.88}$$

Under the action of T:

$$T|a\rangle = s^2 |a\rangle, \tag{13.89}$$

$$T|\omega\rangle = s^{\varepsilon} |\omega\rangle, \tag{13.90}$$

$$\varepsilon = 4 - d, \tag{13.91}$$

where I have introduced the all-important parameter $\varepsilon = 4 - d$.

If we bring in the magnetic field, we have another eigenvector:

$$T|b\rangle = s^{1+\frac{d}{2}} |b\rangle. \tag{13.92}$$

This result is deduced as follows:

$$h\phi(0) = h\phi'(0/s)s^{1+\frac{d}{2}} \quad \text{(from Eq. (13.66))} \tag{13.93}$$

$$\equiv h_s \phi'(0). \tag{13.94}$$

So the Gaussian fixed point always has two relevant eigenvalues associated with temperature and magnetic field. The third eigenvalue $\omega d = 4 - d = |\varepsilon|$ is irrelevant for $d > 4$ and relevant for $d < 4$. It follows that for $d < 4$, the Gaussian fixed point does not describe the Ising class which can be driven to criticality by tuning just two parameters: $h = 0$, $t = 0$. It does describe critical phenomena with three relevant directions, but we will not go there. So, with one brief exception, we will study the Gaussian fixed point only for $d > 4$.

13.2 Gaussian Model Exponents for $d > 4$, $\varepsilon = 4 - d = -|\varepsilon|$

Figure 13.2 depicts the situation for $d > 4$ in the (r_0, u_0) plane, where r_0 is the coefficient of ϕ^2 and u_0 that of ϕ^4. The magnetic field is associated with a coordinate h and an eigenvector coming out of the page.

To attain criticality, we must first set $h = 0$. Next, in the (r_0, u_0) plane we need to tune just one parameter to hit the critical surface, the line

$$r_0 + \frac{Au_0}{d-2} = 0, \tag{13.95}$$

which just follows the irrelevant eigenvector flowing into the fixed point $K^* = (0,0)$. Equation 13.95 says that if we start with a non-zero u_0 we must tune r_0 to be $-\frac{Au_0}{d-2}$ to be critical.

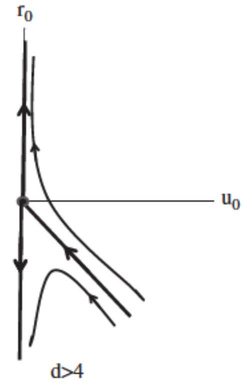

Figure 13.2 Flow in the Gaussian model for $d > 4$. The one-dimensional critical surface is shown by the attractive flow line into $K^* = (0,0)$.

In Section 12.4, Eqs. (12.143) and (12.145), we performed an abstract study of a flow with three parameters, two relevant and one irrelevant. I repeat those equations below with one minor change in notation: I replace L, the factor by which we change the spatial lattice size, with the factor s which produces the equivalent reduction in Λ:

$$|\Delta K\rangle = t|a\rangle + h|b\rangle + g|\omega\rangle, \tag{13.96}$$

$$T(s)|\Delta K\rangle = ts^{ad}|a\rangle + hs^{bd}|b\rangle + gs^{-|\omega|d}|\omega\rangle \tag{13.97}$$

$$= ts^{\frac{1}{\nu}}|a\rangle + hs^{d-\frac{\beta}{\nu}}|b\rangle + gs^{-|\omega|d}|\omega\rangle. \tag{13.98}$$

To evade relevant flow in the t and h directions, to retain or attain criticality, we just have to tune to $t = h = 0$.

What are h, t, and g in terms of h, r_0, and u_0?

Now h is just h. It gets rescaled as

$$h \to h_s = hs^{1+\frac{d}{2}}. \tag{13.99}$$

In the absence of u_0, the role of t is played by r_0, the coefficient of ϕ^2, since a non-zero r_0 takes us off the Gaussian fixed point. However, in the presence of u_0, t should be measured vertically up (in the r_0 direction) *from* the critical line $r_0 + \frac{Au_0}{d-2}u_0 = 0$. To understand this systematically we do the following:

- Express the initial position vector $r_0|1\rangle + u_0|2\rangle$, associated with the action

$$S = r_0\phi^2 + u_0\phi^4, \tag{13.100}$$

in terms of the eigenvectors of T:

$$|a\rangle = |1\rangle, \tag{13.101}$$

$$|\omega\rangle = -\frac{A}{d-2}|1\rangle + |2\rangle. \tag{13.102}$$

- Find the effect of an RG by scale factor s:

$$|a\rangle \to T|a\rangle = s^2|a\rangle, \tag{13.103}$$
$$|\omega\rangle \to T|\omega\rangle = s^{-|\varepsilon|}|\omega\rangle. \tag{13.104}$$

- Find the renormalized action in the canonical basis of ϕ^2 and ϕ^4.

The result of this exercise is that the action evolves as follows:

$$r_0\phi^2 + u_0\phi^4 \to \left[\left(r_0 + \frac{Au_0}{d-2}\right)s^2 - \frac{Au_0}{d-2}s^{-|\varepsilon|}\right]\phi^2 + u_0 s^{-\varepsilon}\phi^4. \tag{13.105}$$

Exercise 13.2.1 *Derive Eq. (13.105).*

Exercise 13.2.2 *Show that the initial point $(r_0 = 0,\ u_0 = 1) = |2\rangle$ does not flow to the fixed point even though u_0 is termed irrelevant. Do this by writing the initial state in terms of $|a\rangle$ and $|\omega\rangle$.*

As $s \to \infty$, we may drop the $s^{-|\varepsilon|}$ part compared to the s^2 part in the first term and identify

$$t = \left(r_0 + \frac{Au_0}{d-2}\right). \tag{13.106}$$

As expected, when $t = 0$ we lie on the critical line. The second term has the scaling form already and u_0 plays the role of g, the irrelevant coupling. But remember this: u_0 being irrelevant does not mean that if we add a tiny bit of it, the action will flow to the fixed point. Instead, u_0 will generate some r_0 and the final point will run off along the $|a\rangle$ axis, as discussed in Exercise 13.2.2.

To find the other exponents we need to begin with f, the free energy per unit volume,

$$f = -\frac{\ln Z}{\text{Volume}}, \tag{13.107}$$

and take various derivatives. Now the RG, in getting rid of fast variables, does not keep track of their contribution to f. These were the $\ln Z_0(\text{fast})$ factors which were dropped along the way as unimportant for the averages of the slow modes. Fortunately, these contributions were analytic in all the parameters, coming as they did from fast modes. What we want is f_s, the *singular part of the free energy*, which is controlled by the yet to be integrated soft modes near $k = 0$. This remains unaffected as we eliminate modes with one trivial modification: due to the change in scale that accompanies the RG, unit volume after RG corresponds to volume s^d before RG. Consequently, the free energy *per unit volume* behaves as follows in d dimensions:

$$f_s(t, h, u_0) = s^{-d} f_s(ts^2, hs^{1+\frac{1}{2}d}, u_0 s^{4-d} + \cdots), \tag{13.108}$$

where the ellipsis refers to even more irrelevant couplings like $w_0(\phi^*\phi)^3$, which can be safely set to zero and ignored hereafter.

Following the familiar route,

$$m(-|t|,h,u_0) \simeq \left.\frac{\partial f}{\partial h}\right|_{h=0} \tag{13.109}$$

$$= s^{1-\frac{1}{2}d} m(-|t|s^2, 0, u_0 s^{4-d}) \tag{13.110}$$

$$= |t|^{(-\frac{1}{2})(1-\frac{1}{2}d)} m(-1,0,u_0 t^{(d-4)/2}) \tag{13.111}$$

$$= |t|^{(-\frac{1}{2})(1-\frac{1}{2}d)} m(-1,0,0) \ \text{ when } t \to 0, \tag{13.112}$$

$$\beta = \frac{d-2}{4}. \tag{13.113}$$

Consider the arguments of m in Eq. (13.112). Starting with a small *negative* $t = -|t|$ (required for non-zero m) I have renormalized to a point where it has grown to a robust value of -1. The middle argument $h = 0$ once the h derivative has been taken. Finally, I have set $u_{0s} = u_0|t|^{(d-4)/2} = 0$ in the limit $t \to 0$.

Taking another h derivative,

$$\chi(t,0,u_0) = t^{-1}\chi(1,0,u_0 t^{(d-4)/2}) \simeq t^{-1}\chi(1,0,0), \ \text{ i.e.,} \tag{13.114}$$

$$\gamma = 1. \tag{13.115}$$

(Unlike m, which exists only for $t < 0$, χ and hence γ can be computed in the $t > 0$ region. Thus we can let t grow under RG to $+1$.)

To find C_V, we take two derivatives of f with respect to t and find, as usual,

$$\alpha = 2 - \frac{1}{2}d. \tag{13.116}$$

Finally, to find δ we begin with Eq. (13.108), take an h-derivative, and then set $t = 0$ to obtain

$$m(0,h,u_0) \simeq \left.\frac{\partial f_s}{\partial h}\right|_{t=0} = s^{1-\frac{1}{2}d} m(0, hs^{1+\frac{1}{2}d}, u_0 s^{4-d}) \tag{13.117}$$

$$= h^{\frac{d-2}{d+2}} m(0,1,u_0 h^{(d-4)/(1+\frac{1}{2}d)}) \tag{13.118}$$

$$= h^{\frac{d-2}{d+2}} m(0,1,0) \ \text{ when } h \to 0, \tag{13.119}$$

$$\delta = \frac{d+2}{d-2}. \tag{13.120}$$

Table 13.1 compares the preceding exponents of the Gaussian model for $d > 4$ to Landau theory.

The exponents β and δ agree only at $d = 4$ but not above. So which one is right? It turns out it is Landau's. Let me show you what was wrong with the way these two Gaussian exponents were derived.

Look at the passage from Eq. (13.111) to Eq. (13.112) for m. Using the RG scaling, we arrived at

$$m(-|t|, h = 0, u_0) = |t|^{\frac{d-2}{4}} m(-1,0,u_0|t|^{(d-4)/2}), \tag{13.121}$$

Table 13.1 *Gaussian model for $d > 4$ versus Landau theory.*

Exponent	Landau	Gaussian $d > 4$
α	jump	$\frac{4-d}{2} < 0$
β	$\frac{1}{2}$	$\frac{d-2}{4}$
γ	1	1
δ	3	$\frac{d+2}{d-2}$
ν	$\frac{1}{2}$	$\frac{1}{2}$
η	0	0

which relates the magnetization in the critical region with a tiny negative t to its value at a point far from criticality, where t had been renormalized to -1, where fluctuations are negligible, and where we can use Landau's derivation with impunity. Landau's analysis gives, in the magnetized phase,

$$m(-|r|,0,u) \simeq \sqrt{\frac{|r|}{u}}. \tag{13.122}$$

Applying this general result to our case,

$$m(-1,0,u_0|t|^{(d-4)/2}) \simeq \sqrt{\frac{1}{u_0|t|^{(d-4)/2}}} \simeq |t|^{-\frac{d-4}{4}}, \tag{13.123}$$

and this means we cannot simply set $u_0|t|^{(d-4)/2}$ to zero as $t \to 0$, because it enters the denominator in the expression for m computed far from the critical point. Thus, $m(-1,0,0)$ is not some ignorable constant prefactor, but a factor with a divergent t dependence. Incorporating this singularity from Eq. (13.123) into Eq. (13.121), we are led to

$$m(-|t|,h=0,u_0) = |t|^{\frac{d-2}{4}}|t|^{-\frac{d-4}{4}} \simeq |t|^{\frac{1}{2}}, \tag{13.124}$$

which is Landau's result.

So the error was in setting the irrelevant variable to 0 when it appeared in the *denominator* of the formula for m in the region far from criticality. For this reason, u_0 is called a *dangerous irrelevant variable*. A dangerous irrelevant variable is one which cannot be blindly set to zero even if it renormalizes to 0 far from the critical region. The singularity it produces must be incorporated with care in ascertaining the true critical behavior.

Likewise, given Landau's answer far from criticality,

$$m(0,h,u) \simeq \left[\frac{h}{u}\right]^{1/3}, \tag{13.125}$$

the derivation of δ must be modified as follows:

$$m(0,h,u_0) \simeq \left.\frac{\partial f_s}{\partial h}\right|_{t=0} \tag{13.126}$$
$$= s^{1-\frac{1}{2}d} m(0, hs^{1+\frac{1}{2}d}, u_0 s^{4-d}) \tag{13.127}$$
$$= h^{\frac{d-2}{d+2}} m(0,1,u_0 h^{(d-4)/(1+\frac{1}{2}d)}) \tag{13.128}$$
$$= h^{\frac{d-2}{d+2}} \left(\frac{1}{u_0 h^{(d-4)/(1+\frac{1}{2}d)}}\right)^{1/3}, \tag{13.129}$$
$$= h^{\frac{1}{3}} \tag{13.130}$$
$$\delta = 3. \tag{13.131}$$

Now for why Landau theory works for $d > 4$ even though it ignores fluctuations about the minimum of the action. These fluctuations are computed perturbatively in u_0. For $d > 4$ these correction terms are given by convergent integrals. They do not modify the singularities of the theory at $u_0 = 0$.

Conversely, perturbation theory in u_0 fails in $d < 4$ near criticality *no matter how small* u_0 *is*. The true expansion parameter that characterizes the Feynman graph expansion ends up being $u_0 r_0^{\frac{1}{2}(d-4)}$ and not u_0. Thus, no matter how small u_0 is, the corrections due to it will blow up as we approach criticality.

One can anticipate this on dimensional grounds: since r_0 always has dimension 2 and u_0 has dimension $4-d$ in momentum units, the dimensionless combination that describes interaction strength is $u_0 r_0^{\frac{1}{2}(d-4)}$.

13.3 Wilson–Fisher Fixed Point $d < 4$

If $d < 4$, both directions become relevant at the Gaussian fixed point. (We have set $h = 0$.) It is totally unstable. *Two* parameters, namely r_0 and u_0, have to be tuned to hit criticality. This does not correspond to any Ising-like transition. This fixed point will be of interest later on, when we consider renormalization of quantum field theories, but for now let us move on to a fixed point in $d < 4$ that has only one relevant eigenvalue and describes Ising and Ising-like transitions.

Here is the trick due to Wilson and Fisher [3] for finding and describing it perturbatively in the small parameter

$$\varepsilon = 4 - d. \tag{13.132}$$

Their logic is that if mean-field theory works at $d = 4$, it should work near $d = 4$ with small controllable fluctuations. But this requires giving a meaning to the calculation for continuous d. As mentioned earlier, we just need to deal with the measure $d^d k$. The idea is to compute the exponents as series in ε and then set $\varepsilon = 1$, hoping to get reliable results for $d = 3$. Here we consider just the terms to order ε.

First for the renormalization of r_0. We already know that

$$\frac{dr_0}{dl} = 2r_0 + \frac{u_0 A}{1 + r_0}. \tag{13.133}$$

The denominator is just $k^2 + r_0$ when $k = \Lambda = 1$.

We already have part of the flow for u_0 from Eq. (13.79):

$$\frac{du_0}{dl} = (d-4)u_0 + \mathcal{O}(u_0^2) = \varepsilon u_0 + \mathcal{O}(u_0^2). \tag{13.134}$$

We need the $\mathcal{O}(u_0^2)$ term to find the fixed point u to order ε. This term describes how u_0 renormalizes itself to order u_0^2.

We have to take two powers of the quartic interaction, each with 16 possible monomials, and perform the averages we need to get what we want: a term of the form

$$-\int \phi_s^*(k_4)\phi_s^*(k_3)\phi_s(k_2)\phi_s(k_1)u(4321), \tag{13.135}$$

which can be added on to the existing quartic term to renormalize it.

Look at Figure 13.3. Part (a) shows a contribution in which the two slow fields at the left vertex are part of the quartic term generated, and the two fast ones are averaged with their counterparts in the right vertex. Part (b) is identical except for the way the external momenta are attached to the vertices. Part (c) has a factor of $\frac{1}{2}$ to compensate for the fact that the vertical internal lines are both particles whose exchange does not produce a new contribution, in contrast to the internal lines in (a) and (b) which describe particle–antiparticle pairs. Part (d) describes a disconnected diagram which does not contribute to the flow and in fact cancels in the cumulant expansion.

Since we are looking only for the change in the marginal coupling $u_0 = u(0,0,0,0)$, we may assume the slow fields are all at $k = 0$. With no momentum flowing into the loops, all propagators have the same momentum $k = \Lambda$. All diagrams make the same contribution except for the $\frac{1}{2}$ in Figure 13.3(c).

You have three choices here. You can accept what I say next, or go through the 256 terms and collect the relevant pieces, or use Feynman diagrams which automate the process. For those of you who are interested, I show at the end of this section how the following result,

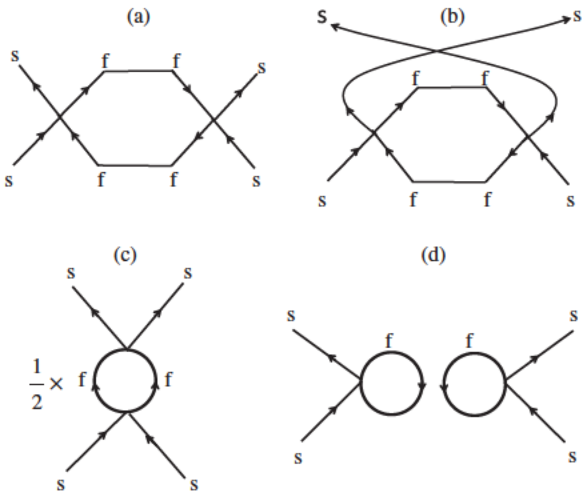

Figure 13.3 (a), (b), and (c) describe the three diagrams that contribute to the flow of u_0. Diagram (c) has a factor of $\frac{1}{2}$ due to the identity of two particles in the loop. Diagram (d) is a disconnected diagram. All external momenta vanish and all loop momenta are infinitesimally close to Λ.

Eq. (13.136), is obtained from Feynman diagrams:

$$u'_0 = (1+\varepsilon dl)\left[u_0 - \frac{5}{2}\frac{u_0^2 A dl}{(1+r_0)^2}\right], \tag{13.136}$$

$$\frac{du_0}{dl} = \varepsilon u_0 - \frac{5}{2}\frac{u_0^2 A}{(1+r_0)^2}, \tag{13.137}$$

where A is the same constant as in the tadpole graph that renormalized r_0 because it involves the same loop integral over one momentum.

Here, then, are our flow equations [Eqs. (13.133) and (13.137)]:

$$\beta_r = \frac{dr_0}{dl} = 2r_0 + \frac{u_0 A}{1+r_0} \simeq 2r_0 + u_0 A(1-r_0), \tag{13.138}$$

$$\beta_u = \frac{du_0}{dl} = \varepsilon u_0 - \frac{5}{2}\frac{u_0^2 A}{(1+r_0)^2} = \varepsilon u_0 - \frac{5}{2}u_0^2 A, \tag{13.139}$$

where in the last step I have set $r_0 = 0$ in the denominator with errors of order $u_0^2 r_0$.

From Eq. (13.139), we learn that to order ε the fixed point values u^* are

$$u_0^* = 0 \quad \text{Gaussian}, \tag{13.140}$$

$$u_0^* = \frac{2\varepsilon}{5A} \quad \text{Wilson–Fisher (WF)}. \tag{13.141}$$

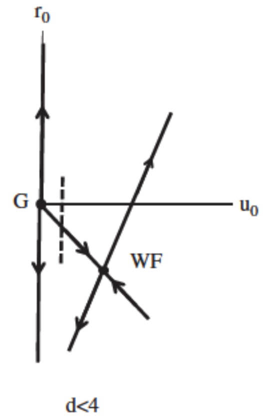

Figure 13.4 The Gaussian fixed point (with two relevant directions) and the Wilson–Fisher fixed point (with one relevant direction). Ignore the vertical dotted line for now.

Applying this to Eq. (13.138), we learn that the fixed point values r_0^* are, to order ε,

$$r_0^* = 0 \quad \text{Gaussian,} \tag{13.142}$$

$$r_0^* = -\frac{\varepsilon}{5} \quad \text{WF.} \tag{13.143}$$

The situation for $d < 4$ is shown in Figure 13.4. I will not consider the Gaussian fixed point at the origin, other than to note what we already know: that it is repulsive in both directions. The linearized flow near the WF fixed points is

$$\begin{pmatrix} \frac{d\delta r_0}{dl} \\ \frac{d\delta u_0}{dl} \end{pmatrix} = \begin{pmatrix} \frac{\partial \beta_r}{\partial r_0} & \frac{\partial \beta_r}{\partial u_0} \\ \frac{\partial \beta_u}{\partial r_0} & \frac{\partial \beta_u}{\partial u_0} \end{pmatrix}_{r_0^*, u_0^*} \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix} \tag{13.144}$$

$$= \begin{pmatrix} 2 - \frac{2}{5}\varepsilon & A(1+\frac{\varepsilon}{5}) \\ 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} \delta r_0 \\ \delta u_0 \end{pmatrix}, \tag{13.145}$$

where the matrix is evaluated at the fixed point and terms of higher order than ε have been dropped. The most interesting consequence is

$$\nu = \frac{1}{2 - \frac{2}{5}\varepsilon} = \frac{1}{2} + \frac{\varepsilon}{10} = 0.6 \text{ for } d = 3. \tag{13.146}$$

Observe that the answer does not depend on A.

To get the Ising result from this $U(1) = O(2)$ result we replace the $\frac{5}{2}$ in Eq. (13.139) and above by $\frac{6}{2}$ above because:

- the factor $\frac{1}{2!2}$ which is replaced by $\frac{1}{4!}$ is exactly canceled by the multiplicities;
- for real scalars there is no distinction between particles and antiparticles and all three graphs contribute equally.

Table 13.2 *The ε expansion versus others (# denotes numerical results).*

Exponent	Landau	Ising ($\mathcal{O}(\varepsilon)$)	Ising (#)	U(1)
α	jump	$\frac{\varepsilon}{6}=0.17$	0.110	$\frac{\varepsilon}{10}$
β	$\frac{1}{2}$	$\frac{1}{2}-\frac{\varepsilon}{6}=0.33$	0.326	$\frac{1}{2}-\frac{3\varepsilon}{20}$
γ	1	$1+\frac{\varepsilon}{6}=1.17$	1.24	$1+\frac{\varepsilon}{5}$
δ	3	$3+\varepsilon=4$	4.79	$3+\varepsilon$
ν	$\frac{1}{2}$	$\frac{1}{2}+\frac{\varepsilon}{12}=0.58$	0.630	$\frac{1}{2}+\frac{\varepsilon}{5}$
η	0	$\mathcal{O}(\varepsilon^2)$	0.036	$\mathcal{O}(\varepsilon^2)$

The result is:

$$\nu = \frac{1}{2-\frac{2}{6}\varepsilon} = \frac{1}{2} + \frac{\varepsilon}{12} \ = 0.58 \text{ for } d=3. \tag{13.147}$$

Now for the other exponents at $d = 4-\varepsilon$. The scaling of h is still

$$h_s = hs^{1+\frac{1}{2}d} \tag{13.148}$$

for real and complex fields because the field rescaling needed to obtain a fixed point action has not been altered to order ε: the coefficient of the k^2 term receives no corrections to order ε. The other exponents then follow from the usual analysis. Only the *deviation* δu_0 from the fixed point is irrelevant and flows to 0, but the fixed point itself is not at $u_0 = 0$. So u_0 is not dangerous and the extraction of exponents has no pitfalls. Table 13.2 compares the Ising exponents to order ε with Landau theory and the best numerically known answers in $d = 3$. I also show the $U(1)$ answers to $\mathcal{O}(\varepsilon)$.

Observe from the table that every correction to the Landau exponents is in the right direction, toward the numerical answers. The exponents obey $\alpha + 2\beta + \gamma = 2$, as is assured by the scaling arguments used in their derivation.

At higher orders in ε one faces two problems. The first is that the ε expansion is asymptotic: it does not represent a convergent series; beyond some point, agreement will worsen and fancy resummation techniques will have to be invoked. The second is that the Wilson approach gets very complicated to the next order. We need to include more operators (like $(\phi^*\phi)^3$). The kinematics can get messy. Consider, for example, the one-loop diagrams in Figure 13.3. The momenta in the loop have to be limited to a narrow sliver of width $d\Lambda$ near the cut-off. Since the external momenta were chosen to be zero, if one of the momenta was at Λ then so was the other one automatically, by momentum conservation. Had we been interested in irrelevant corrections to $u(k)$, we would have had to consider non-zero external momenta. In this case, if one of the momenta k were within the shell at Λ, restricting the other to the shell would have been very cumbersome. For these reasons one employs what is called the "field theory approach," which is less intuitive but more efficient. This will be described later.

Another feature we see at higher orders in ε is that the factor for field rescaling changes from

$$\zeta = s^{1+\frac{1}{2}d}, \tag{13.149}$$

which was derived in the non-interacting theory to keep the kinetic term with coefficient $\phi^* k^2 \phi$. At higher orders, the $\phi^* k^2 \phi$ term can get corrections as modes are eliminated. To bring the coefficient back to unity, a different ζ will be needed.

13.3.1 Digression on Feynman Diagrams

The diagrams of Figure 13.3 may be deduced from the Feynman diagrams shown in Figure 13.5. I show them in $d = 4$ for convenience. In quantum field theory, they describe

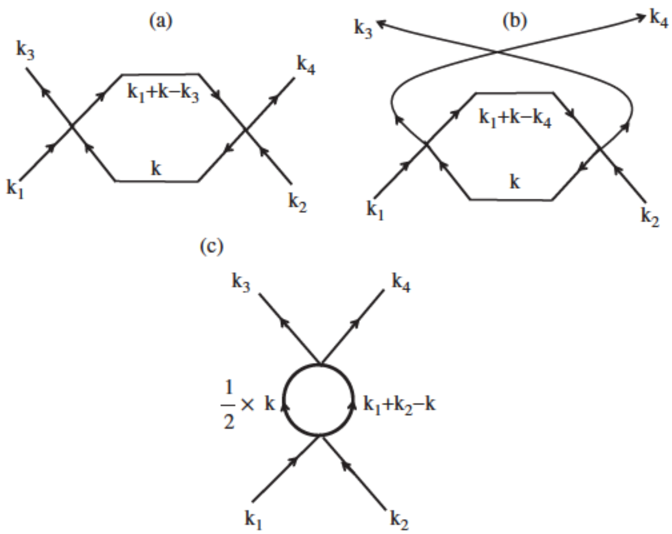

Figure 13.5 The Feynman diagrams that renormalize the quartic interaction at one loop. To get Eq. (13.136), all external momenta are set to 0 and loop integrals are restricted to lie between $k = \Lambda = 1 - dl$ and $k = \Lambda$.

the one-loop correction to the coupling u and are given by

$$\begin{aligned}
u_0' = u_0 - u_0^2 \Bigg[&\int_0^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+r_0)(|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_3|^2+r_0)} \\
&+ \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+r_0)(|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_4|^2+r_0)} \\
&+ \frac{1}{2}\int_0^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+r_0)(|-\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2|^2+r_0)} \Bigg].
\end{aligned} \tag{13.150}$$

The field theory diagrams agree on multiplicity and topology with the Wilsonian ones, but differ as follows:

- The loop momenta go from 0 to Λ.
- The external momenta have some general values $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4$.

We can borrow the corresponding integrals for the WF calculation if we set all external momenta to 0, set $k = \Lambda = 1$ in every loop, and multiply by $dk = dl$ to represent the integration over an infinitesimal shell of thickness dl at the cut-off. Finally, in $d = 4 - \varepsilon$ we must rescale the coupling by $1 + \varepsilon dl$, as in Eq. (13.136), because it is dimensionful. With these values, and $S_4 = 2\pi^2$, all three loops contribute an equal amount

$$-\frac{u_0^2 dl}{8\pi^2(1+r_0)^2}, \tag{13.151}$$

which we then multiply by $\frac{5}{2}$ to account for the three diagrams, the last of which contributes with a relative size of $\frac{1}{2}$. As an aside, we note that the constant $A = \frac{1}{8\pi^2}$.

13.4 Renormalization Group at $d = 4$

The case of $d = 4$ is very instructive. (Remember that d is the number of *spatial dimensions* and usually $d = 3$. However, as described in [4–6], certain systems in $d = 3$ have low-energy propagators that resemble those from $d = 4$ in the infrared. The following analysis applies to them.)

$$u_0' = u_0 - \frac{5u_0^2}{2} \int_{d\Lambda} \frac{k^3 dk d\Omega}{(2\pi)^4 k^4}, \tag{13.152}$$

$$\frac{du_0}{dl} = -\frac{5u_0^2}{16\pi^2}, \tag{13.153}$$

where I have again used the fact that the area of a unit sphere in $d = 4$ is $S_4 = 2\pi^2$. To one-loop accuracy we have the following flow:

$$\frac{dr_0}{dl} = 2r_0 + au_0, \tag{13.154}$$

$$\frac{du_0}{dl} = -bu_0^2, \tag{13.155}$$

where a and b are positive constants whose precise values will not matter.

We shall now analyze these equations. First, besides the Gaussian fixed point at the origin, there are no other points where both derivatives vanish. Next, the equation for u_0 is readily integrated to give

$$u_0(l) = \frac{u_0(0)}{1 + bu_0(0)l}. \tag{13.156}$$

This means that if we start with a positive coupling $u_0(0)$ at $\Lambda = \Lambda_0$ and renormalize to $\Lambda = \Lambda_0 e^{-l}$, the effective coupling renormalizes to zero as $l \to \infty$. One says that u_0 *is marginally irrelevant*. In the present case it vanishes as follows as $l \to \infty$:

$$u(t) \lim_{t\to\infty} \frac{1}{bl}. \tag{13.157}$$

This statement needs to be understood properly. *In particular, it does not mean that if we add a small positive u_0 to the Gaussian fixed point, we will renormalize back to the Gaussian fixed point.* This is because the small u_0 will generate an r_0, and that will quickly grow under renormalization. What is true is that ultimately u_0 will decrease to zero, but r_0 can be large. To flow to the Gaussian fixed point, we must start with a particular combination of r_0 and u_0 which describes the critical surface. All this comes out of Eq. (13.154) for r_0, which is integrated to give

$$r_0(l) = e^{2l}\left[r_0(0) + \int_0^l e^{-2l'} \frac{au_0(0)}{1 + bu_0(0)l'} dl'\right]. \tag{13.158}$$

Let us consider large l. Typically, r_0 will flow to infinity exponentially fast due to the exponential prefactor, unless we choose r_0 such that the object in brackets vanishes as $l \to \infty$:

$$r_0(0) + \int_0^\infty e^{-2l'} \frac{au_0(0)}{1 + bu_0(0)l'} dl' = 0. \tag{13.159}$$

If we introduce this relation into Eq. (13.158), we find that

$$r_0(l) = e^{2l}\left[-\int_l^\infty e^{-2l'} \frac{au_0(0)}{1 + bu_0(0)l'} dl'\right] \simeq -\frac{a}{2bl}. \tag{13.160}$$

Combined with the earlier result Eq. (13.157), we find, for large l,

$$r_0(l) \lim_{l\to\infty} -\frac{a}{2bl}, \tag{13.161}$$

$$u_0(l) \lim_{l\to\infty} \frac{1}{bl}, \tag{13.162}$$

which defines the critical surface (a line) in the r_0–u_0 plane:

$$r_0(l) = -\frac{au_0(l)}{2}. \tag{13.163}$$

This flow into the fixed point at $d = 4$ resembles the flow in $d > 4$ depicted in Figure 13.2, except that the approach to the fixed point is logarithmic and not power law.

Look at Eq. (13.156). It tells us that in the deep infrared, the coupling actually vanishes as $1/l$, and that in the large-l region we can make reliable weak coupling calculations. This is, however, thanks to the RG. In simple perturbation theory, we would have found the series

$$u_0(l) = u_0(0) - u_0^2 bl + \mathcal{O}(l^2) \tag{13.164}$$

with ever increasing terms as $l \to \infty$. The RG sums up the series for us [Eq. (13.156)] and displays how the coupling flows to 0 in the infrared.

References and Further Reading

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14

Two Views of Renormalization

Here I discuss the relationship between two approaches to renormalization: the older one based on removing infinities in the quest for field theories in the continuum, and the more modern one due to Wilson based on obtaining effective theories. My focus will be on a few central questions. No elaborate calculations will be done.

14.1 Review of RG in Critical Phenomena

Let us recall the problem of critical phenomena and its resolution by the RG. Suppose we have some model on a lattice with some parameters, like $K_1, K_2, \ldots$ of the Ising model. At very low and very high temperatures ($K \to \infty$ or $K \to 0$) we can employ perturbative methods like the low-temperature or high-temperature expansions to compute correlation functions. These series are predicated on the smooth change of physics as we move away from these extreme end points. By definition, these methods will fail at the critical point (and show signs of failing as we approach it) because there is a singular change of phase. One signature of trouble is the diverging correlation length ξ. The RG beats the problem by trading the original system near the critical point for one that is comfortably away from it (and where the series work) and things like ξ can be computed. The RG then provides a dictionary for translating quantities of original interest in terms of new ones. For example,

$$\xi(r_0) = 2^N \xi(r_N), \tag{14.1}$$

where $r_0 \simeq t$ is the deviation from criticality, N is the number of factor-of-2 RG transformations performed, and r_N the coupling that r_0 evolves into. At every step,

$$r_0 \to r_0 2^{ad} = r_0 2^{1/\nu}. \tag{14.2}$$

We keep renormalizing until r_N has grown to a safe value far from criticality, say

$$r_N = r_0 2^{N/\nu} = 1, \quad \text{that is} \tag{14.3}$$

$$2^N = r_0^{-\nu}. \tag{14.4}$$

Then, from Eq. (14.1),

$$\xi(r_0) = r_0^{-\nu}\xi(1). \tag{14.5}$$

The divergence in ξ is translated into the divergence in N, the number of steps needed to go from r_0 to $r_N = 1$ as r_0 approaches the critical value of 0.

In terms of the continuous scale s (which replaces 2^N), these relations take the form

$$\xi(r_0) = s\xi(r_s), \tag{14.6}$$

$$r_s = r_0 s^{1/\nu}. \tag{14.7}$$

Typically one finds some *approximate* flow equations, their fixed points $\boldsymbol{K}^*$, the linearized flow near $\boldsymbol{K}^*$, and, eventually, the exponents.

14.2 The Problem of Quantum Field Theory

Consider the field theory with action (with $c = 1 = \hbar$)

$$S = \int \left[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m_0^2\phi^2(x) + \frac{\lambda_0}{4!}\phi^4(x)\right]d^4x \tag{14.8}$$

$$= S_0 + S_{\mathrm{I}}, \tag{14.9}$$

where S_0 is the quadratic part. I have chosen $d = 4$, which is relevant to particle physics and serves to illustrate the main points, and a real scalar field to simplify the discussion. The parameters m_0 and λ_0 are to be determined by computing some measurable quantities and comparing to experiment.

To this end, we ask what is typically computed, how it is computed, and what information it contains.

Consider the two-point correlation function

$$G(x) = \langle\phi(x)\phi(0)\rangle \tag{14.10}$$

$$= \frac{\int[\mathcal{D}\phi]\phi(x)\phi(0)e^{-S}}{\int[\mathcal{D}\phi]e^{-S}}. \tag{14.11}$$

First, let us assume that $\lambda_0 = 0$. Doing the Gaussian functional integral we readily find

$$G(r) \simeq \frac{e^{-m_0 r}}{r^2}. \tag{14.12}$$

How do we determine m_0 from experiment? In the context of particle physics, m_0 would be the particle mass, measured the way masses are measured. If, instead, the ϕ^4 theory were being used to describe spins on a lattice of spacing a, we would first measure the dimensionless correlation length ξ (in lattice units) from the exponential decay of correlations and relate it to m_0 by the equation

$$m_0 = \frac{1}{a\cdot\xi}. \tag{14.13}$$

Sometimes I will discuss correlations of four ϕ's. They will also be called G, but will be shown with four arguments. If not, assume we are discussing the two-point function.

In momentum space we would consider the Fourier transform

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\rangle = (2\pi)^4 \delta^4(\boldsymbol{k}_1 - \boldsymbol{k}_2) G(k), \quad \text{where} \tag{14.14}$$

$$G(k) = G_0(k) = \frac{1}{k^2 + m_0^2}; \tag{14.15}$$

the subscript on G_0 reminds us that we are working with a free-field theory. Correlations with more fields can be computed as products of two-point functions $G_0(k)$ using Wick's theorem. If this explains the data, we are done.

14.3 Perturbation Series in λ_0: Mass Divergence

Let us say the $\lambda_0 = 0$ theory does not explain the data. For example, the particles could be found to scatter. The $\lambda_0 = 0$ theory cannot describe that. So we toss in a λ_0 and proceed to calculate correlation functions, and fit the results to the data to determine m_0 and λ_0.

When $\lambda_0 \neq 0$, we resort to perturbation theory. We bring the $\lambda_0 \phi^4$ term in S downstairs as a power series in λ_0 and do the averages term-by-term using Wick's theorem. To order λ_0, we find

$$G(x) = \langle \phi(x)\phi(0)\rangle \tag{14.16}$$

$$= \frac{\int [\mathcal{D}\phi]\phi(x)\phi(0) e^{-S_0(\phi)} \left[1 - \frac{\lambda_0}{4!}\int \phi^4(y) d^4y\right]}{\int [\mathcal{D}\phi] e^{-S_0(\phi)} \left[1 - \frac{\lambda_0}{4!}\int \phi^4(y) d^4y\right]}. \tag{14.17}$$

In the denominator, we pair the four $\phi(y)$'s two-by-two to obtain

$$\text{denominator} = 1 - \frac{\lambda_0}{8}\int G_0^2(0) d^4y. \tag{14.18}$$

In the numerator, one option is to pair $\phi(x)$ and $\phi(0)$, which are being averaged, and pair the fields inside the y integral with each other. This will give $G_0(x) \cdot \left(1 - \frac{\lambda_0}{8}\int G_0^2(0) d^4y\right)$. The factor in parentheses will get canceled by the normalizing partition function in the denominator. This happens in general: any contribution in which the fields being averaged do not mingle with the ones in the interaction, the so-called *disconnected terms*, may be dropped.

This leaves us with contributions where $\phi(x)$ and $\phi(0)$ are paired with the $\phi(y)$'s. The result is, *to order* λ_0,

$$G(x) = G_0(x) - \frac{1}{2}\lambda_0 \int G_0(x-y) G_0(y-y) G_0(y-0) d^4y. \tag{14.19}$$

Since the second term is of order λ_0, we may set the normalizing denominator $1 - \frac{\lambda_0}{8}\int G_0^2(0) d^4y$ to 1.

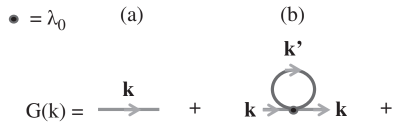

Figure 14.1 (a) $G_0(k) = G(k)$ in free-field theory. (b) One-loop correction to m_0^2. The lines with arrows denote the free propagator $G_0(k) = \frac{1}{k^2+m_0^2}$.

In momentum space,

$$G(k) = \frac{1}{k^2+m_0^2} - \frac{1}{k^2+m_0^2} \underbrace{\left[\frac{1}{2} \int_0^\infty \frac{\lambda_0}{k'^2+m_0^2} \frac{d^4k'}{(2\pi)^4} \right]}_{\delta m_0^2} \frac{1}{k^2+m_0^2} \cdots \tag{14.20}$$

This series is represented in Figure 14.1.

To this order in λ_0 we may rewrite this as

$$G(k) = \frac{1}{k^2+m_0^2+\delta m_0^2}. \tag{14.21}$$

We conclude that the mass squared in the interacting theory is

$$m^2 = m_0^2 + \delta m_0^2. \tag{14.22}$$

The next natural thing to do is compare the measured m^2 to this result and find a relation constraining m_0^2 and λ_0.

It is here we encounter the serious trouble with continuum field theory: δm_0^2 *is quadratically divergent in the ultraviolet*:

$$\delta m_0^2 = \frac{1}{2} \int_0^\infty \frac{\lambda_0}{k'^2+m_0^2} \frac{d^4k'}{(2\pi)^4}. \tag{14.23}$$

So no matter how small λ_0 is, the change in mass δm_0^2 is infinite. The infinity comes from working in the continuum with no limit on the momenta in Fourier expansions. The theory seems incapable of describing the experiment with a finite m, assuming m_0 and λ_0 are finite.

Let us set this aside and compute the scattering amplitude, to compare it with experiment to constrain m_0^2 and λ_0.

14.4 Scattering Amplitude and the Γ's

We must clearly begin with the correlation of four fields, two each for the incoming and outgoing particles. The momenta are positive flowing inwards and there is no difference between particles and antiparticles.The correlation function $G(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4)$ is depicted in

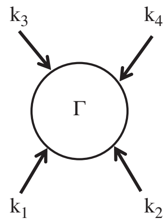

Figure 14.2 The scattering amplitude Γ is a function of the particle momenta, all chosen to point inwards. Their vector sum is zero.

Figure 14.2 and is defined as follows after pulling out the momentum-conserving δ function:

$$\langle\phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\phi(\boldsymbol{k}_3)\phi(\boldsymbol{k}_4)\rangle = (2\pi)^4\delta^4(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4). \tag{14.24}$$

To lowest order in λ_0, we get, upon pairing the four external ϕ's with the four ϕ's in the $\lambda_0\phi^4$ interaction,

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = G(\boldsymbol{k}_1)G(\boldsymbol{k}_2)G(\boldsymbol{k}_3)G(\boldsymbol{k}_4)\lambda_0. \tag{14.25}$$

However, $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is not the scattering amplitude which we should square to get the cross section. The four external propagators do not belong there. (In Minkowski space, the propagators will diverge because $k^2 = m^2$.) The scattering amplitude $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is defined as follows:

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = G(\boldsymbol{k}_1)G(\boldsymbol{k}_2)G(\boldsymbol{k}_3)G(\boldsymbol{k}_4)\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4). \tag{14.26}$$

That is,

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = G^{-1}(\boldsymbol{k}_1)G^{-1}(\boldsymbol{k}_2)G^{-1}(\boldsymbol{k}_3)G^{-1}(\boldsymbol{k}_4)G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4). \tag{14.27}$$

To lowest order,

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda_0. \tag{14.28}$$

Do we really need to bring in another function $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ if it is just $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ with the four external legs chopped off? Actually, we could get by with just the $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$'s, but in doing so would miss some important part of quantum field theory (QFT). First, $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is not alone, it is part of a family of functions, as numerous as the G's. That is, there are entities $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ for all n. They provide an alternate, equally complete, description of the theory to the G's, just like the Hamiltonian formalism is an alternative to the Lagrangian formalism. They are better suited than the $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$'s for discussing renormalization. And they are not just $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$'s with the external legs amputated.

In view of time and space considerations, I will digress briefly to answer just two questions:

- Where do the Γ's come from?
- What are the Feynman diagrams that contribute to them?

Consider the partition function $Z(J)$ with a source:

$$Z(J) = \int [\mathcal{D}\phi]\, e^{-S} e^{\int J(x)\phi(x)dx} \equiv e^{-\boldsymbol{W}(J)}. \tag{14.29}$$

The *generating functional* $\boldsymbol{W}(J)$ yields $G_{\mathrm{c}}(x_1,\ldots,x_n)$ upon repeated differentiation by $J(x)$, where the subscript c stands for *connected*:

$$\boldsymbol{W}(J) = -\int \frac{dx_1 \cdots dx_n}{n!} G_{\mathrm{c}}(x_1,\ldots,x_n) J(x_1)\cdots J(x_n). \tag{14.30}$$

In particular,

$$\bar{\phi}(x) \equiv \langle \phi(x) \rangle = -\frac{\partial \boldsymbol{W}}{\partial J(x)}. \tag{14.31}$$

(It is understood here and elsewhere that the derivatives are taken at $J = 0$.) Taking one more derivative gives

$$\langle \phi(x)\phi(y) \rangle_{\mathrm{c}} = -\frac{\partial^2 \boldsymbol{W}}{\partial J(x)\partial J(y)} = G_{\mathrm{c}}(x,y). \tag{14.32}$$

Given this formalism, in which $\boldsymbol{W}(J)$ is a functional of J and $\bar{\phi}$ is its derivative, it is natural to consider a Legendre transform to a functional $\boldsymbol{\Gamma}(\bar{\phi})$ with J as its derivative. By the familiar route one follows to go from the Lagrangian to the Hamiltonian or from the energy to the free energy, we are led to

$$\boldsymbol{\Gamma}(\bar{\phi}) = \int J(y)\bar{\phi}(y)dy + \boldsymbol{W}(J). \tag{14.33}$$

By the usual arguments,

$$\frac{\partial \boldsymbol{\Gamma}(\bar{\phi})}{\partial \bar{\phi}(y)} = J(y). \tag{14.34}$$

The Taylor expansion

$$\boldsymbol{\Gamma}(\bar{\phi}) \stackrel{\text{def}}{=} \int \frac{dx_1 \cdots dx_n}{n!} \Gamma(x_1,\ldots,x_n)\bar{\phi}(x_1)\cdots\bar{\phi}(x_n) \tag{14.35}$$

defines the Γ's with n arguments. A similar expansion in terms of $\bar{\phi}(\boldsymbol{k})$ defines $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$.

Given this definition, and a lot of work, one can show that $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ will have the following diagrammatic expansion:

- Draw the connected diagrams that contribute to $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ with the same incoming lines, except for those diagrams that can be split into two disjoint parts by cutting just one internal line. For this reason the Γ's are called 1*PI* or *one-particle irreducible* correlation functions.
- Append a factor $G^{-1}(\boldsymbol{k})$ for every incoming particle of momentum $\boldsymbol{k}$.

To get acquainted with this formalism, let us derive the relation between $\Gamma(k)$ and $G(k)$ that it implies. Given that $J(x)$ and $J(y)$ are independent, it follows that

$$\delta(x-y) = \frac{\partial J(x)}{\partial J(y)} \tag{14.36}$$

$$= \frac{\partial^2 \mathbf{\Gamma}}{\partial J(y)\partial\bar{\phi}(x)} \tag{14.37}$$

$$= \frac{\partial^2 \mathbf{\Gamma}}{\partial\bar{\phi}(x)\partial J(y)} \tag{14.38}$$

$$= \int dz \frac{\partial^2 \mathbf{\Gamma}}{\partial\bar{\phi}(x)\partial\bar{\phi}(z)} \frac{\partial\bar{\phi}(z)}{\partial J(y)} \tag{14.39}$$

$$= -\int dz \frac{\partial^2 \mathbf{\Gamma}}{\partial\bar{\phi}(x)\partial\bar{\phi}(z)} \frac{\partial^2 \mathbf{W}}{\partial J(z)\partial J(y)} \tag{14.40}$$

$$= \int dy \Gamma(x,z) G(z,y), \tag{14.41}$$

which leads to the very interesting result that the matrices Γ and G with elements $\Gamma(x,z)$ and $G(z,y)$ are inverses:

$$\Gamma = G^{-1}. \tag{14.42}$$

This agrees with the rules given above for computing the two-point function $\Gamma(k)$ from $G(k)$: If we take the two-point function $G(k)$ and multiply by two inverse powers of $G(k)$ (one for each incoming line) we get $\Gamma(k) = G^{-1}(k)$.

Upon further differentiation with respect to $\bar{\phi}(\boldsymbol{k})$, one can deduce the relation between the G's and Γ's and the Feynman rules stated above.

14.4.1 Back to Coupling Constant Renormalization

Let us now return to the scattering amplitude $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$. To lowest order in λ_0,

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda_0. \tag{14.43}$$

It is $|\lambda_0|^2$ you must use to compute cross sections.

In general, $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ will depend on the external momenta. However, to this order in λ_0, we find Γ does not have any momentum dependence and coincides with the coupling λ_0 in the action.

As we go to higher orders, $\Gamma(0,0,0,0)$ will be represented by a power series in λ_0. *We will then define* $\Gamma(0,0,0,0)$ *as the coupling* λ*, not the* λ_0 *in the action.* This fixes the interaction strength completely. I am not saying that the external momenta vanish in every scattering event, but that in any one theory, given $\Gamma(0,0,0,0)$, a unique $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is given by Feynman diagrams.

The trick of comparing the observed scattering rate to the one calculated from Eq. (14.43) to extract λ_0 will work only if λ_0 is small and higher-order corrections are negligible. Let us assume that λ_0 is very small, just like in electrodynamics where the analog of $\lambda_0 \simeq \frac{1}{137}$.

We will now consider scattering to order λ_0^2, even though it is one order higher than the correction to m_0^2. The reason is that it is also given by a one-loop graph, as shown in Figure 14.3, and *the systematic way to organize perturbation theory is in the number of loops*. If we restore the $\frac{1}{\hbar}$ in front of the action, we will find (Exercise 14.4.1) that the tree-level diagram, which is zeroth order in the loop expansion, is of order $\frac{1}{\hbar}$ and that each additional loop brings in one more positive power of $\hbar$. The loop expansion is therefore an $\hbar$ expansion. (During Christmas, we have a tree in our house but no wreath on the door, making us Christians at tree level but not one-loop level.)

Exercise 14.4.1 *Introduce* $\hbar^{-1}$ *in front of the action and see how this modifies* G_0 *and* λ_0*. Look at the diagrams for* G *and* Γ *to one loop and see how the loop brings in an extra* $\hbar$*.*

The one-loop corrections to scattering are depicted in Figure 14.3. They correspond to the following expression:

$$\Gamma(0,0,0,0) = \lambda_0 - 3\lambda_0^2 \int_0^\infty \frac{1}{(k^2+m_0^2)^2} \frac{d^4k}{(2\pi)^4} \equiv \lambda_0 + \delta\lambda_0 \equiv \lambda. \tag{14.44}$$

This defines the coupling λ to next order.

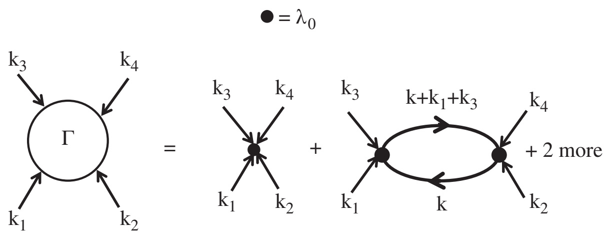

Figure 14.3 One-loop correction to Γ and $\lambda = \Gamma(0,0,0,0)$. Two more diagrams with external momenta connected to the vertices differently are not shown. They make the same contributions when external momenta vanish. The incoming arrows denote momenta and not propagators of that momentum (which have been amputated).

The factor of 3 comes from three loops with different routing of external momenta to the interaction vertices. Since all external momenta vanish, the graphs make identical contributions. *Unfortunately, $\delta\lambda_0$ is logarithmically divergent.*

14.5 Perturbative Renormalization

How do we reconcile these infinities in mass and coupling with the fact that actual masses and cross sections are finite? We employ the notion of *renormalization*.

First, we introduce a large momentum cut-off Λ in the loop integrals so that everything is finite but Λ-*dependent*:

$$\delta m_0^2(\Lambda) = \frac{\lambda_0}{2}\int_0^\Lambda \frac{1}{k^2+m_0^2}\frac{d^4k}{(2\pi)^4}, \tag{14.45}$$

$$\delta\lambda_0(\Lambda) = -3\lambda_0^2\int_0^\Lambda \frac{1}{(k^2+m_0^2)^2}\frac{d^4k}{(2\pi)^4}. \tag{14.46}$$

Then we identify the perturbatively corrected quantities with the measured ones. That is, we say

$$m^2 = m_0^2(\Lambda) + \delta m_0^2(\Lambda) \tag{14.47}$$

is the finite measured or *renormalized mass*, and that $m_0^2(\Lambda)$ is the *bare mass*, with an Λ-dependence chosen to ensure that m^2 equals the measured value. This means that we must choose

$$m_0^2(\Lambda) = m^2 - \frac{\lambda_0}{2}\int_0^\Lambda \frac{1}{k^2+m_0^2}\frac{d^4k}{(2\pi)^4} \tag{14.48}$$

$$= m^2 - \frac{\lambda}{2}\int_0^\Lambda \frac{1}{k^2+m^2}\frac{d^4k}{(2\pi)^4}, \tag{14.49}$$

where I have replaced the bare mass and coupling by the physical mass and coupling with errors of higher order.

Likewise, we must go back to Eq. (14.44) and choose

$$\lambda_0(\Lambda) = \lambda + 3\lambda_0^2\int_0^\Lambda \frac{1}{(k^2+m_0^2)^2}\frac{d^4k}{(2\pi)^4} \tag{14.50}$$

$$= \lambda + 3\lambda^2\int_0^\Lambda \frac{1}{(k^2+m^2)^2}\frac{d^4k}{(2\pi)^4}, \tag{14.51}$$

where I have replaced the bare mass squared by the physical mass squared and ł_0^2 by λ^2 with errors of higher order.

Equations (14.49) and 14.51 specify the requisite bare mass $m_0^2(\lambda, m, \Lambda)$ and bare coupling $\lambda_0(\lambda, m, \Lambda)$ corresponding to the experimentally determined values of λ and m

for any given Λ. If we choose the bare parameters as above, we will end up with physical mass and coupling that are finite and independent of Λ, *to this order*.

What about the scattering amplitude for non-zero external momenta? What about its divergences? We find that

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda_0 - \lambda_0^2 \left[\int_0^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m_0^2)(|\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_3|^2+m_0^2)} \right.$$
$$\left. + \text{ two more contributions} \right] \tag{14.52}$$

is logarithmically divergent as $\Lambda \to \infty$. Don't panic yet! We first replace m_0^2 by m^2 everywhere, due to the λ_0^2 in front of the integral. Next, we use Eq. (14.51) to replace the first λ_0 by

$$\lambda_0 = \lambda + 3\lambda^2 \int_0^\Lambda \frac{1}{(k^2+m^2)^2} \frac{d^4k}{(2\pi)^4}, \tag{14.53}$$

and the λ_0^2 in front of the integral by λ^2 (with errors of higher order), to arrive at

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda + \lambda^2 \left[\int_0^\Lambda \left[\frac{1}{(k^2+m^2)(|\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_3|^2+m^2)} - \frac{1}{(k^2+m^2)^2} \right] \frac{d^4k}{(2\pi)^4} \right.$$
$$\left. + \text{ two more contributions} \right]. \tag{14.54}$$

I have divided the $3\lambda^2$ term in Eq. (14.53) into three equal parts and lumped them with the three integrals in large square brackets.

The integrals are now convergent because as $k \to \infty$, the integrand in the diagram shown goes as

$$\frac{(q^2 + 2\boldsymbol{k}\cdot\boldsymbol{q})k^3}{k^6}, \tag{14.55}$$

where $\boldsymbol{q} = \boldsymbol{k}_1 + \boldsymbol{k}_3$ is the external momentum flowing in. Because the $\boldsymbol{k}\cdot\boldsymbol{q}$ term does not contribute due to rotational invariance, the integrand has lost two powers of k due to renormalization. The other two diagrams are also finite for the same reason. In short, *once* $\Gamma(0,0,0,0)$ *is rendered finite, so is* $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$.

The moral of the story is that, *to one-loop order*, the quantities considered so far are free of divergences when written in terms of the renormalized mass and coupling.

14.6 Wavefunction Renormalization

However, at next order a new kind of trouble pops up that calls for more renormalization. I will describe this in terms of

$$\Gamma(k) = G^{-1}(k). \tag{14.56}$$

In free-field theory,

$$\Gamma(k) = k^2 + m_0^2 \tag{14.57}$$

and, to one-loop order (consult Figure 14.4(a) and (b)),

$$\Gamma(k) = k^2 + m_0^2 + \delta m_0^2, \tag{14.58}$$

where δm_0^2 is the one-loop contribution that we encountered in Eq. (14.20).

To next order in the loop expansion, we see two more diagrams shown in Figure 14.4(c) and (d). (Check that the two-loop diagram has one more power of $\hbar$ than the one-loop diagram.) Let us represent their (divergent) contributions as follows:

$$\Gamma(k) = k^2 + m_0^2 + \delta m_0^2 + \lambda_0^2 A(m_0, \Lambda) + \lambda_0^2 B(m_0, \Lambda, k). \tag{14.59}$$

The term $\lambda_0^2 A(m_0, \Lambda)$, being k-independent, makes a contribution to mass renormalization and we can deal with it as before.

By contrast, B depends on the external momentum k and is given, up to constants, by

$$B(m_0^2, \Lambda, k) = \int_0^\Lambda \frac{d^4k_1 d^4k_2}{(k_1^2 + m_0^2)(k_2^2 + m_0^2)(|k_1 + k_2 + k|^2 + m_0^2)}. \tag{14.60}$$

Consider the expansion of B in a series in k^2. The zeroth-order term $\lambda_0^2 B(m_0^2, \Lambda, 0)$ also contributes to mass renormalization.

The next term, proportional to k^2, modifies the k^2 term from free-field theory:

$$k^2 \to k^2 + \lambda_0^2 \frac{dB(m_0, \Lambda, k^2)}{dk^2}\bigg|_0 k^2 \equiv k^2 \left(1 + c\lambda_0^2 \ln \frac{\Lambda^2}{m_0^2}\right) \equiv k^2 Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right), \tag{14.61}$$

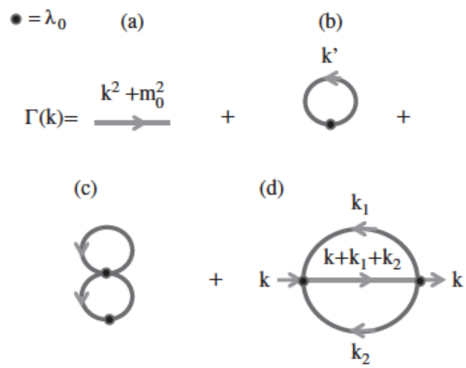

Figure 14.4 (a) $\Gamma(k)$ in free-field theory. (b) One-loop correction to m_0^2. (c) A k-independent two-loop correction $\lambda_0^2 A$ that renormalizes the mass. (d) A k-dependent two-loop correction $\lambda_0^2 B$ that renormalizes the k^2 part. The external legs have been amputated.

where c is some constant and I have introduced the *field renormalization factor*:

$$Z^{-\frac{1}{2}}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right) = \left(1 + c\lambda_0^2 \ln \frac{\Lambda^2}{m_0^2}\right)^{\frac{1}{2}}. \tag{14.62}$$

Because Z^{-1} diverges, the k^2 term in $\Gamma(k)$ now has a divergent coefficient.

Let us first handle this divergence and then interpret our actions. We begin with

$$\Gamma(k) = Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right)k^2 + k\text{-independent term } m_1^2 + \mathcal{O}(k^4). \tag{14.63}$$

Multiplying both sides by Z, we arrive at

$$Z\Gamma(k) = k^2 + Zm_1^2 + \mathcal{O}(k^4) \equiv k^2 + m^2 + \mathcal{O}(k^4), \tag{14.64}$$

where we have finally defined the quantity m^2 that is identified with the experimentally measured renormalized mass to this order.

The renormalized function

$$\Gamma_{\mathrm{R}} = Z\Gamma \tag{14.65}$$

now has a finite value and finite derivative at $k^2 = 0$:

$$\Gamma_{\mathrm{R}}(0) = m^2, \tag{14.66}$$

$$\left.\frac{d\Gamma_{\mathrm{R}}(k^2)}{dk^2}\right|_{k^2=0} = 1. \tag{14.67}$$

What does $\Gamma \to \Gamma_{\mathrm{R}}$ imply for G? Since $\Gamma = G^{-1}$, it follows that the *renormalized propagator*

$$G_{\mathrm{R}}(k) = Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right)G(k) \tag{14.68}$$

is divergence free. As Z is independent of momentum we may also assert that the Fourier transform to real space given by

$$G_{\mathrm{R}}(\boldsymbol{r}) = Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right)G(\boldsymbol{r}) \tag{14.69}$$

is also divergence free. But

$$G(\boldsymbol{r}) = \langle \phi(\boldsymbol{r})\phi(0)\rangle, \tag{14.70}$$

which means that

$$G_{\mathrm{R}}(\boldsymbol{r}) = \langle Z^{-\frac{1}{2}}\phi(\boldsymbol{r})Z^{-\frac{1}{2}}\phi(0)\rangle \equiv \langle \phi_{\mathrm{R}}(\boldsymbol{r})\phi_{\mathrm{R}}(0)\rangle \tag{14.71}$$

is divergence free. Above, we have defined a *renormalized field*

$$\phi_R = Z^{-\frac{1}{2}}\phi \tag{14.72}$$

in coordinate or momentum space, which has divergence-free correlations when everything is expressed in terms of renormalized mass and coupling (except for the unavoidable momentum-conservation δ-function in front of $G(k)$). One refers to Eq. (14.72) as *field renormalization*.

Several questions arise at this point:

- Since our original task was to compute correlations of ϕ, what good is it to have correlations of ϕ_R, even if the latter are finite?
- Renormalization looks like a Ponzi scheme, wherein we keep shoving problems to higher and higher orders. How many more new infinities will arise as we go to higher orders in λ_0 and k^2 and consider correlation functions of more than two fields? Will all the infinities be removed by simply renormalizing the mass, coupling, and field?

As to the first point, it turns out that the overall scale of ϕ does not affect any physical quantity: one will infer the same particle masses and physical scattering matrix elements before and after rescaling. This is not obvious, and I will not try to show that here.

As for the second set of points, it is the central claim of renormalization theory that no more quantities need to be renormalized (though the amount of renormalization will depend on the order of perturbation theory), and that the renormalized correlation function of rescaled fields

$$\phi_R = Z^{-\frac{1}{2}}\phi, \tag{14.73}$$

expressed in terms of the renormalized mass and coupling,

$$G_R(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,m,\lambda) = Z^{-M/2}G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,m_0,\lambda_0,\Lambda), \tag{14.74}$$

are finite and independent of Λ as $\Lambda \to \infty$.

(New divergences arise if the spatial arguments of any two or more ϕ's in G_R coincide to form the operators like ϕ^2. We will not discuss that here.)

The proof of renormalizability is very complicated. To anyone who has done the calculations, it is awesome to behold the cancellation of infinities in higher-loop diagrams as we rewrite everything in terms of quantities renormalized at lower orders. It seems miraculous and mysterious.

While all this is true for the theory we just discussed, ϕ^4 interaction in $d=4$, referred to as ϕ^4_4, there are also *non-renormalizable theories*. For example, if we add a ϕ^6 interaction in $d=4$, the infinities that arise cannot be fixed by renormalizing any finite number of parameters. Here it should be borne in mind that in quantum field theory one adds this term with a coefficient, $\lambda_6 = w_6/\mu^2$, where μ is some fixed mass (say 1 GeV) introduced to define a dimensionless w_6. In the post-Wilson era one adds the ϕ^6 term with coupling $\lambda_6 = w_6/\Lambda^2$, which is more natural. Its impact is benign and will be explained later.

What is the diagnostic for renormalizability? The answer is that *any interaction that requires a coupling constant with inverse dimensions of mass is non-renormalizable*. The couplings of ϕ^2 and ϕ^4 have dimensions m^2 and m^0, while a ϕ^6 coupling would have dimension m^{-2} in $d = 4$. These dimensions are established (in units of $\hbar = 1 = c$) by demanding that the kinetic term $\int(\nabla\phi)^2 d^d x$ be dimensionless and using that to fix the dimension of ϕ as

$$[\phi(x)] = \left(\frac{d}{2} - 1\right). \tag{14.75}$$

I invite you to show that

$$[\lambda] = 4 - d, \tag{14.76}$$

which means that λ is marginal in $d = 4$ and renormalizable in $d < 4$. Likewise, try showing that λ_6, the coupling for the ϕ^6 interaction, has dimension

$$[\lambda_6] = 6 - 2d, \tag{14.77}$$

which makes it non-renormalizable in $d = 4$ but renormalizable for $d \leq 3$.

You must have noticed the trend: *The renormalizable couplings are the ones which are relevant or marginal at the Gaussian fixed point.*

That the Gaussian fixed point plays a central role is to be expected in all old treatments of QFT because they were based on perturbation theory about the free-field theory. These topics are treated nicely in many places; a sample [1–6] is given at the end of this chapter. The relation between relevance and renormalizability can be readily understood in Wilson's approach to renormalization, which I will now describe. His approach gives a very transparent non-perturbative explanation of the "miracle" of canceling infinities in renormalizable theories.

14.7 Wilson's Approach to Renormalizing QFT

Compared to the diagrammatic and perturbative proof of renormalization in QFT, Wilson's approach [7, 8] is simplicity itself.

Recall our goal: to define a QFT in the continuum with the following properties:

- All quantities of physical significance – correlation functions, masses, scattering amplitudes, and so on – must be finite.
- There should be no reference in the final theory to a lattice spacing a or an ultraviolet momentum cut-off Λ.

Of course, at intermediate stages a cut-off will be needed and the continuum theory will be defined as the $\Lambda \to \infty$ limit of such cut-off theories.

Wilson's approach is structured around a fixed point of the RG. Every relevant direction will yield an independent parameter.

It is assumed that we know the eigenvectors and eigenvalues of the flow near this fixed point.

Even if we cannot find such fixed points explicitly, the RG provides a *framework* for understanding renormalizability, just as it provides a framework for understanding critical phenomena and demystifying universality in terms of flows, fixed points, scaling operators, and so on, even without explicit knowledge of these quantities.

Consider a scalar field theory. By assumption, we are given complete knowledge of a fixed point action S^* that lives in some infinite-dimensional space of *dimensionless couplings* such as r_0, u_0, and so forth. The values of these couplings are what we previously referred to as $\boldsymbol{K}^*$. Let the fixed point have one relevant direction, labeled by a coordinate t. As t increases from 0, the representative point moves from S^* to $S^* + tS_{\text{rel}}$, where S_{rel} is the relevant perturbation, a particular combination of ϕ^2,ϕ^4, and so on. Once we go a finite distance from S^* the flow may not be along the direction of the relevant eigenvector at S^*, but along its continuation, a curve called the *renormalized trajectory* (RT).

Let us say that our goal is to describe physics in the 1 GeV scale using a continuum theory. (In terms of length, 1 GeV corresponds to roughly 1 fermi, a natural unit for nuclear physics. More precisely, $1\,\text{GeV} \cdot 1\text{fermi} \simeq 5 \simeq 1$ in units $\hbar = c = 1$.) Although we limit our interest to momenta within the cut-off of 1 GeV, we want the correlations to be exactly those of an underlying theory with a cut-off that approaches infinity, a theory that knows all about the extreme-short-distance physics. The information from very short distances is not discarded, but encoded in the renormalized couplings that flow under the RG.

Notice the change in language: we are speaking of a very large cut-off Λ. We are therefore using laboratory units in contrast to the Wilsonian language in which the cut-off is always unity. (For example, when we performed decimation, the new lattice size a served as the unit of length in terms of which the dimensionless correlation length ξ was measured.)

To make contact with QFT, we too will carry out the following discussion *in fixed laboratory units*. In these units the allowed momenta will be reduced from a huge sphere of radius Λ GeV to smaller and smaller spheres of radius Λ/s GeV. The surviving momenta will range over smaller and smaller values, and they will be a small subset of the original set $k < \Lambda$.

We have had this discussion about laboratory versus running units before in discussing the continuum limit of a free-field theory. If we want the continuum correlation to fall by $1/e$ over a distance of 1 fermi, we fix the two points a fermi apart in the continuum and overlay lattices of smaller and smaller sizes a. As $a \to 0$, the number of lattice sites within this 1 fermi separation keeps growing and the dimensionless correlation length has to keep growing at the same rate to keep the decay to $1/e$.

So, we are not going to rescale momenta as modes are eliminated. How about the field? In the Wilson approach the field gets rescaled even in free-field theory because k gets rescaled to $k' = sk$. We will not do that anymore. However, we will rescale by the factor Z introduced in connection with the renormalized quantities Γ_{R} and G_{R}. This Z was needed in perturbation theory to avert a blow-up of the k^2 term in Γ due to the loop correction. [Recall the appearance of Z in the two-loop diagram, Eq. (14.61)]. In the Wilsonian RG there will also be a correction to the k^2 term from loop diagrams (now integrated over the eliminated modes), and these will modify the coefficient of the k^2 term. We will bring in a Z to keep the coefficient of k^2 fixed at 1. The reason is not to cancel divergences, for there are none, but because *the strength of the interaction is measured relative to the free-field term*. For example, in a ϕ^4 theory if we rescale $\phi(x)$ by 5 this will boost the coefficients ϕ^2 and $(\nabla\phi)^2$ by 25 and that of the quartic term by 625. But it is still the same theory. For this reason, to compare apples to apples, one always rescales the k^2 coefficient to unity, *even if there are no infinities*.

Let us now begin the quest for the continuum theory.

Say we want a physical mass of 1 GeV or a correlation length of 1 fermi. First we pick a point t_0 on the RT where the dimensionless correlation length $\xi_0 = 2^0 = 1$, as indicated in Figure 14.5. We refer to the action at t_0 as $S(0)$.

No cut-off or lattice size has been associated with the point t_0, since everything is dimensionless in Wilson's approach. All momenta are measured in units of the cut-off, and the cut-off is unity at every stage in the RG. We now bring in *laboratory units* and assign to t_0 a momentum cut-off of $\Lambda_0 = 2^0 = 1$ GeV.

What is the mass corresponding to this ξ_0 in GeV? For this, we need to recall the connection between ξ and m:

$$G(r) \simeq e^{-mr} = \exp\left[-\frac{r}{a\xi}\right] = \exp\left[-\frac{r\Lambda}{\xi}\right], \tag{14.78}$$

which means that the mass is related to the cut-off and ξ as follows:

$$m = \frac{\Lambda}{\xi}. \tag{14.79}$$

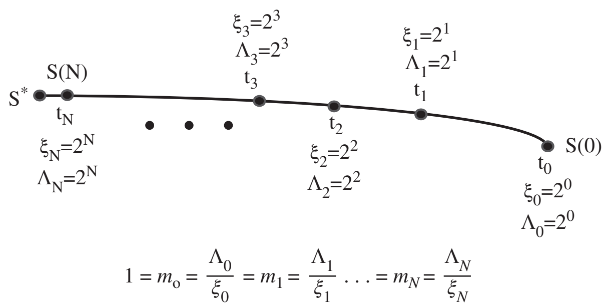

Figure 14.5 Points on the renormalized trajectory emanating from the fixed point S^*. To end up at the theory with cut-off $\Lambda_0 = 1$ GeV and action $S(0)$ after N RG steps of factor of 2 reduction of Λ, we must begin with the point labeled N, cut-off $\Lambda_N = 2^N$ GeV, $\xi_N = 2^N$ (dimensionless), and action $S(N)$. The sequence of points $S(N)$, $N \to \infty$ defines the continuum limit.

Thus, the mass corresponding to $S(0)$ is

$$m_0 = \frac{\Lambda_0}{\xi_0} = \frac{1\,\text{GeV}}{1} = 1\,\text{GeV}. \tag{14.80}$$

Imagine that we got to the point t_0 by performing N RG steps of size 2, starting with the point t_N where $\xi_N = 2^N$ and $\Lambda_N = 2^N$ GeV. *At every stage, the dimensionful mass is* 1 *GeV:*

$$m_N = \frac{\Lambda_N}{\xi_N} = 1. \tag{14.81}$$

Thus we have a sequence of actions, $S(n) : n = 0, 1, \ldots, N$, defined on smaller and smaller length scales or larger and larger momentum cut-offs, which produce the requisite physical mass. Not only is the mass fixed, the complete interaction is fixed to be $S(0)$. We have reverse-engineered it so that the theory at 1 GeV stays fixed at $S(0)$ while the underlying theory is defined on a sequence of actions $S(N)$ for which $\xi_N = 2^N$, and cut-off 2^N GeV, with $N \to \infty$. *We can make N as large as we like because ξ diverges as we approach S^*.*

We have managed to renormalize the theory by providing for each cut-off 2^N an action $S(t_N) \equiv S(N)$ that yields the theory $S(0)$ at low energies. This is the *continuum limit*.

This discussion also makes it obvious how to obtain a theory with a cut-off of 2 GeV: we just stop the RG one step earlier, at $S(1)$.

We can be more explicit about the continuum limit by invoking our presumed knowledge of ν. Near the fixed point we know that

$$\xi = t^{-\nu}. \tag{14.82}$$

This means that

$$2^N = t_N^{-\nu} \tag{14.83}$$

$$t_N = 2^{-N/\nu}, \tag{14.84}$$

which specifies the bare coupling or action $S(t_N) \equiv S(N)$ as a function of the cut-off $\Lambda_N = 2^N$ and the critical exponent ν. Just to be explicit: the bare action for cut-off $\Lambda = 2^N$ GeV is $S = S^* + 2^{-N/\nu} S_{\text{rel}}$, where S_{rel} is the relevant eigenoperator (some linear combination of ϕ^2, ϕ^4, etc.) that moves us along the RT starting at S^*.

We have managed to send the cut-off of the underlying theory to 2^N GeV with $N \to \infty$ holding fixed the action $S(0)$ for a theory with a cut-off of 1 GeV, but we need more. We need to ensure that not only does the low-energy action have a limit $S(0)$, as $\Lambda_N \to \infty$, but so do all the M-point correlation functions $G(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_M)$ defined by

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\cdots\phi(\boldsymbol{k}_M)\rangle = (2\pi)^d \delta\left(\sum_i \boldsymbol{k}_i\right) G(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_M). \tag{14.85}$$

Since we measure momentum in fixed laboratory units, the surviving momenta and fields $\phi(k)$ in the $\Lambda_0 = 1$ GeV theory are a subset of the momenta and fields in the underlying $\Lambda_N = 2^N$ GeV theory.

This may suggest that

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)). \tag{14.86}$$

However, Eq. (14.86) is incorrect. The reason is that the fields that appear in $S(0)$ are different from the ones we began with in $S(N)$, because we rescale the field to keep the coefficient of the k^2 term fixed in the presence of higher-loop corrections.

So, at every RG step we define a renormalized ϕ_{R} as follows:

$$\phi_{\mathrm{R}}(\boldsymbol{k}) = Z^{-\frac{1}{2}}\phi(\boldsymbol{k}), \tag{14.87}$$

and write S in terms of that field. If there are N steps in the RG the same equation would hold, with Z being the product of the Z's from each step. *So, the fields entering* $S(0)$ *are rescaled versions of the original fields entering* $S(N)$.

This means that, for the M-point correlation,

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = Z(N)^{\frac{M}{2}}G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)), \tag{14.88}$$

where $Z(N)$ is the net renormalization factor after N RG steps starting with cut-off 2^N.

Look at the $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N))$ on the left-hand side. This is the correlation function of a theory with a growing cut-off. The coupling is chosen as a function of cut-off that grows like 2^N. If G is finite as $N\to\infty$, we have successfully renormalized. The equation above expresses G as the product of two factors. The second factor is a correlation function evaluated in a theory with action $S(0)$ which remains fixed as $N\to\infty$ by construction. It has a finite non-zero mass and a finite cut-off, and is thus free of ultraviolet and infrared divergences. So we are good there. But, this need not be true of the Z-factor in front, because it is the result of (product over) Z's from N steps, with $N\to\infty$. Let us take the Z factor to the left-hand side:

$$Z(N)^{-\frac{M}{2}}G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)). \tag{14.89}$$

The left-hand side is now finite as $N\to\infty$, namely $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0))$. In other words, *the correlation functions of the renormalized fields are finite and cut-off independent as the cut-off approaches* ∞. This is the continuum limit.

In this approach it is obvious how, by choosing just one coupling (the initial value t_N of the distance from the fixed point along the RT) as a function of the cut-off ($\Lambda = 2^N$), we have an expression for finite correlation functions computed in terms of the finite renormalized interaction $S(0)$. Renormalizability is not a miracle if we start with an RG fixed point with a relevant coupling (or couplings) and proceed as above.

14.7.1 Possible Concerns

You may have some objections or concerns at this point.

What about $t<0$? Is there not a flow to the left of S^*? There is, and it defines another continuum theory. In the magnetic case the two sides would correspond to the ordered and disordered phases. However, the rest of the discussion would be similar. (There are some cases, like Yang–Mills theory, where the fixed point is at the origin and the region of negative coupling is unphysical [9, 10].)

You may object that we have found a smooth limit for the correlation of the renormalized fields, whereas our goal was to find the correlations of the original fields. Have we not found a nice answer to the wrong question? No. As mentioned earlier (without proof), the physical results of a field theory – masses, scattering amplitudes, and so on – are unaffected by such a k- and x-independent rescaling of the fields. So what we have provided in the end are finite answers to all physical questions pertaining to the low-energy physics in the continuum.

Another very reasonable objection is that the preceding diagram and discussion hide one important complexity. Even though the flow along the RT is one-dimensional, it takes place in an infinite-dimensional space of all possible couplings. As we approach the fixed point S^* along the RT, we have to choose the couplings of an infinite number of terms like the ϕ^2, ϕ^4, ϕ^6, $\phi^2(\nabla\phi)^2$, and so on of the short-distance interaction. This seems impractical. It also seems to have nothing to do with standard renormalization, where we vary one or two couplings to banish cut-off dependence.

14.7.2 Renormalization with Only Relevant and Marginal Couplings

We resolve this by bringing in the irrelevant directions and seeing what they do to the preceding analysis. Look at Figure 14.6.

Besides the RT, I show one irrelevant trajectory that flows into the fixed point. This is a stand-in for the *entire multidimensional critical surface*, which includes every critical system of this class. Somewhere in the big $\boldsymbol{K}$ space is an axis describing a simple coupling, which I call r_0. It could be the nearest-neighbor coupling K of an Ising model or some combination of the elementary couplings $r_0\phi^2$ and $u_0\phi^4$ of a scalar field theory which can be varied to attain criticality. We will see how to define the continuum limit by taking a sequence of points on the r_0 axis.

Though the interaction is simple, we can hit criticality by varying its strength. The critical point, where the r_0 axis meets the critical surface, is indicated by r^*.

Now, r^* is a *critical* point while S^* is a *fixed point*. The two differ by irrelevant terms. This means that the correlation functions at r^* will not have the scaling forms of S^* in general. To see the ultimate scaling forms associated with the fixed point S^*, we do not have to renormalize: if we evaluate the correlation functions at r^* in the limit $k \to 0$ or $r \to \infty$, they will exhibit these laws. For example, at the Ising critical point, $G(k) \simeq 1/k^{2-\eta}$ will result as $k \to 0$, or $G(r) \simeq 1/r^{\frac{1}{4}}$ will follow as $r \to \infty$, despite being formulated on a lattice with just the symmetry of a square.

Of course, we can understand this in terms of the RG. If we limit ourselves to $k \to 0$, we are permitted to trade our initial theory with a large Λ for one with $\Lambda \simeq k$, which is related by RG flow to S^*.

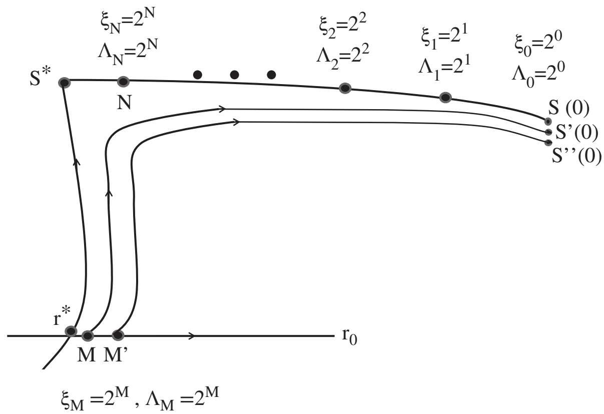

Figure 14.6 Flow with one relevant direction (the RT) and one irrelevant direction, which is a stand-in for the entire critical surface. The axis labeling the simple coupling r_0 (which could stand for $r_0\phi^2$) cuts the critical surface at r^*. Look at the points on the trajectory emanating from the point M on the r_0 axis. At point M, $\Lambda_M = 2^M$ and $\xi_M = 2^M$. We will end up at the theory with cut-off $\Lambda_0 = 1$ GeV and action $S'(0)$ after M RG steps of factor of 2 reduction of Λ. The sequence of points $S(M)$, $M \to \infty$ defines the continuum limit defined using just a single simple relevant coupling like r_0. If we start at M' we will reach $S''(0)$ (equivalent in the infrared to $S(0)$ and $S'(0)$) after $M-1$ steps. This is how one renormalizes in quantum field theory, by choosing simple couplings as a function of cut-off. The coupling M' corresponds to $\Lambda = 2^{M-1}$.

To define the continuum theory starting on this axis corresponding to a simple coupling, we pick a point M such that after M RG steps (of powers of 2) we arrive at the point $S'(0)$ that differs from $S(0)$, the theory generated from S^*, by a tiny amount *in the irrelevant direction*. The tiny irrelevant component will vanish asymptotically, and even when it is non-zero will make negligible corrections in the infrared. This result is inevitable given the irrelevance of the difference between r^* and S^*. We can go to the continuum limit by starting closer and closer to the critical surface (raising M) and reaching the target $S'(0)$ after more and more steps. As $M \to \infty$, our destination $S'(0)$ will coalesce with $S(0)$, which lies on the RT.

As a concrete example, consider Figure 13.4. Look at the dotted line parallel to the r_0 axis that comes straight down and crosses the critical line joining the Gaussian and WF fixed points. By starting closer and closer to the critical point where the dotted line crosses the critical line, we can renormalize the continuum theory based on the WF fixed point. The flow will initially flow toward the WF fixed point, and eventually will run alongside the RT. We can arrange to reach a fixed destination on the RT (the analog of $S(0)$) by starting at the appropriate distance from the critical line. You can also vary u_0 at fixed (negative) r_0 to approach the critical line with the same effect.

Now we can see the answer to a common question: how does a field theorist manage to compensate for a change in cut-off by renormalizing (i.e., varying with Λ) one or two couplings, whereas in Wilson's scheme, it takes a change in an infinite number of couplings? In other words, when we flow along the RT, i.e., vary one parameter t, we are actually varying an infinite number of elementary couplings in $\boldsymbol{K}$-space. How can a field theorist achieve the same result varying one or two couplings? The answer is that the field theorist does not really compensate for all the changes a changing cut-off produces. This is simply impossible. Whereas in Wilson's approach all correlation functions right up to the cut-off are preserved under the RG, in the field theory, *only correlations in the limit* $k/\Lambda \to 0$ *are preserved.*

Let us dig a little deeper into this. Suppose we begin at the point M, where $\Lambda = 2^M$, and reach the point $S'(0)$ in the figure after M RG steps of size 2. Say we ask what bare coupling with a cut-off 2^{M-1} will reproduce the answers of M with $\Lambda = 2^M$. It does not exist in general. Suppose, however, that we ask only about correlations in the infrared limit, $k/\Lambda \to 0$. Now we may trade the initial couplings for those on the RG trajectory. The point M flows to $S'(0)$ after M steps, i.e., when $\Lambda = 1$. The difference between $S(0)$ and the $S'(0)$ are technically and literally irrelevant in the infrared limit. If we start on the r_0 axis at M', at a suitably chosen point a little to the right of M, we can, after $M-1$ steps, reach the point $S''(0)$ that agrees with $S(0)$ and $S'(0)$ up to irrelevant corrections. It follows that if we reduce the cut-off by 2 we must change M to M', and if we increase the cut-off by 2 we must change M' to M. In other words, for each cut-off 2^M there is a point on the r_0 axis that has the same long-distance physics as the point M does with $\Lambda = 2^M$. This is how one renormalizes in QFT.

In QFT, one does not apologize for considering only the limit $k/\Lambda \to 0$ because there, Λ is an artifact that must be sent to ∞ at the end. So, $k/\Lambda \to 0 \; \forall \, k$.

Suppose I add a tiny irrelevant coupling, say $w_6\phi^6$, to the simple interaction of the starting point M. (Imagine the point is shifted slightly out of the page by w_6.) After M steps, the representative point again has to end up close to the RT. It may now end up slightly to the left or right of $S'(0)$ (ignore the component outside the page, which must have shrunk under the RG). Say it is to the right. This is what would have happened had we started with no w_6 but with a slightly bigger r_0 (a little to the right of M). A similar thing is true if the end point with w_6 in the mix is to the left of S'_0. In either case, *the effect of an irrelevant perturbation is equivalent to a different choice of the initial relevant coupling.*

It is understood above that w_6 is finite in units of the cut-off, and hence is very small in laboratory units, scaling as Λ^{-2} in $d = 4$. Had it been of order μ^{-2}, where μ is some fixed mass, it would not have been possible to absorb its effects by renormalization because it could correspond to an infinite perturbation in the natural units, namely Λ. But this is what field theorist tend to do in declaring it a non-renormalizable theory.

14.8 Theory with Two Parameters

Consider next the Gaussian fixed point in $d < 4$ when it has *two* relevant directions. Look at the flow in Figure 14.7. A generic point near the fixed point (the origin) will run away

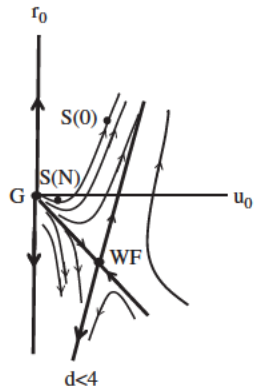

Figure 14.7 The situation in $d < 4$ when the Gaussian fixed point has two relevant directions. One can arrange to end up with a continuum theory with action $S(0)$, containing *two* free parameters, by starting closer and closer to the origin on the RT that passes through the point $S(0)$. Two bare couplings will have to be tuned, based on two relevant exponents that describe their growth under RG.

along the curves shown. We can make any point on any of those flow lines our destination $S(0)$ describing the continuum theory with 1 GeV cut-off, and arrange to get there after N RG steps by a suitable choice of initial coordinates $S(N)$ close to the origin. The flow away from the fixed point will now be controlled by two eigenvalues. We get a two-parameter family of continuum theories here. (The discussion near the fixed point is in terms of the simple interactions $r_0\phi^2$ and $u_0\phi^4$ because $\boldsymbol{K}^*$ is at the origin.)

In the relevant space of the Gaussian fixed point, there is a line connecting it to the Wilson–Fisher fixed point WF. If you pick a generic point on that line you will flow to WF and end up with its exponents. (In laboratory units, such a starting point will have a very large dimensionful coupling $\lambda = \Lambda^{4-d}u_0$.) But there is a way to fight that flow to WF: start closer and closer to G in such a way that after N steps you reach a fixed destination on the line. That would be a continuum theory that is massless but has one free parameter. (This is not a *natural* theory because the bare coupling is unnaturally small, being of order μ^{4-d}, where μ is some fixed mass, rather than of order Λ^{4-d}.)

14.8.1 Triviality of ϕ_4^4

Finally, consider the ϕ_4^4 theory based on the Gaussian fixed point that has one relevant coupling (mass or r_0) and interaction u_0 which is marginal at tree level but flows logarithmically slowly to the Gaussian fixed point at one loop. For this reason, this is not a suitable fixed point for constructing an interacting theory in the continuum. But suppose we try anyway. Since we can always make a theory massive, let us focus on getting an interacting field theory. So we begin with a point on the marginally attractive direction depicted in Figure 14.8.

Figure 14.8 The flow of coupling in ϕ_4^4. The origin is marginally attractive. To end up at some $u(1)$ in a theory with, say, a 1 GeV cut-off, we need to begin at *larger* bare values $u_0(\Lambda)$, which in fact diverge as $\Lambda \to \infty$. It has been shown numerically by Wilson that any $u(\Lambda)$, including $u(\Lambda) = \infty$, flows to the origin, rendering the continuum theory trivial. The only way to define an interacting ϕ_4^4 is to construct one based on a strong coupling fixed point u^*. If u^* is the bare coupling, it will not move under the RG and define an interacting massless theory. We can also arrange to end up with u at a fixed distance to the left or right of u^* by starting out closer and closer to it in a way determined by the relevant eigenvalue at u^*.

We can parametrize this point by u_0 and assume r_0 is adjusted to put us on the critical line. Say we want a final coupling $u(1)$ in a 1 GeV cut-off theory. Let this target coupling be in the weak coupling regime where we have established marginal irrelevance. To get to $u(1)$ in the long-distance theory, we need to begin with a *larger* bare value $u(\Lambda)$ because the coupling is irrelevant. In perturbation theory, the desired bare value grows without limit as $\Lambda \to \infty$ (see Eq. (14.51)), rendering perturbation theory meaningless. The only legitimate way to construct an interacting ϕ_4^4 theory is to base it on a fixed point u^*, if we can find one. If we begin there at the bare level, we will stay there under the RG and define a massless interacting theory. We could also begin slightly to its left so as to end at our target value $u(1)$ after N steps, starting closer and closer to u^* as $N \to \infty$. This would be an interacting theory. (We can also get a theory with a fixed u to the right of u^*.) So now we are back to relevant flow coming out of the strong coupling fixed point u^*. However, Wilson has verified by thorough numerical analysis that such a fixed point does not exist anywhere on the u axis, including the point at infinity. The general consensus now is that ϕ_4^4 is trivial, i.e., non-interacting.

14.9 The Callan–Symanzik Equation

I will now provide a very brief introduction to this equation due to Callan [11] and Symanzik [12]. It is used extensively in quantum field theory as well as critical phenomena. It is mostly used to study the behavior of correlation functions in some extreme kinematical region: large momenta to describe asymptotic freedom in QCD [9, 10] or small momenta to describe critical phenomena. In the latter case it is the only practical way to deal with higher orders in the ε expansion.

14.9.1 Basis for the Callan–Symanzik Equation

Recall that in the Wilson approach, by construction, a theory with a cut-off Λ and couplings $\boldsymbol{K}(1)$ is equivalent to a theory with a cut-off Λ/s and couplings $\boldsymbol{K}(s)$ as long we ask questions below the new cut-off Λ/s. (We use laboratory units in which the cut-off shrinks by a factor s and momenta are not rescaled.) Correlation functions are, however, not invariant under this change of cut-off due to the change in the scale of the field to keep the k^2 term fixed after every iteration. The original ϕ we started with is related to the ϕ_R that appears in the theory with the new cut-off as

$$\phi(\boldsymbol{k}) = Z^{\frac{1}{2}}\phi_R(\boldsymbol{k}). \tag{14.90}$$

Consequently,

$$Z(N)^{-\frac{M}{2}} G(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_M, S(N)) = G(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_M, S(0)), \tag{14.91}$$

where the action $S(0)$ and the corresponding coupling $\boldsymbol{K}(0)$ are reached after N RG steps of cut-off reduction by 2.

The Callan–Symanzik equation is derived in quantum field theory from a similar relation which, however, holds only in the limit $\Lambda \to \infty$, or more precisely $k/\Lambda \to 0$, where k is any fixed momentum. The reason for the restriction is that a cut-off change can be compensated by changing a handful of (marginal and relevant) couplings only in this limit, in which irrelevant corrections vanish as positive powers of k/Λ. The Callan–Symanzik equation is not limited to the study of correlation functions as $\Lambda \to \infty$ in QFT. We can also use it in critical phenomena where Λ is some finite number $\Lambda \simeq 1/a$, provided we want to study the limit $k/\Lambda \to 0$, i.e., at distances far greater than the lattice size a. All that is required in both cases is that $k/\Lambda \to 0$.

We begin with the central claim of renormalization theory that the correlations of

$$\phi_R = Z^{-\frac{1}{2}}\phi, \tag{14.92}$$

expressed in terms of the renormalized mass and coupling,

$$G_R(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_M, m, \lambda) = \lim_{\Lambda\to\infty} Z^{-M/2}(\lambda_0, \Lambda/m_0) G(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_M, m_0(\Lambda), \lambda_0(\Lambda), \Lambda), \tag{14.93}$$

are finite and independent of Λ.

For a theory with a mass m we have seen that the renormalized *inverse* propagator Γ and four-point amplitude $\Gamma_R(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4)$ can be made to obey

$$\Gamma_R(0) = m^2, \tag{14.94}$$

$$\left.\frac{d\Gamma_R(k)}{dk^2}\right|_{k=0} = 1, \tag{14.95}$$

$$\Gamma_R(0,0,0,0) = \lambda. \tag{14.96}$$

We are going to study a critical (massless) theory in what follows. Although we can impose

$$\Gamma_R(0) = 0 \tag{14.97}$$

to reflect zero mass, we cannot impose Eqs. (14.95) and (14.96). This is because in a massless theory both these quantities have infrared divergences at $k = 0$. These are physical, just like the diverging Coulomb cross section. So we pick some point $k = \mu > 0$ where these quantities can be finite, and demand that

$$\left.\frac{d\Gamma_{\mathrm{R}}(k)}{dk^2}\right|_{k=\mu} = 1, \tag{14.98}$$

$$\Gamma_{\mathrm{R}}(\mu,\mu,\mu,\mu) = \lambda = \mu^{\varepsilon} u_{\mathrm{R}}. \tag{14.99}$$

This calls for some explanation.

First, μ is arbitrary, and any choice of μ can be used to specify a theory. If you change μ you will have to change λ accordingly if you want to describe the same theory.

Next, we are working in $d = 4 - \varepsilon$ dimensions, where λ has dimension ε. It is expressed as the product of a dimensionless parameter u_{R} and the factor μ^{ε}, which restores the right engineering dimension.

Finally, $\Gamma(\mu,\mu,\mu,\mu)$ is a schematic: it stands for a symmetric way to choose the momenta all of the scale μ:

$$\boldsymbol{k}_i \cdot \boldsymbol{k}_j = \frac{\mu^2}{3}(4\delta_{ij} - 1). \tag{14.100}$$

We will not need this expression from now on.

It is to be noted that the theory is not renormalizable in $d = 4 - \varepsilon$ due to the power-law infrared divergences that arise. However, if we expand everything in a double series in u and ε, the infinities (which will be logarithmic) can be tamed order by order, i.e., renormalized away. This double expansion will be understood from now on.

14.9.2 Massless $M = 2$ Correlations in $d = 4 - \varepsilon$

I will illustrate the Callan–Symanzik approach with the case $M = 2$, that is, two-point correlations, and study just the critical (massless) case in $d = 4 - \varepsilon$. Consider the system at point P in Figure 14.9 lying on the critical line joining the Gaussian and WF fixed points. It has a cut-off Λ and a coordinate $u(\Lambda) \equiv u$. We are interested in $\Gamma(k,u,\Lambda)$ in the limit $k/\Lambda \to 0$. We cannot use simple perturbation theory, even if u is small, because the expansion parameter will turn out to be $u \ln \frac{\Lambda}{k}$. The trick is to move the cut-off to a value of the order of k, thereby avoiding large logarithms, and work with the coupling $u(k)$ rather than $u = u(\Lambda)$. It is during this cut-off reduction that the coupling will flow from $u(\Lambda)$ to $u(k)$. We expect that $u(k) \to u^*$, the WF fixed point, as $k \to 0$.

It is convenient to work with the *inverse* propagator $\Gamma = G^{-1}$, which obeys

$$\Gamma_{\mathrm{R}}(k,u_{\mathrm{R}},\mu) = \lim_{\Lambda\to\infty} \Big[Z^1(u(\Lambda),\Lambda/\mu)\Gamma(k,u(\Lambda),\Lambda) \Big]. \tag{14.101}$$

The key to the Callan–Symanzik equation approach is the observation that since the left-hand side is independent of Λ (in the limit $\Lambda \to \infty$), so must be the right-hand side,

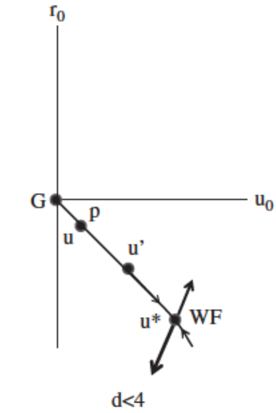

Figure 14.9 We want $\Gamma(k,u(\Lambda),\Lambda)$ as $k \to 0$ at a point P with coupling $u(\Lambda) = u$. The RG flow takes us to the WF fixed point u^* via the point u'.

which means

$$\lim_{\Lambda\to\infty} \Lambda \frac{d}{d\Lambda}\left[Z(u(\Lambda),\Lambda/\mu)\Gamma(k,u(\Lambda),\Lambda)\right] = 0. \tag{14.102}$$

Writing out the explicit and implicit Λ derivatives, we find that

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u,\Lambda/\mu)\frac{\partial}{\partial u} - \gamma(u,\Lambda/\mu)\right]\Gamma(k,u(\Lambda),\Lambda) = 0, \tag{14.103}$$

where

$$\beta(u,\Lambda/\mu) = \Lambda\frac{\partial u(\Lambda)}{\partial\Lambda}\bigg|_{\mu,u_R}, \tag{14.104}$$

$$\gamma(u,\Lambda/\mu) = -\Lambda\frac{\partial \ln Z(u(\Lambda),\Lambda/\mu)}{\partial\Lambda}\bigg|_{\mu,u_R}. \tag{14.105}$$

Next, we argue that since μ does not enter Γ, it cannot enter the dimensionless functions γ or β, which must therefore be functions only of $u(\Lambda)$. Thus we arrive at the *Callan–Symanzik equation*:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u)\frac{\partial}{\partial u} - \gamma(u)\right]\Gamma(k,u(\Lambda),\Lambda) = 0. \tag{14.106}$$

The solution, derived by the method of characteristics, is

$$\Gamma(k,u(\Lambda_1),\Lambda_1) = \exp\left[\int_{\ln\Lambda_2}^{\ln\Lambda_1}\gamma(u(\ln\Lambda))d\ln\Lambda\right]\Gamma(k,u(\Lambda_2),\Lambda_2). \tag{14.107}$$

The solution is readily understood in Wilson's picture. The correlation function with cut-off Λ_2 is the same as that with Λ_1, provided we use the renormalized coupling in going from Λ_1 to Λ_2 and account for the field rescaling factor Z. Imagine doing the mode elimination in stages. Each stage will contribute a factor to Z, and the final Z will be a product of the Z's in each step depending on the coupling u at that stage. We reason as follows:

$$\Gamma(\Lambda_1)Z(\Lambda_1) = \Gamma(\Lambda_2)Z(\Lambda_2) = \Gamma_{\mathrm{R}} \tag{14.108}$$

$$\Gamma(\Lambda_1) = \frac{Z(\Lambda_2)}{Z(\Lambda_1)}\Gamma(\Lambda_2) \tag{14.109}$$

$$= e^{(\ln Z(\Lambda_2)-\ln Z(\Lambda_1))}\Gamma(\Lambda_2) \tag{14.110}$$

$$= \exp\left[\int_{\ln\Lambda_1}^{\ln\Lambda_2} \frac{d\ln Z}{d\ln\Lambda} d\ln\Lambda\right]\Gamma(\Lambda_2) \tag{14.111}$$

$$= \exp\left[\int_{\ln\Lambda_2}^{\ln\Lambda_1} \gamma(u(\ln\Lambda)) d\ln\Lambda\right]\Gamma(\Lambda_2), \text{ with} \tag{14.112}$$

$$\gamma = -\frac{d\ln Z}{d\ln\Lambda}. \tag{14.113}$$

We verify that the solution Eq. (14.107) satisfies Eq. (14.106) by taking $\Lambda_1 \frac{\partial}{\partial\Lambda_1}$ of both sides:

$$\Lambda_1 \frac{\partial\Gamma(k,u(\Lambda_1),\Lambda_1)}{\partial\Lambda_1} + \beta(u(\Lambda_1))\frac{\partial\Gamma(k,u(\Lambda_1),\Lambda_1)}{\partial u(\Lambda_1)} = \gamma(u(\ln\Lambda_1))\Gamma(k,u(\Lambda_1),\Lambda_1). \tag{14.114}$$

Sometimes Eq. (14.107) is written in terms of an integral over the running coupling $u(\Lambda)$:

$$\Gamma(k,u(\Lambda_1),\Lambda_1) = \exp\left[\int_{u_2\equiv u(\Lambda_2)}^{u_1\equiv u(\Lambda_1)} \gamma(u)\frac{du}{\beta(u)}\right]\Gamma(k,u(\Lambda_2),\Lambda_2). \tag{14.115}$$

This version comes in handy if the integral over u is dominated by a zero of the β-function. We will have occasion to use it.

14.9.3 Computing the β-Function

The first step in using the Callan–Symanzik equation is the computation of β, which we will do to one loop. We begin with the renormalization condition,

$$u_{\mathrm{R}}\mu^{\varepsilon} = \Lambda^{\varepsilon}\left[u(\Lambda) - \frac{3u^2(\Lambda)}{16\pi^2}\ln\frac{\Lambda}{\mu}\right], \tag{14.116}$$

where the right-hand side was encountered earlier for the case $d = 4$ where $\varepsilon = 0$. Now we have to introduce the Λ^{ε} in front as part of the definition of the coupling. Setting to zero the $\ln\Lambda$-derivative of both sides (at fixed μ and $u_{\rm R}$), we have (keeping only terms of order εu and u^2),

$$0 = \varepsilon u(\Lambda) + \underbrace{\frac{du(\Lambda)}{d\ln\Lambda}}_{\beta(u)} - \frac{3u^2(\Lambda)}{16\pi^2}. \tag{14.117}$$

(We anticipate that β will be of order εu or u^2, and do not take the $\ln\Lambda$-derivative of the $3u^2$ term, for that would lead to a term of order u^3 or $u^2\varepsilon$.) The result is

$$\beta(u) = -\varepsilon u + \frac{3u^2}{16\pi^2}. \tag{14.118}$$

The way β is defined, as Λ increases (more relevant to QFT), u flows toward the origin, while if Λ decreases (more relevant to us), it flows away and hits a zero at

$$u^* = \frac{16\varepsilon\pi^2}{3}. \tag{14.119}$$

That is,

$$\beta(u^*) = 0. \tag{14.120}$$

This is the WF fixed point. For future use, note that the slope of the β-function at the fixed point is

$$\omega = \left.\frac{d\beta(u)}{du}\right|_{u^*} = -\varepsilon + \frac{6u^*}{16\pi^2} = \varepsilon. \tag{14.121}$$

This irrelevant exponent $\omega = \varepsilon$ determines how quickly we approach the fixed point as we lower the cut-off. Here are the details.

14.9.4 Flow of $u - u^*$

Let us write a variable cut-off as

$$\Lambda(s) = \frac{\Lambda}{s}, \qquad s > 1. \tag{14.122}$$

It follows that

$$\frac{d}{d\ln\Lambda} = -\frac{d}{d\ln s}. \tag{14.123}$$

The coupling

$$u(s) \equiv u(\Lambda/s) \tag{14.124}$$

flows as follows:

$$\frac{du(s)}{d\ln s} = -\frac{du(\Lambda)}{d\ln\Lambda} = \varepsilon u(s) - \frac{3u_s^2}{16\pi^2} \equiv \bar{\beta}(u) = -\beta(u). \tag{14.125}$$

Integrating the flow of the coupling as a function of s, starting from $u(1) = u$, gives

$$\int_{u(1)=u}^{u(s)} \frac{du'}{\bar{\beta}(u')} = \ln s. \tag{14.126}$$

Now we expand $\bar{\beta}$ near the fixed point:

$$\bar{\beta}(u') = \bar{\beta}(u^*) - \omega(u' - u^*) = 0 - \omega(u' - u^*) = (-\omega)(u' - u^*). \tag{14.127}$$

(The minus in front of ω reflects the switch from β to $\bar{\beta} = -\beta$.) Substituting this into the previous equation, we get

$$\int_{u(1)=u}^{u(s)} \frac{du'}{(-\omega)(u' - u^*)} = \ln s, \tag{14.128}$$

with the solution

$$u(s) - u^* = (u - u^*)s^{-\omega} = (u - u^*)s^{-\varepsilon}. \tag{14.129}$$

That is, *the initial deviation from the fixed point* $(u - u^*)$ *shrinks by a factor* $s^{-\varepsilon} = s^{-\omega}$ *under the RG transformation* $\Lambda \to \Lambda/s$. Equation (14.129) will be recalled shortly.

14.9.5 Computing γ

The function γ begins at two loops. Armed with the two-loop result

$$Z = 1 + \frac{u^2}{6(4\pi)^4} \ln \frac{\mu}{\Lambda} + \cdots, \tag{14.130}$$

we find

$$\gamma = -\frac{d \ln Z}{d \ln \Lambda} = \frac{u^2}{6(4\pi)^4}. \tag{14.131}$$

At the fixed point

$$u^* = \frac{16\pi^2 \varepsilon}{3}, \tag{14.132}$$

we have

$$\gamma(u^*) \equiv \gamma^* = \frac{\varepsilon^2}{54}. \tag{14.133}$$

For later use, note that

$$\gamma' = \left.\frac{d\gamma}{du}\right|_{u^*} = \frac{\varepsilon}{144\pi^2}. \tag{14.134}$$

14.9.6 Computing $\Gamma(k,u,\Lambda)$

Now we are ready to confront the correlation function $\Gamma(k,u,\Lambda)$, which is the two-point function on the critical line shown in Figure 14.9. We want to know its behavior as a function of k as $k \to 0$. We expect it to be controlled by the WF fixed point.

The equation obeyed by $\Gamma(k,u,\Lambda)$ is

$$\left[-\frac{\partial}{\partial \ln s} - \bar{\beta}(u(s))\frac{\partial}{\partial u} - \gamma(u(s))\right]\Gamma(k,u(s),\Lambda/s) = 0. \tag{14.135}$$

Suppose we are at the fixed point, where $\bar{\beta} = 0$ and

$$\gamma = \gamma(u^*) \equiv \gamma^*. \tag{14.136}$$

The equation to solve is

$$\frac{\partial \Gamma(k,u^*,\Lambda/s)}{\partial \ln s} = -\gamma(u^*)\Gamma(k,u^*,\Lambda/s), \tag{14.137}$$

with an obvious solution

$$\Gamma(k,u^*,\Lambda) = s^{\gamma^*}\Gamma(k,u^*,\Lambda/s). \tag{14.138}$$

By dimensional analysis,

$$\Gamma(k,u^*,\Lambda/s) = k^2 f\left(\frac{k}{\Lambda/s}\right) = k^2 f\left(\frac{ks}{\Lambda}\right). \tag{14.139}$$

Substituting this into Eq. (14.138), we arrive at

$$\Gamma(k,u^*,\Lambda) = s^{\gamma^*}k^2 f\left(\frac{ks}{\Lambda}\right). \tag{14.140}$$

Now we choose

$$s = \frac{\Lambda}{k}, \tag{14.141}$$

which just means

$$\frac{\Lambda}{s} = k, \tag{14.142}$$

i.e., the new cut-off equals the momentum of interest. With this choice,

$$\Gamma(k,u^*,\Lambda) = \left(\frac{\Lambda}{k}\right)^{\gamma^*} k^2 f(1) \simeq k^{2-\gamma^*}. \tag{14.143}$$

Comparing to the standard form

$$\Gamma(k) \simeq k^{2-\eta}, \tag{14.144}$$

we find that

$$\eta = \gamma(u^*) \equiv \gamma^*. \tag{14.145}$$

In case you wondered how $\Gamma(k)$ can go as $k^{2-\eta}$ when it has engineering dimension 2, the answer is given above: $k^{-\eta}$ is really $\left(\frac{k}{\Lambda}\right)^{-\eta}$.

Finally, we ask how subleading corrections to the fixed point behavior arise if we start at some $u \neq u^*$ with a k that is approaching zero. For this, we return to the solution to the Callan–Symanzik equation

$$\Gamma(k, u(\Lambda_1), \Lambda_1) = \exp\left[\int_{\ln\Lambda_2}^{\ln\Lambda_1} \gamma(u(\ln\Lambda'))d\ln\Lambda'\right]\Gamma(k, u(\Lambda_2), \Lambda_2). \tag{14.146}$$

Let

$$\Lambda_1 = \Lambda, \tag{14.147}$$
$$\Lambda_2 = \Lambda/s, \tag{14.148}$$
$$u(\Lambda/s) \equiv u(s), \tag{14.149}$$
$$u(\Lambda) \equiv u(1). \tag{14.150}$$

Then

$$\Gamma(k, u(1), \Lambda) = \exp\left[\int_s^1 \gamma(u'(s'))\frac{-ds'}{s'}\right]\Gamma(k, u(s), \Lambda/s). \tag{14.151}$$

Corrections are going to arise from both the exponential factor and $\Gamma(k, u(s), \Lambda/s))$, due to the fact that at any non-zero $\frac{\Lambda}{s} = k$, the coupling $u(s)$ is close to, but not equal to, u^*, which is reached only asymptotically.

Consider first the exponential factor. Expanding γ near u^* as

$$\gamma(u') = \gamma^* + \gamma'(u' - u^*) + \cdots, \tag{14.152}$$
$$\gamma' = \frac{\varepsilon}{144\pi^2} \quad \text{[Eq. (14.134)]}, \tag{14.153}$$

we have, in the exponent,

$$\begin{aligned}\int_s^1 \gamma(u'(s'))\frac{-ds'}{s'} &= \int_1^s (\gamma^* + \gamma'(u(s') - u^*))\frac{ds'}{s'} \\ &= \gamma^* \ln s + \gamma'(u - u^*)\int_1^s (s')^{-\omega}\frac{ds'}{s'} \\ &= \gamma^* \ln s + \frac{\gamma'}{\omega}(u - u^*)(1 - s^{-\omega}).\end{aligned} \tag{14.154}$$

Thus the exponential factor becomes

$$\exp[\cdots] = s^{\gamma^*}\left(1+\frac{\gamma'}{\omega}(u-u^*)(1-s^{-\omega})\cdots\right). \tag{14.155}$$

Next, consider

$$\begin{aligned}\Gamma(k,u(s),\Lambda/s)) &= \Gamma(k,u^*+u(s)-u^*,\Lambda/s)\\ &= k^2 f\left[\frac{ks}{\Lambda},u^*+(u(s)-u^*)\right]\\ &= k^2 f\left[\frac{ks}{\Lambda},u^*+(u-u^*)s^{-\omega}\right].\end{aligned} \tag{14.156}$$

If we now set

$$s = \frac{\Lambda}{k} \tag{14.157}$$

and recall that $\omega = \varepsilon$, we find, upon putting the two factors in Eqs. (14.155) and (14.156) together, an irrelevant correction of the form $\left(\frac{k}{\Lambda}\right)^{\varepsilon}$:

$$\Gamma(k,u(\Lambda),\Lambda) = k^2\left(\frac{\Lambda}{k}\right)^{\gamma^*}\left(a+c\left(\frac{k}{\Lambda}\right)^{\varepsilon}\right), \tag{14.158}$$

where a and c are some constants.

14.9.7 Variations of the Theme

The preceding introduction was aimed at giving you an idea of how the Callan–Symanzik machine works by focusing on Γ, corresponding to two-particle correlations, and only for the critical case. There are so many possible extensions and variations.

The first variation is to go to the non-critical theory, where, in addition to the marginal coupling u, we have a relevant coupling, denoted by t, which as usual measures deviation from criticality. It multiplies the operator ϕ^2, whose presence calls for additional renormalization. The final result will be quite similar: as $k/\Lambda \to 0$, the flow will first approach the fixed point and then follow the renormalized trajectory.

Next, we can go from correlations of two fields to M fields and work with $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M)$.

Finally, let us go back to the relation between bare and renormalized Γ's:

$$\Gamma_{\mathrm{R}}(k,u_{\mathrm{R}}(\mu),\mu) = \lim_{\Lambda\to\infty} Z(u(\Lambda),\Lambda/\mu)\Gamma(k,u(\Lambda),\Lambda). \tag{14.159}$$

We got the Callan–Symanzik equation by saying that since the Γ_{R} on the left-hand side had no knowledge of Λ, i.e., was cut-off independent, we could set the $\ln\Lambda$-derivative of the right-hand side to zero. This equation describes how the bare couplings and correlations have to change with the cut-off to keep fixed some renormalized quantities directly related to experiment.

Instead, we could argue that since Γ does not know about μ, the $\ln\mu$-derivative of the left-hand side must equal the same derivative acting on just the Z on the right-hand side (which has been expressed as $Z(u_{\mathrm{R}}(\mu),\Lambda/\mu)$). The resulting equation,

$$\left[\frac{\partial}{\partial \ln\mu}+\beta\frac{\partial}{\partial u_{\mathrm{R}}}-\gamma\right]\Gamma_{\mathrm{R}}(k,u_{\mathrm{R}}(\mu),\mu)=0, \tag{14.160}$$

where

$$\beta(u_{\mathrm{R}})=\left.\frac{\partial u_{\mathrm{R}}}{\partial \ln\mu}\right|_{u(\Lambda),\Lambda}, \tag{14.161}$$

$$\gamma(u_{\mathrm{R}})=\left.\frac{\partial \ln Z}{\partial \ln\mu}\right|_{u(\Lambda),\Lambda}, \tag{14.162}$$

dictates how the renormalized coupling and correlations must change with μ in order to represent the same underlying bare theory. (Again, the dimensionless functions β and γ cannot depend on μ/Λ because they are determined by Γ_{R}, which does not know about Λ.)

We can use either approach to get critical exponents, flows, and Green's functions (because Γ and Γ_{R} differ by Z, which is momentum and position independent), but there are cultural preferences. In statistical mechanics, the bare correlations are physically significant and describe underlying entities like spins. The cut-off is real and given by $\Lambda \simeq 1/a$. To particle physicists, the cut-off is an artifact, and the bare Green's functions and couplings are crutches to be banished as soon as possible so that they can work with experimentally measurable, finite, renormalized quantities defined on the scale μ. They prefer the second version based on Γ_{R}.

References and Further Reading

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